

## Three Integral Representations for the Trigamma Function and Some Special Identities

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**Abstract.** I proved three new integral representations for the trigamma function.

### 1. INTRODUCTION

Utilizing a classical integral representation and unorthodox method of substitution, I demonstrated that:

$$\psi_1(s) = \int_0^{\infty} \frac{(u+1)^{-s} \ln(u+1)}{u} du$$

and

$$\psi_1(s) = \int_0^{\infty} \frac{te^{(1-s)t}}{e^t - 1} dt;$$

as well as proved, using an integral representation of the natural logarithm function, that:

$$\psi_1(s) = \int_0^1 \int_0^1 \frac{x^{s-1}}{y(\ln y - 1)(\ln y - x)} dx dy.$$

### 2. THEOREM

Theorem 1. For  $\Re(s) > 0$ , then

$$\psi_1(s) = \int_0^{\infty} \frac{(u+1)^{-s} \ln(u+1)}{u} du, \quad (1)$$

where  $\psi_1(s)$  denotes the trigamma function and  $\ln s$  denotes the natural logarithm.

*Proof.* In [1], the trigamma function is defined by

$$\psi_1(s) := - \int_0^1 \frac{t^{s-1}}{1-t} \ln t dt. \quad (2)$$

Take  $t = \frac{a+bu}{1+u}$  and  $dt = \left[ \frac{b}{1+u} - \frac{a+bu}{(1+u)^2} \right] du$  in (2), I obtain

$$\psi_1(s) = - \int_{-\frac{a}{b}}^{\frac{a-1}{b-1}} \left[ \frac{\left(\frac{a+bu}{1+u}\right)^{s-1}}{1 - \frac{a+bu}{1+u}} \ln \left(\frac{a+bu}{1+u}\right) \right] \left[ \frac{b}{1+u} - \frac{a+bu}{(1+u)^2} \right] du, \quad (3)$$

Let  $-\frac{a-1}{b-1} = \tan(\pi\theta)$  and  $-\frac{a}{b} = \cot(\pi\theta)$ , therefore,

$$a = \frac{(\tan(\pi\theta)+1) \cot(\pi\theta)}{\cot(\pi\theta) - \tan(\pi\theta)}, \quad b = - \frac{\tan(\pi\theta)+1}{\cot(\pi\theta) - \tan(\pi\theta)}. \quad (4)$$

Put (4) in (3) and simplifying, I have

$$\psi_1(s) = - \int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u, \theta) du, \quad (5)$$

where

$$f(u, \theta) = \frac{2\cot(2\pi\theta) \left( \frac{\cos(\pi\theta) - u\sin(\pi\theta)}{(u+1)(\cos(\pi\theta) - \sin(\pi\theta))} \right)^s \ln \left( \frac{\cos(\pi\theta) - u\sin(\pi\theta)}{(u+1)(\cos(\pi\theta) - \sin(\pi\theta))} \right)}{(u - \tan(\pi\theta))(u - \cot(\pi\theta))}.$$

Consider the limit  $\theta \rightarrow 1$  in (5)

$$\lim_{\theta \rightarrow 1} \psi_1(s) = - \lim_{\theta \rightarrow 1} \int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u, \theta) du, \tag{6}$$

then

$$\begin{aligned} \lim_{\theta \rightarrow 1} \psi_1(s) &= \psi_1(s), \\ \lim_{\theta \rightarrow 1} f(u, \theta) &= - \frac{\left( \frac{1}{u+1} \right)^s \ln \left( \frac{1}{u+1} \right)}{u}, \\ \lim_{\theta \rightarrow 1} \tan(\pi\theta) &= 0, \\ \lim_{\theta \rightarrow 1} \cot(\pi\theta) &= \infty. \end{aligned} \tag{7}$$

Hence, from (6) and (7), it follows that

$$\psi_1(s) = \int_0^\infty \frac{(u+1)^{-s} \ln(u+1)}{u} du. \square$$

Corollary 1. For  $\Re(s) > 0$ , then

$$\psi_1(s) = \int_0^\infty \frac{te^{(1-s)t}}{e^t - 1} dt.$$

*Proof.* I substitute  $u + 1 = e^t$  and  $du = e^t dt$  in Theorem 1.  $\square$

3.1.11. THEOREM. For  $\Re(s) > 0$ , then

$$\psi_1(s) = \int_0^1 \int_0^1 \frac{x^{s-1}}{y(\ln y - 1)(\ln y - x)} dx dy, \tag{8}$$

where  $\psi_1(z)$  denotes the trigamma function and  $\ln s$  denotes the natural logarithm.

