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$f(R)$ view of Lyttleton-BondiCosmological Model in Peres Spacetime

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Abstract Over the last few years, among various alternatives to the Einstein theory of gravity, especially $f(R)$ theories of gravity have received more importance due to number of interesting results in cosmology and astrophysics. Pandey [10] gave an $f(R)$ theory of gravity to obtain conformally invariant gravitational waves in which field equations have the form given by (3). In this paper we have investigated Lyttleton Bondi Cosmological model in view of field equations of $f(R)$ theory of gravity for Generalized Peres spacetime and finally a wave like solution is obtained.

1 Introduction

General Relativity as formulated by Einstein, only shortly be a century old, is the first purely geometric theory of Gravity. At its core is one of the most beautiful and revolutionary conceptions of modern science the idea that gravity is geometry of four dimensional curved spacetime. In this theory, matter dynamics is prescribed by the geometry of space and, conversely, geometry is determined by matter, so that notion of absolute space of classical mechanics is definitely dropped. One of the most remarkable non-Newtonian features of the relativistic theory of gravitation is the possibility of existence of gravitational waves. General relativity predicts that ripples in space-time curvature can propagate with the speed of light through otherwise empty space. These ripples are gravitational waves. Any mass in non-spherical, non rectilinear motion produces gravitational waves, but gravitational waves are produced most copiously in events such as the coalescence of two compact stars, the merger of massive black holes, or the big bang. They are meaningfully comparable with electromagnetic waves except for their non-conformal invariance. This is because the gravitational wave equations in the background of Friedman universe can be reduced to [2, 3]

$$\mu'' + \mu\left(n^2 - \frac{a''}{a}\right) = 0\,, \tag{1}$$

where $a(\eta)$ the scale factor of Friedmann universe and n is the wave number such that the physical

wave length is given by $\left(\frac{2\pi a}{n}\right)$. The effective potential $\frac{a''}{a}$ in (7) where prime denotes derivatives with respect to η and is related to cosmic time t by $cdt = a(\eta)\,d\eta$ distinguishes this equation from the ordinary wave equations in the Minkowski world. The fact that $\frac{a''}{a}$ is non-zero except for $a = const,$ and $a = a_0\eta$ is a manifestation of the so-called conformal non-invariance of gravitational wave equations.

Hence gravitational waves are an inevitable consequence of Einstein theory of gravitation derivable from Hilbert Lagrangian. It is, therefore, necessary to change Hilbert Lagrangian to modify Einstein field equations to obtain conformally invariant gravitational waves equations.

Over the years alternative theories have been postulated. For example, Weyl [4] suggested the invariant R^2 to make the field action scale invariant (and to unify gravitation with electromagnetism). It is a conformally invariant theory but it does not reduce to Newtonian theory in linearized limit. Another attractive alternative suggested is $R^{3/2}$ so that the coupling constant in matter Lagrangian is dimensionless. Breizman *et.al.* [5] Studied the behavior of homogenous isotropic universe whose underlying Lagrangian density depends on R^n (here *n* is a numerical constant), and Nariai [6] considered this action in studying the problem of gravitational instability in an expanding universe.

Because of non-conformal invariance of gravitational wave equations, to nullify the manifestation of gravitation as evident from eq. (1) without any special choice of scale factor, Pandey [1] gave an $f(R)$ theory of gravity considering Lagrangian in the form [7, 8]

$$\int = R + \sum_{n=2}^{N} C_n \frac{(l^2 R)^n}{6l^2} \quad \text{or, equivalently,} \quad \int = R + \sum_{n=2}^{N} a_n R^n \ , \tag{2}$$

where *l* is the characteristic length and C_n are the dimensionless arbitrary coefficients corresponding the values of *n*. They are introduced to nullify the manifestation of gravitation. The values $n = 0$ and $n = 1$ result in Hilbert Lagrangian, that is, the Einstein theory. Therefore *n* begins from *n=2* onwards. The characteristic radii of curvature of the background world are to be large compared with the gravitational wavelength. Usually (see, for example, Sokolov [9] one writes different coupling constants for each term, for instance, $\alpha R + \beta R^2 + \gamma R_{ik} R^{ik}$. This leads to confusion in analyzing the predictions of the theory. But here only one coupling constant as in the Hilbert case enters all the terms.

