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An Integral Representation for the Polylogarithm Function and Some Special Values

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Abstract. I proved a new integral representation for the polylogarithm function.

1. INTRODUCTION

Using an integral representation and unorthodox method of substitution, I demonstrated that:

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)}\int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)}du.$$

2. THEOREM

Theorem 1. *For* $\Re(z) < 1$ *or* $\Im(z) \neq 0$, *then*

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)}\int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)}du, \tag{1}$$

where $\mathrm{Li}_s(z)$ *denotes the polylogarithm function,* $\Gamma(s)$ *denotes the gamma function and* $\ln s$ *denotes the natural logarithm.*

Proof. In [1], the polylogarithm function is defined by

$$\mathrm{Li}_s(z) := \frac{z}{\Gamma(s)}\int_0^1 \frac{[\ln(1/t)]^{s-1}}{1-zt}dt, \tag{2}$$

for $\Re(s) > 0$.

Take $t = \frac{a+bu}{1+u}$ and $dt = \left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2}\right]du$ in (2), I obtain

$$\mathrm{Li}_s(z) := \frac{z}{\Gamma(s)}\int_{-\frac{a}{b}}^{-\frac{a-1}{b-1}} \left[\frac{\left[\ln\left(\frac{1+u}{a+bu}\right)\right]^{s-1}}{1-z\frac{a+bu}{1+u}}\right]\left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2}\right]du, \tag{3}$$

Let $-\frac{a-1}{b-1} = \tan(\pi\theta)$ and $-\frac{a}{b} = \cot(\pi\theta)$, therefore,

$$a = \frac{(\tan(\pi\theta)+1)\cot(\pi\theta)}{\cot(\pi\theta)-\tan(\pi\theta)}, \quad b = -\frac{\tan(\pi\theta)+1}{\cot(\pi\theta)-\tan(\pi\theta)}. \tag{4}$$

Put (4) in (3) and simplifying, I have

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)}\int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u,\theta)du, \tag{5}$$

where

$$f(u,\theta) = \frac{(-\cot(\pi\theta)-1)\ln^{s-1}\left(-\frac{(u+1)(\cot(\pi\theta)-1)}{u-\cot(\pi\theta)}\right)}{(u+1)(\cot(\pi\theta)(u-z+1)+u(z-1)-1)}.$$

Consider the limit $\theta \to 1$ in (5)

$$\lim_{\theta\to 1} \mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \lim_{\theta\to 1} \int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u,\theta)du, \tag{6}$$

then

$$\lim_{\theta\to 1} \mathrm{Li}_s(z) = \mathrm{Li}_s(z),$$

$$\lim_{\theta\to 1} f(u,\theta) = -\frac{\ln(1+u)^{s-1}}{(1+u)(1+u-z)},$$

$$\lim_{\theta\to 1} \tan(\pi\theta) = 0,$$

$$\lim_{\theta\to 1} \cot(\pi\theta) = \infty. \tag{7}$$

Hence, from (6) and (7), it follows that

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{\ln(1+u)^{s-1}}{(1+u)(1+u-z)} du . \square$$

Corollary 1. *For* $\Re(z) < 1$ *or* $\Im(z) \neq 0$, *then*

$$6\mathrm{Li}_2(z) = -\pi^2 + 3\ln^2(1-z) - 6\ln(1-z)\ln(-z) + 6\,\mathrm{Li}_2\left(\frac{1}{1-z}\right), \tag{8}$$

where $\mathrm{Li}_2(z)$ *denotes the dilogarithm function and* $\ln z$ *denotes the natural logarithm.*

Proof. I let $s = 2$ in Theorem 1. □

Special Values. Let $z = \frac{1}{2}$ in Corollary 1, then

$$6\mathrm{Li}_2\left(\frac{1}{2}\right) = -\pi^2 + 3\ln^2\left(\frac{1}{2}\right) - 6\ln\left(\frac{1}{2}\right)\ln\left(-\frac{1}{2}\right) + 6\,\mathrm{Li}_2(2). \tag{9}$$

On the other hand, in [2], I find

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}. \tag{10}$$

I substitute (10) in the left hand side of the (9), and encounter

$$\mathrm{Li}_2(2) = \frac{\pi^2}{4} - \mathrm{i}\pi\ln 2. \tag{11}$$

Let $z = \frac{2}{3}$ in Corollary 1, then

$$\mathrm{Li}_2(3) - \mathrm{Li}_2\left(\frac{2}{3}\right) = \frac{\pi^2}{6} + \frac{\ln^2 3}{2} - \ln 2\ln 3 - \mathrm{i}\,\pi\ln 3. \tag{12}$$

Let $z = \frac{3}{4}$ in Corollary 1, then

$$\mathrm{Li}_2(4) - \mathrm{Li}_2\left(\frac{3}{4}\right) = \frac{\pi^2}{6} + 2\ln^2 2 - 2\ln 2\ln 3 - 2\,\mathrm{i}\,\pi\ln 2. \tag{13}$$

From (13), I conclude that

$$\operatorname{Li}_2(4) = 2 \cdot {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1 \\ \frac{3}{2}, \frac{3}{2}\end{matrix}\right| \frac{1}{4}\right) - 2\,\mathrm{i}\,\pi\,\ln 2$$

$$= \frac{\pi^2}{6} - \ln^2 2 - 2\operatorname{Li}_2\left(-\tfrac{1}{2}\right) - 2\,\mathrm{i}\,\pi\,\ln 2 \tag{14}$$

and

$$\operatorname{Li}_2\left(\tfrac{3}{4}\right) = 2\ln 2\ln 3 - 3\ln^2 2 - 2\operatorname{Li}_2\left(-\tfrac{1}{2}\right). \tag{15}$$

