

An Integral Representation for the Polylogarithm Function and Some Special Values

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Abstract. I proved a new integral representation for the polylogarithm function.

1. INTRODUCTION

Using an integral representation and unorthodox method of substitution, I demonstrated that:

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)} du.$$

2. THEOREM

Theorem 1. For $\Re(z) < 1$ or $\Im(z) \neq 0$, then

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)} du, \quad (1)$$

where $\text{Li}_s(z)$ denotes the polylogarithm function, $\Gamma(s)$ denotes the gamma function and $\ln s$ denotes the natural logarithm.

Proof. In [1], the polylogarithm function is defined by

$$\text{Li}_s(z) := \frac{z}{\Gamma(s)} \int_0^1 \frac{[\ln(1/t)]^{s-1}}{1-zt} dt, \quad (2)$$

for $\Re(s) > 0$.

Take $t = \frac{a+bu}{1+u}$ and $dt = \left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2} \right] du$ in (2), I obtain

$$\text{Li}_s(z) := \frac{z}{\Gamma(s)} \int_{-\frac{a}{b}}^{-\frac{a-1}{b-1}} \left[\frac{[\ln(\frac{1+u}{a+bu})]^{s-1}}{1-z\frac{a+bu}{1+u}} \right] \left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2} \right] du, \quad (3)$$

Let $-\frac{a-1}{b-1} = \tan(\pi\theta)$ and $-\frac{a}{b} = \cot(\pi\theta)$, therefore,

$$a = \frac{(\tan(\pi\theta)+1) \cot(\pi\theta)}{\cot(\pi\theta)-\tan(\pi\theta)}, \quad b = -\frac{\tan(\pi\theta)+1}{\cot(\pi\theta)-\tan(\pi\theta)}. \quad (4)$$

Put (4) in (3) and simplifying, I have

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u, \theta) du, \quad (5)$$

where

$$f(u, \theta) = \frac{(-\cot(\pi\theta) - 1) \ln^{s-1} \left(-\frac{(u+1)(\cot(\pi\theta) - 1)}{u - \cot(\pi\theta)} \right)}{(u+1)(\cot(\pi\theta)(u-z+1) + u(z-1) - 1)}.$$

Consider the limit $\theta \rightarrow 1$ in (5)

$$\lim_{\theta \rightarrow 1} \text{Li}_s(z) = \frac{z}{\Gamma(s)} \lim_{\theta \rightarrow 1} \int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u, \theta) du, \tag{6}$$

then

$$\begin{aligned} \lim_{\theta \rightarrow 1} \text{Li}_s(z) &= \text{Li}_s(z), \\ \lim_{\theta \rightarrow 1} f(u, \theta) &= -\frac{\ln(1+u)^{s-1}}{(1+u)(1+u-z)}, \\ \lim_{\theta \rightarrow 1} \tan(\pi\theta) &= 0, \\ \lim_{\theta \rightarrow 1} \cot(\pi\theta) &= \infty. \end{aligned} \tag{7}$$

Hence, from (6) and (7), it follows that

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{\ln(1+u)^{s-1}}{(1+u)(1+u-z)} du. \square$$

Corollary 1. For $\Re(z) < 1$ or $\Im(z) \neq 0$, then

$$6\text{Li}_2(z) = -\pi^2 + 3 \ln^2(1-z) - 6 \ln(1-z) \ln(-z) + 6 \text{Li}_2\left(\frac{1}{1-z}\right), \tag{8}$$

where $\text{Li}_2(z)$ denotes the dilogarithm function and $\ln z$ denotes the natural logarithm.

Proof. I let $s = 2$ in Theorem 1. \square

Special Values. Let $z = \frac{1}{2}$ in Corollary 1, then

$$6\text{Li}_2\left(\frac{1}{2}\right) = -\pi^2 + 3 \ln^2\left(\frac{1}{2}\right) - 6 \ln\left(\frac{1}{2}\right) \ln\left(-\frac{1}{2}\right) + 6 \text{Li}_2(2). \tag{9}$$

On the other hand, in [2], I find

$$\text{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}. \tag{10}$$

I substitute (10) in the left hand side of the (9), and encounter

$$\text{Li}_2(2) = \frac{\pi^2}{4} - i\pi \ln 2. \tag{11}$$

Let $z = \frac{2}{3}$ in Corollary 1, then

$$\text{Li}_2(3) - \text{Li}_2\left(\frac{2}{3}\right) = \frac{\pi^2}{6} + \frac{\ln^2 3}{2} - \ln 2 \ln 3 - i\pi \ln 3. \tag{12}$$

Let $z = \frac{3}{4}$ in Corollary 1, then

$$\text{Li}_2(4) - \text{Li}_2\left(\frac{3}{4}\right) = \frac{\pi^2}{6} + 2 \ln^2 2 - 2 \ln 2 \ln 3 - 2 i\pi \ln 2. \tag{13}$$

From (13), I conclude that

$$\begin{aligned} \text{Li}_2(4) &= 2 \cdot {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1 \\ 3, 3 \end{matrix} \middle| \frac{1}{4}\right) - 2i\pi \ln 2 \\ &= \frac{\pi^2}{6} - \ln^2 2 - 2 \text{Li}_2\left(-\frac{1}{2}\right) - 2i\pi \ln 2 \end{aligned} \quad (14)$$

and

$$\text{Li}_2\left(\frac{3}{4}\right) = 2 \ln 2 \ln 3 - 3 \ln^2 2 - 2 \text{Li}_2\left(-\frac{1}{2}\right). \quad (15)$$