By applying variational principle to this action, Pandey [10] obtained the following field
equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \sum_{n=2}^{N} n a_n \ R^{n-1} \left[R_{\mu\nu} - \frac{R g_{\mu\nu}}{2n} - \frac{n(n-1)}{R} (R_{;\mu;\nu} - g_{\mu\nu} \Box R) - \left\{ \frac{(n-1)(n-2)}{R^2} \right\} (R_{;\mu} R_{;\nu} - g_{\mu\nu} R_{;\alpha} R^{;\alpha}) \right] = \kappa T_{\mu\nu}. \tag{3}$$

Here $T_{\mu\nu} = (-g)^{\frac{1}{2}} (\delta \int_s / \delta g^{\mu\nu})$ stands for the energy momentum tensor responsible for the production of the gravitational potential $g_{\mu\nu}$. It can be seen that $T^{\nu}_{\mu;\nu}=0$ holds for these field equations as it is in case of Einstein general relativity.

2 Peres Spacetime

Peres [11] considered a space-time represented by the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x, y, z - t)(dz - dt)^2 \ , \tag{4}$$

which can also be put in the form

$$ds^2 = -dx^2 - dy^2 - (1 - E)dz^2 - 2E \ dz \, dt + (1 + E)dt^2, \tag{5}$$

by replacing $-2f$ by E. Evidently E is a function of x, y and $(z - t)$. The space-time of Peres belong to second class of Petrov's classification and the source of f (or E), as interpreted by Peres, is a null electromagnetic field. This space-time has been studied by Takeno [12] in detail concerning its phase velocity, coordinate conditions, energy momentum pseudo tensor, etc. , and also the solutions of various field equations obtained in this space-time have been called ‘plane wave-like’.

In consideration of the above Pandey [1] proposed a generalization of Peres space-time represented by the metric (4), (5), above as

$$ds^2 = -Adx^2 - Bdy^2 - (1-E)dz^2 - 2E\,dz\,dt + (1+E)dt^2, \tag{6}$$

where $A = A(z,t), \quad B = B(z,t), \quad E = E(x,y,z,t)$.

We consider the metric (6) in the following form

$$ds^2 = -dx^2 - dy^2 - (1-E)dz^2 - 2E\,dz\,dt + (1+E)dt^2 \quad , \tag{7}$$

where $A = 1, \quad B = 1, \quad E = E(x,y,z,t)$.

Lyttleton-Bondi Cosmological Model

A cosmological model was developed by Lyttleton and Bondi [13] based on the assumption that there is continuous creation of matter due to net imbalance of the charge which may arise from the difference in magnitude of the charge of proton and that of an electron, or from the difference in number of protons as compared to the number of electrons. Hence Maxwell field equations were

modified to incorporate this idea of creation as

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \tag{8}$$

$$F^{\mu\nu}_{;\nu} = J^{\mu} - \lambda A^{\mu}, \tag{9}$$

$$J^{\mu}_{;\mu} = q, \tag{10}$$

where λ is a constant, A_μ and J_μ denote the four potential and current density four vector respectively, $F_{\mu\nu}$ denotes the anti-symmetric electromagnetic field tensor and q the rate of creation of charge per unit proper volume. A semi colon denotes here covariant differentiation.

The energy momentum of the field is

$$T^{\mu}_{\nu} = \left(F_{\nu\alpha}F^{\alpha\mu} + \frac{1}{4}g^{\mu}_{\nu}F_{\alpha\beta}F^{\alpha\beta} \right) + \lambda \left(A_{\nu}A^{\mu} - \frac{1}{2}g^{\mu}_{\nu}A_{\alpha}A^{\alpha} \right). \tag{11}$$

When $F_{\mu\nu} = 0$ (with zero electromagnetic field) relation (9) and (11) becomes respectively

$$J^{\mu} = \lambda A^{\mu}, \tag{12}$$

and

$$T^{\mu}_{\nu} = \lambda \left(A_{\nu}A^{\mu} - \frac{1}{2}g^{\mu}_{\nu}A_{\alpha}A^{\alpha} \right) \tag{13}$$

Lyttleton and Bondi [13] have investigated the nature of the field in a Newtonian frame work with zero electromagnetic field and non zero potentials. The same assumption have also been utilized by them to study the de Sitter metric. Burman [14] has studied several aspects of the Lyttleton Bondi field. Nduka [15] obtains a solution of the relativistic field equations for a static Lyttleton-Bondi sphere. He establishes that a static Lyttleton-Bondi sphere is a charged dust and that under certain conditions the sphere cannot exist.