Let $z = \frac{4}{5}$ in Corollary 1, then

$$\operatorname{Li}_2(5) - \operatorname{Li}_2\left(\tfrac{4}{5}\right) = \frac{\pi^2}{6} + \frac{\ln^2 5}{2} - 2\ln 2\ln 5 - \mathrm{i}\,\pi\,\ln 5\,, \tag{16}$$

Let $z = \frac{5}{6}$ in Corollary 1, then

$$\operatorname{Li}_2(6) - \operatorname{Li}_2\left(\tfrac{5}{6}\right) = \frac{\pi^2}{6} + \frac{\ln^2 6}{2} - \ln 5\ln 6 - \mathrm{i}\,\pi\,\ln 6. \tag{17}$$

Theorem 2. *For* $\Re(s) > 1$, *then*

$$\operatorname{Li}_2(z) + \operatorname{Li}_2\left(\tfrac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - \mathrm{i}\,\pi\ln z, \tag{18}$$

where $\operatorname{Li}_s(z)$ *denotes the polylogarithm function and* $\ln s$ *denotes the natural logarithm.*

Proof. In Theorem 1, the polylogarithm function is defined by

$$\operatorname{Li}_s(z) := \frac{z}{\Gamma(s)}\int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)}\,du, \tag{19}$$

for $\Re(s) > 0$.

Let $s = 2$; thus, the dilogarithm function is

$$\operatorname{Li}_2(z) = z\int_0^\infty \frac{\ln(1+u)}{(1+u)(1+u-z)}\,du. \tag{20}$$

On the other hand, in [3], I prove that

$$\ln z = (z-1)\int_0^\infty \frac{1}{(t+z)(t+1)}\,dt, \tag{21}$$

for $\Re(z) > 0$.

From (20) and (21), it follows that

$$\operatorname{Li}_2(z) = z\int_0^\infty \frac{u}{(1+u)(1+u-z)}\int_0^\infty \frac{1}{(1+u+t)(1+t)}\,dt\,du$$

$$= z\int_0^\infty \frac{1}{1+t}\int_0^\infty \frac{u}{(1+u)(1+u+t)(1+u-z)}\,du\,dt$$

$$= z\int_0^\infty \frac{1}{t+1}\frac{(t+1)z\ln(t+1) - t\,(z-1)\ln(1-z)}{t\,z\,(t+z)}\,dt$$

$$= z\int_0^\infty \frac{\ln(t+1)}{t\,(t+z)}\,dt - (z-1)\ln(1-z)\int_0^\infty \frac{1}{(t+1)(t+z)}\,dt$$

$$= \frac{2\pi^2 + 6\ln(z-1)\ln z - 3\ln^2 z - 6\mathrm{Li}_2\left(\frac{1}{z}\right)}{6} - \ln(1-z)\ln z$$

$$= \frac{\pi^2}{3} + [\ln(z-1) - \ln(1-z)]\ln z - \frac{\ln^2 z}{2} - \mathrm{Li}_2\left(\frac{1}{z}\right), \tag{22}$$

ergo,

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - \mathrm{i}\,\pi\ln z\,.\square$$

Theorem 2. *For* $\Re(s) > 1$, *then*

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - \mathrm{i}\,\pi\ln z, \tag{23}$$

where $\mathrm{Li}_s(z)$ *denotes the polylogarithm function and* $\ln s$ *denotes the natural logarithm.*

Proof. In Theorem 1, the polylogarithm function is defined by

$$\mathrm{Li}_s(z) := \frac{z}{\Gamma(s)}\int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)}\,du, \tag{24}$$

for $\Re(s) > 0$.

Let $s = 2$; thus, the dilogarithm function is

$$\mathrm{Li}_2(z) = z\int_0^\infty \frac{\ln(1+u)}{(1+u)(1+u-z)}\,du. \tag{25}$$

On the other hand, in [3], I prove that

$$\frac{\ln x}{x-1} = \int_0^1 \frac{t^{-1}}{(\ln t - x)(\ln t - 1)}\,dt. \tag{26}$$

for $\Re(z) > 1$. I set $x = 1 + u$ in (26), and obtain

$$\ln(1+u) = u\int_0^1 \frac{t^{-1}}{(\ln t - u - 1)(\ln t - 1)}\,dt. \tag{27}$$

From (25) and (27), it follows that

$$\mathrm{Li}_2(z) = z\int_0^\infty \frac{u}{(1+u)(1+u-z)}\int_0^1 \frac{t^{-1}}{(\ln t - u - 1)(\ln t - 1)}\,dt\,du$$

$$= z\int_0^1 \frac{t^{-1}}{\ln t - 1}\int_0^\infty \frac{u}{(1+u)(1+u-z)(\ln t - u - 1)}\,du\,dt$$

$$= z \int_0^\infty \frac{1}{t+1} \frac{(t+1)z \ln(t+1) - t\,(z-1)\ln(1-z)}{t\,z\,(t+z)}\,dt$$

$$= z \int_0^\infty \frac{\ln(t+1)}{t\,(t+z)}\,dt - (z-1)\ln(1-z) \int_0^\infty \frac{1}{(t+1)(t+z)}\,dt$$

$$= \frac{2\pi^2 + 6\ln(z-1)\ln z - 3\ln^2 z - 6\mathrm{Li}_2\left(\frac{1}{z}\right)}{6} - \ln(1-z)\ln z$$

$$= \frac{\pi^2}{3} + [\ln(z-1) - \ln(1-z)]\ln z - \frac{\ln^2 z}{2} - \mathrm{Li}_2\left(\frac{1}{z}\right), \tag{28}$$

ergo,

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - \mathrm{i}\,\pi \ln z\,.\square$$

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