Let $z = \frac{4}{5}$ in Corollary 1, then

$$\text{Li}_2(5) - \text{Li}_2\left(\frac{4}{5}\right) = \frac{\pi^2}{6} + \frac{\ln^2 5}{2} - 2 \ln 2 \ln 5 - i\pi \ln 5, \quad (16)$$

Let $z = \frac{5}{6}$ in Corollary 1, then

$$\text{Li}_2(6) - \text{Li}_2\left(\frac{5}{6}\right) = \frac{\pi^2}{6} + \frac{\ln^2 6}{2} - \ln 5 \ln 6 - i\pi \ln 6. \quad (17)$$

Theorem 2. For $\Re(s) > 1$, then

$$\text{Li}_2(z) + \text{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - i\pi \ln z, \quad (18)$$

where $\text{Li}_s(z)$ denotes the polylogarithm function and $\ln s$ denotes the natural logarithm.

Proof. In Theorem 1, the polylogarithm function is defined by

$$\text{Li}_s(z) := \frac{z}{\Gamma(s)} \int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)} du, \quad (19)$$

for $\Re(s) > 0$.

Let $s = 2$; thus, the dilogarithm function is

$$\text{Li}_2(z) = z \int_0^\infty \frac{\ln(1+u)}{(1+u)(1+u-z)} du. \quad (20)$$

On the other hand, in [3], I prove that

$$\ln z = (z-1) \int_0^\infty \frac{1}{(t+z)(t+1)} dt, \quad (21)$$

for $\Re(z) > 0$.

From (20) and (21), it follows that

$$\text{Li}_2(z) = z \int_0^\infty \frac{u}{(1+u)(1+u-z)} \int_0^\infty \frac{1}{(1+u+t)(1+t)} dt du$$

$$\begin{aligned}
 &= z \int_0^\infty \frac{1}{1+t} \int_0^\infty \frac{u}{(1+u)(1+u+t)(1+u-z)} du dt \\
 &= z \int_0^\infty \frac{1}{t+1} \frac{(t+1)z \ln(t+1) - t(z-1)\ln(1-z)}{t z (t+z)} dt \\
 &= z \int_0^\infty \frac{\ln(t+1)}{t(t+z)} dt - (z-1) \ln(1-z) \int_0^\infty \frac{1}{(t+1)(t+z)} dt \\
 &= \frac{2\pi^2 + 6 \ln(z-1) \ln z - 3 \ln^2 z - 6\text{Li}_2\left(\frac{1}{z}\right)}{6} - \ln(1-z) \ln z \\
 &= \frac{\pi^2}{3} + [\ln(z-1) - \ln(1-z)] \ln z - \frac{\ln^2 z}{2} - \text{Li}_2\left(\frac{1}{z}\right), \tag{22}
 \end{aligned}$$

ergo,

$$\text{Li}_2(z) + \text{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - i \pi \ln z. \square$$

Theorem 2. For $\Re(s) > 1$, then

$$\text{Li}_2(z) + \text{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - i \pi \ln z, \tag{23}$$

where $\text{Li}_s(z)$ denotes the polylogarithm function and $\ln s$ denotes the natural logarithm.

Proof. In Theorem 1, the polylogarithm function is defined by

$$\text{Li}_s(z) := \frac{z}{\Gamma(s)} \int_0^\infty \frac{[\ln(1+u)]^{s-1}}{(1+u)(1+u-z)} du, \tag{24}$$

for $\Re(s) > 0$.

Let $s = 2$; thus, the dilogarithm function is

$$\text{Li}_2(z) = z \int_0^\infty \frac{\ln(1+u)}{(1+u)(1+u-z)} du. \tag{25}$$

On the other hand, in [3], I prove that

$$\frac{\ln x}{x-1} = \int_0^1 \frac{t^{-1}}{(\ln t-x)(\ln t-1)} dt. \tag{26}$$

for $\Re(z) > 1$. I set $x = 1 + u$ in (26), and obtain

$$\ln(1+u) = u \int_0^1 \frac{t^{-1}}{(\ln t-u-1)(\ln t-1)} dt. \tag{27}$$

From (25) and (27), it follows that

$$\begin{aligned}
 \text{Li}_2(z) &= z \int_0^\infty \frac{u}{(1+u)(1+u-z)} \int_0^1 \frac{t^{-1}}{(\ln t-u-1)(\ln t-1)} dt du \\
 &= z \int_0^1 \frac{t^{-1}}{\ln t-1} \int_0^\infty \frac{u}{(1+u)(1+u-z)(\ln t-u-1)} du dt
 \end{aligned}$$

$$\begin{aligned}
&= z \int_0^{\infty} \frac{1}{t+1} \frac{(t+1)z \ln(t+1) - t(z-1)\ln(1-z)}{t z (t+z)} dt \\
&= z \int_0^{\infty} \frac{\ln(t+1)}{t(t+z)} dt - (z-1) \ln(1-z) \int_0^{\infty} \frac{1}{(t+1)(t+z)} dt \\
&= \frac{2\pi^2 + 6 \ln(z-1) \ln z - 3 \ln^2 z - 6\text{Li}_2\left(\frac{1}{z}\right)}{6} - \ln(1-z) \ln z \\
&= \frac{\pi^2}{3} + [\ln(z-1) - \ln(1-z)] \ln z - \frac{\ln^2 z}{2} - \text{Li}_2\left(\frac{1}{z}\right), \tag{28}
\end{aligned}$$

ergo,

$$\text{Li}_2(z) + \text{Li}_2\left(\frac{1}{z}\right) = \frac{\pi^2}{3} - \frac{\ln^2 z}{2} - i \pi \ln z. \square$$

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