In a static plane symmetric situation, A_μ must have the form given by

$$A_\mu = [\phi, a, 0, 0]. \tag{14}$$

But since $F_{\mu\nu} = 0$, ø must be a constant. Now we have from (14) and (7)

$$A^\mu = [(1-E)\phi, -a, 0, -\phi E], \tag{15}$$

such that

$$A_\mu A^\mu = (1-E)\phi^2 - a^2. \tag{16}$$

3 Field equations

Higher order field equation (3) for $n = 2$ for the metric (7) in presence of electromagnetic energy tensor $T_{\mu\nu}$ given by (13) with A^μ from (15) yields

$$\frac{1}{2}\left[-\partial^2{}_y E - \partial^2{}_x E - 4a_2 U \left(\begin{array}{l} \frac{1}{4}(1+E)U + \frac{1}{2}\left(-\partial^2{}_y E - \partial^2{}_x E - (1+E)U\right) + \\ \left[\begin{array}{l} -((1+E)\partial_z E + (2+E)\partial_t E)(-\partial_z U) - 2\partial^2{}_t U + \partial_y E \partial_y U \\ +\partial_x E \partial_x U + (\partial_t E + E(\partial_z E + \partial_t E))(\partial_t U) \\ +2(1+E)\left(\partial^2{}_y U + \partial^2{}_x U + (\partial^2{}_z - \partial^2{}_t)U\right) \end{array}\right] / U \end{array}\right)\right] = \frac{k\lambda}{2}\left[\begin{array}{l} a^2(1+E) \\ +\phi^2(1+E^2) \end{array}\right], \tag{17}$$

$$\frac{1}{2}\left[-\partial_x\partial_z E - \partial_x\partial_t E + 2a_2 \left(\begin{array}{l} -2\partial^3{}_z E \partial_x E + 2(\partial_x\partial_z E)(\partial_t\partial_z E) - 6(\partial_x E)(\partial_t\partial^2{}_z E) \\ +2(\partial_t\partial_z E)(\partial_t\partial_x E) + (\partial^2{}_z E)(\partial_x\partial_z E + \partial_x\partial_t E) + 4\partial_t\partial_x\partial^2{}_z E \\ +(\partial_x\partial_z E)(\partial^2{}_t E) + (\partial_t\partial_x E)(\partial^2{}_t E) - 6(\partial_x E)(\partial_z\partial^2{}_t E) \\ +8(\partial_x\partial_z\partial^2{}_t E) - 2(\partial_x E)(\partial^3{}_t E) + 4\partial^3{}_t\partial_x E \end{array}\right)\right] = k\lambda\varphi a, \tag{18}$$

$$\frac{1}{2}(-U)\left(1 + a_2\left(-U - 8\left(\partial^2{}_y U + 2\partial^2{}_x U + (\partial^2{}_z - \partial^2{}_t)U\right)/U\right)\right) = \frac{k\lambda}{2}\left[a^2 - \phi^2(-1+E)\right], \tag{19}$$

$$\frac{1}{2}\left[-\partial_y\partial_z E - \partial_y\partial_t E + 2a_2 \left(\begin{array}{l} -2\partial^3{}_z E \partial_y E + 2(\partial_y\partial_z E)(\partial_t\partial_z E) - 6(\partial_y E)(\partial_t\partial^2{}_z E) \\ +2(\partial_t\partial_z E)(\partial_t\partial_y E) + (\partial^2{}_z E)(\partial_y\partial_z E + \partial_y\partial_t E) + 4\partial_t\partial_y\partial^2{}_z E \\ +(\partial_y\partial_z E)(\partial^2{}_t E) + (\partial_t\partial_y E)(\partial^2{}_t E) - 6(\partial_y E)(\partial_z\partial^2{}_t E) \\ +8\partial_y\partial_z\partial^2{}_t E - 2(\partial_y E)(\partial^3{}_t E) + 4\partial^3{}_t\partial_y E \end{array}\right)\right] = 0, \tag{20}$$

