

Real Dynamics of Family of Functions $\lambda \frac{xe^x}{e^x-1}$ for Positive λ

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Abstract: In this paper, the real dynamics of one parameter family of functions $h_\lambda(x) = \lambda \frac{xe^x}{e^x-1}$, $\lambda > 0$ is investigated. It is shown that the real fixed point of $h_\lambda(x)$ exists only for $0 < \lambda < 1$ which is attracting and there is no real fixed point for $\lambda > 1$. It is found that the whole real line converges to real attracting fixed point of $h_\lambda(x)$ for $0 < \lambda < 1$.

The dynamics of functions in complex plane using real dynamics near to real fixed points have been induced by many researchers. The fixed points play vital role to explore the dynamical behavior of functions. Some related investigation can be found in [1, 3, 4]. The theory of fixed points and dynamics or iteration of transcendental functions can be seen in [2].

A point x is said to be a fixed point of function $f(x)$ if $f(x) = x$. A fixed point x_0 is called an attracting, neutral (indifferent) or repelling if $|f'(x_0)| < 1$, $|f'(x_0)| = 1$ or $|f'(x_0)| > 1$ respectively.

In this paper, we study one parameter family of function $\frac{xe^x}{e^x-1}$ which arises from the generalized Bernoulli generating function, Apostol-Bernoulli generating function and Stirling generating function. To study this family of functions, we consider

$$H = \left\{ h_\lambda(x) = \lambda \frac{xe^x}{e^x-1} h_\lambda(0) = \lambda : \lambda > 0, x \in \mathbb{R} \right\}$$

The existence of real fixed point, their nature and real dynamics of the function $h_\lambda \in H$ are investigated in the present paper.

It is shown that the function $h_\lambda \in H$ has a unique real fixed point in the following theorem:

Theorem 1: Let $h_\lambda \in H$. Then, the function $h_\lambda(x)$ has a unique real fixed point x_λ for $\lambda < 1$ and no real fixed points for $\lambda > 1$. The fixed point x_λ is positive if $0 < \lambda < 1$ and negative if $\lambda < 0$.

Proof: For fixed points of $h_\lambda \in H$, $h_\lambda(x_\lambda) = x_\lambda$. The solution of this equation is $x_\lambda = -\ln(1-\lambda)$. The solution x_λ is real if $\lambda < 1$, otherwise solution is not real if $\lambda > 1$. Therefore, it follows that x_λ is a unique real fixed point for $\lambda < 1$. Moreover, if $0 < \lambda < 1$, then $x_\lambda > 0$ since $\ln(1-\lambda) < 0$ and if $\lambda < 0$, then $x_\lambda < 0$ since $\ln(1-\lambda) > 0$. Thus, the real fixed point of $h_\lambda(x)$ is positive for $0 < \lambda < 1$ and negative for $\lambda < 0$.

For $\lambda < 0$, the nature of fixed point of $h_\lambda(x)$ is already found in [5]. In the following theorem, the nature of fixed point of $h_\lambda \in H$, for $0 < \lambda < 1$, is determined:

Theorem 2: Let $h_\lambda \in H$. Then, the real fixed point x_λ of the function $h_\lambda(x)$ is attracting for $0 < \lambda < 1$.

Proof: Since x_λ is a real fixed point of $h_\lambda(x)$, then $h_\lambda(x_\lambda) = x_\lambda$ which gives $\lambda = \frac{e^{x_\lambda}-1}{e^{x_\lambda}}$

Using this, we have

$$h'_\lambda(x_\lambda) = \lambda \frac{(e^{x_\lambda} - x_\lambda - 1)e^{x_\lambda}}{(e^{x_\lambda} - 1)^2} = \frac{(e^{x_\lambda} - 1)}{e^{x_\lambda}} \frac{(e^{x_\lambda} - x_\lambda - 1)e^{x_\lambda}}{(e^{x_\lambda} - 1)^2} = 1 - \frac{x_\lambda}{e^{x_\lambda} - 1}$$

It follows that $h'_\lambda(x_\lambda) - 1 = -\frac{x_\lambda}{e^{x_\lambda} - 1} < 0$ since $x_\lambda > 0$. Therefore, $0 < h'_\lambda(x_\lambda) < 1$ since $h'_\lambda(x)$ is positive on \mathbb{R}^+ and consequently, the real fixed point x_λ of $h_\lambda(x)$ is attracting for $0 < \lambda < 1$

The dynamics of $h_\lambda \in H$ on real line is explored in the following theorem:

Theorem 3: Let $h_\lambda \in H$. Then, $h^n(x) \rightarrow x_\lambda$ for $x \in R$, where x_λ is an attracting fixed point of the function $h_\lambda(x)$ for $0 < \lambda < 1$.

Proof: Let $g_\lambda(x) = h_\lambda(x) - x$ for $x \in R$. It is easily seen that $g_\lambda(x)$ is continuously differentiable for $x \in R$. The fixed points of $h_\lambda(x)$ are zeros of $g_\lambda(x)$.

If $0 < \lambda < 1$, by Theorem 2, it follows that $h_\lambda(x)$ has an attracting fixed point x_λ . Since $g'_\lambda(x_\lambda) = h'_\lambda(x_\lambda) - 1 < 0$ and in a neighbourhood of x_λ the function $g'_\lambda(x)$ is continuous, $g'_\lambda(x) < 0$ in some neighbourhood of x_λ . Therefore, $g_\lambda(x)$ is decreasing in a neighbourhood of x_λ . By the continuity of $g_\lambda(x)$, for sufficiently small $\delta > 0$, $g_\lambda(x) > 0$ in $(x_\lambda - \delta, x_\lambda)$ and $g_\lambda(x) < 0$ in $(x_\lambda, x_\lambda + \delta)$. Since $g_\lambda(x) \neq 0$ in $(-\infty, x_\lambda) \cup (x_\lambda, \infty)$, it now follows that $g_\lambda(x) > 0$ in $(-\infty, x_\lambda)$ and $g_\lambda(x) < 0$ in (x_λ, ∞) . Thus, it is seen that

$$h_\lambda(x) - x \begin{cases} < 0 & \text{for } x \in (x_\lambda, \infty) \\ > 0 & \text{for } x \in (-\infty, x_\lambda) \end{cases} \tag{1}$$

By Equation (1), $h_\lambda(x) < x$ for $x \in (x_\lambda, \infty)$, $h_\lambda(x)$ is increasing and $h_\lambda(x) > 0$ for $x \in (x_\lambda, \infty)$, by continuing forward iterations, it follows that

$$x_\lambda < \dots < h_\lambda^n(x) < h_\lambda^{n-1}(x) < \dots < h_\lambda^2(x) < h_\lambda(x) < x$$

Therefore, the $\{h_\lambda^n(x)\}$ sequence is decreasing and bounded below by x_λ . Hence $h_\lambda^n(x) \rightarrow x_\lambda$ as $n \rightarrow \infty$ for $x \in (x_\lambda, \infty)$.

Next, by Equation (1), $h_\lambda(x) > x$ for $x \in (0, x_\lambda)$ and $h_\lambda(x)$ is increasing for $x \in (0, x_\lambda)$, by continuing forward iterations, it shows that

$$x_\lambda > \dots > h_\lambda^n(x) > h_\lambda^{n-1}(x) > \dots > h_\lambda^2(x) > h_\lambda(x) > x$$

It gives that the $\{h_\lambda^n(x)\}$ sequence is increasing and bounded above by x_λ . Hence $h_\lambda^n(x) \rightarrow x_\lambda$ as $n \rightarrow \infty$ for $x \in (0, x_\lambda)$.

Further, since $h_\lambda(x)$ is increasing in the interval $(-\infty, 0)$ and $h_\lambda(x)$ maps the interval $(-\infty, 0)$ into $(0, x_\lambda)$ Consequently, $h_\lambda(x) \rightarrow x_\lambda$ as $n \rightarrow \infty$ for $x \in (-\infty, x_\lambda)$. Thus, $h_\lambda^n(x) \rightarrow x_\lambda$ as $n \rightarrow \infty$ for $x \in (-\infty, \infty)$.

References

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