$$4a_2\left[\partial_x\partial_y\partial^2{}_z E + 2\partial_t\partial_x\partial_y\partial_z E + \partial_x\partial_y\partial^2{}_t E\right] = 0, \tag{21}$$

$$\frac{1}{2}(-U)\left(1 + a_2\left(-U - 8\left(2\partial^2{}_y U + \partial^2{}_x U + (\partial^2{}_z - \partial^2{}_t)U\right)/U\right)\right) = \frac{-k\lambda}{2}\left[a^2 + \phi^2(-1+E)\right], \tag{22}$$

$$\frac{1}{2}\left[\partial^2{}_yE+\partial^2{}_xE-a_2U\left(-EU+2\begin{pmatrix}\partial^2{}_yE\\ +\partial^2{}_xE+EU\end{pmatrix}+\left[4\begin{pmatrix}-2\partial_t\partial_z-(\partial_tE+E(\partial_zE+\partial_tE))\partial_zU\\ -\partial_yE\partial_yU-\partial_xE\partial_xU\\ -\begin{pmatrix}(-1+E)\partial_zE\\ E\partial_tE\end{pmatrix}\partial_tU\\ -2E\left(\partial^2{}_yU+\partial^2{}_xU+\left(\partial^2{}_z-\partial^2{}_t\right)U\right)\end{pmatrix}\right]/U\right)\right]=\frac{-k\lambda}{2}\begin{bmatrix}a^2\\ +\phi^2(-1+E)\end{bmatrix}E, \tag{23}$$

$$\frac{1}{2}\left[\partial_x\partial_zE+\partial_x\partial_tE+2a_2\begin{pmatrix}2\partial^3{}_zE\partial_xE+4\partial^3{}_z\partial_xE-2(\partial_x\partial_zE)(\partial_t\partial_zE)\\ +6(\partial_xE)(\partial_t\partial^2{}_zE)-2(\partial_t\partial_zE)(\partial_t\partial_xE)-(\partial^2{}_zE)\begin{pmatrix}\partial_x\partial_zE\\ +\partial_x\partial_tE\end{pmatrix}\\ +8\partial_t\partial_x\partial^2{}_zE-(\partial_x\partial_zE)(\partial^2{}_tE)-(\partial_t\partial_xE)(\partial^2{}_tE)\\ +6(\partial_xE)(\partial_z\partial^2{}_tE)+4\partial_x\partial_z\partial^2{}_tE+2(\partial_xE)\partial^3{}_tE\end{pmatrix}\right]=0, \tag{24}$$

$$\frac{1}{2}\left[\partial_y\partial_zE+\partial_y\partial_tE+2a_2\begin{pmatrix}2\partial^3{}_zE\partial_yE+4\partial^3{}_z\partial_yE-2(\partial_y\partial_zE)(\partial_t\partial_zE)\\ +6(\partial_yE)(\partial_t\partial^2{}_zE)-2(\partial_t\partial_zE)(\partial_t\partial_yE)-(\partial^2{}_zE)\begin{pmatrix}\partial_y\partial_zE\\ +\partial_y\partial_tE\end{pmatrix}\\ +8\partial_t\partial_y\partial^2{}_zE-(\partial_y\partial_zE)(\partial^2{}_tE)-(\partial_t\partial_yE)(\partial^2{}_tE)\\ +6(\partial_yE)(\partial_z\partial^2{}_tE)+4\partial_y\partial_z\partial^2{}_tE+2(\partial_yE)\partial^3{}_tE\end{pmatrix}\right]=0, \tag{25}$$

$$\frac{1}{2}\left[-\partial^2{}_yE-\partial^2{}_xE-4a_2U\begin{pmatrix}\frac{1}{4}(-1+E)U+\frac{1}{2}\left(-\partial^2{}_yE-\partial^2{}_xE-(-1+E)U\right)\\ +\left[\begin{array}{l}-2\partial^2{}_zU-((-1+E)\partial_zE+E\partial_tE)(-\partial_zU)\\ +\partial_yE\partial_yU+\partial_xE\partial_xU-\begin{pmatrix}(-2+E)\partial_zE\\ +(-1+E)\partial_tE\end{pmatrix}(-\partial_tU)\\ +2(-1+E)\left(\partial^2{}_yU+\partial^2{}_xU+\left(\partial^2{}_z-\partial^2{}_t\right)U\right)\end{array}\right]/U\end{pmatrix}\right]$$
$$=\frac{k\lambda}{2}\begin{bmatrix}a^2\\ +\phi^2(-1+E)\end{bmatrix}(-1+E), \tag{26}$$

where $U=(\partial_z^2+2\partial_z\partial_t+\partial_t^2)E$, $\partial_z=\frac{\partial}{\partial z}$, $\partial_z^2=\partial_{zz}=\frac{\partial^2}{\partial z^2}$, $\partial_t^3=\partial_{ttt}=\frac{\partial^3}{\partial t^3}$
etc. We now determine E such that it satisfies all the above field equations.
Adding equations (25) and (20), since $a_2 \neq 0$ we obtain

$$\partial_y(\partial_z+\partial_t)^3E=0. \tag{27}$$

Equation (27) yields on integration form of E as

$$E=f_1(x,y,Z)+zf_2(x,y,Z)+z^2F_3(x,y,Z)+f(x,z,t) \qquad (Z=z-t). \tag{28}$$

Also equation (21), since $a_2 \neq 0$ becomes

$$\partial_x\partial_yU=0. \tag{29}$$

If we restrict F_3 in its dependence on x and y then the form of E given by equation (28) becomes

$$E = f_1(x,y,Z) + zf_2(x,y,Z) + z^2 f_3(Z) + f(x,z,t), \tag{30}$$

which clearly satisfies equation (29).
Subtracting (22) from (19) we obtain

$$4a_2\left(\partial^2{}_x - \partial^2{}_y\right)U = k\lambda a^2, \tag{31}$$

which in view of (30) yields

$$4a_2\left(\partial_z + \partial_t\right)^2 \partial^2{}_x f = k\lambda a.^2 \tag{32}$$

Adding equations (24) and (18), we obtain

$$4a_2\partial_x\left(\partial_z + \partial_t\right)^3 E = k\lambda\phi a, \tag{33}$$

which in view of (30) yields

$$4a_2\left(\partial_z + \partial_t\right)^3 \partial_x f = k\lambda\phi a. \tag{34}$$

From (32) and (34) we obtain

$$\left(\partial_z + \partial_t\right)a^2 = \phi\partial_x a, \tag{35}$$

and since a is independent of z and t, equation (35) becomes

$$\phi\partial_x a = 0, \tag{36}$$

which yields on integration as

$$a{=}c \tag{37}$$

where c is a constant. Adding twice equation (23), to (17) and (26) we obtain

$$4a_2\left(\partial_z + \partial_t\right)^2 U = k\lambda\left[\phi^2 - 2\left(\phi^2 - a^2\right)E + 2E^2\phi^2\right] \tag{38}$$

Equation (38) when E given by (30) is substituted becomes

$$\frac{4a_2}{k\lambda}\left(\partial_z + \partial_t\right)^4 f = \left[\phi^2 - 2\left(\phi^2 - a^2\right)\left(f_1 + zf_2 + z^2 f_3 + f\right) + 2\left(f_1 + zf_2 + z^2 f_3 + f\right)^2 \phi^2\right]. \tag{39}$$

also, when the form of E is given by (30), a being a constant and the condition (39) hold, then the field equations (17)-(26) are identically satisfied.
Hence we have:
A necessary and sufficient condition that $g_{\mu\nu}$ given by (7), where E have the form given by (30) and a given by (37) constitute the solution of field equations (17)-(26) is that f_1, f_2, f_3, $f, \lambda, \phi,$ and a satisfy (39).

4 Concluding remarks

Equation (38) is obtained as $U = \left(\partial^2{}_z + 2\partial_z\partial_t + \partial^2{}_t\right)E$ is non-zero. It is evident that the value of E given by (30) satisfies this condition.

Also, we have restricted $F_3(x, y, Z)$ as $f_3(Z)$ in (30) which can be considered as one of the forms of $F_3(x, y, Z)$ as

$$F_3(x, y, Z) = x^0 y^0 f_3(Z). \tag{40}$$

We saw that as some of the metric coefficients do depend on x and y the solution can be called plane wave-like following Takeno [16]. Also, in equation (30) assuming f_2, f_3 and f to be zero, we get plane wave-like solutions obtained by Peres [11].

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