*Proof.* In [2], I prove that

$$\frac{\ln x}{x-1} = \int_0^1 \frac{1}{y(\ln y - x)(\ln y - 1)} dy, \tag{9}$$

and I know that trigamma function is defined by

$$\psi_1(s) \stackrel{\text{def}}{=} - \int_0^1 \frac{x^{s-1} \ln x}{1-x} dx. \tag{10}$$

Substituing (9) in (10), then,

$$\psi_1(s) = \int_0^1 \int_0^1 \frac{x^{s-1}}{y(\ln y - 1)(\ln y - x)} dx dy. \square$$

**Special Identities.** Utilizing the Corollary 1, I calculate

$$\psi_1\left(\frac{1}{3}\right) = \frac{4\pi^2}{3} - \psi_1\left(\frac{2}{3}\right), \quad (11)$$

$$\psi_1\left(\frac{1}{4}\right) = \frac{G_{3,3}^{3,2}\left(1 \left| \begin{matrix} 0, \frac{3}{4}, 1 \\ 0, 0, 0 \end{matrix} \right. \right)}{\Gamma\left(\frac{1}{4}\right)}, \quad (12)$$

$$\psi_1\left(\frac{1}{5}\right) = \frac{8\pi^2}{5-\sqrt{5}} - \psi_1\left(\frac{4}{5}\right), \quad (13)$$

$$\psi_1\left(\frac{1}{6}\right) = 4\pi^2 - \psi_1\left(\frac{5}{6}\right), \quad (14)$$

$$\psi_1\left(\frac{1}{8}\right) = \frac{4\pi^2}{2-\sqrt{2}} - \psi_1\left(\frac{7}{8}\right), \quad (15)$$

where  $G_{t,u}^{r,s}\left(x \left| \begin{matrix} a, b, c \\ d, e, f \end{matrix} \right. \right)$  denotes the Meijer G- function and  $\Gamma(x)$  denotes the gamma function.

Using the Theorem 1, I calculate

$$\psi_1\left(\frac{1}{4}\right) = 8C + \pi^2, \quad (16)$$

$$\psi_1\left(\frac{1}{6}\right) = 8\pi^2 + \frac{9}{4}\psi_1\left(\frac{1}{3}\right) - \frac{3}{2}\psi_1\left(\frac{2}{3}\right) - \frac{1}{4}\psi_1\left(\frac{5}{6}\right), \quad (17)$$

$$\psi_1\left(\frac{1}{8}\right) = 32C + 4\pi^2 - \psi_1\left(\frac{5}{8}\right), \quad (18)$$

where  $C$  denotes the Catalan constant.

From (12) and (16), it follows that

$$G_{3,3}^{3,2}\left(1 \left| \begin{matrix} 0, \frac{3}{4}, 1 \\ 0, 0, 0 \end{matrix} \right. \right) = \Gamma\left(\frac{1}{4}\right) (8C + \pi^2). \quad (19)$$

From (14) and (17), it follows that

$$4\pi^2 - \frac{3}{4}\psi_1\left(\frac{5}{6}\right) = 8\pi^2 + \frac{9}{4}\psi_1\left(\frac{1}{3}\right) - \frac{3}{2}\psi_1\left(\frac{2}{3}\right). \quad (20)$$

From (11) and (20), it follows that

$$5\psi_1\left(\frac{2}{3}\right) - \psi_1\left(\frac{5}{6}\right) = \frac{28}{3}\pi^2. \quad (21)$$

From (15) and (18), it follows that

$$\psi_1\left(\frac{5}{8}\right) - \psi_1\left(\frac{7}{8}\right) = 32C + 4\pi^2 - \frac{4\pi^2}{2-\sqrt{2}}. \quad (22)$$

**NOTE.** I leave the reader to prove the following identities

$$\psi_1(s) = \int_0^\infty \frac{(u+1)^{-s} \ln(u+1)}{u} du = \frac{{}_3F_2(1, s, s; s+1, s+1; 1)}{s^2}, \quad (\Re(s) > 0),$$

$$\int_0^\infty \left[ \frac{(u+1)^{-s} \ln(u+1)}{u} \right]^2 du = 2 \frac{{}_4F_3(2, 2s+1, 2s+1, 2s+1; 2s+2, 2s+2, 2s+2; 1)}{(2s+1)^3}, \left( \Re(s) > -\frac{1}{2} \right),$$

$$\int_0^\infty \left[ \frac{(u+1)^{-s} \ln(u+1)}{u} \right]^3 du = 6 \frac{{}_5F_4(3, 3s+2, 3s+2, 3s+2, 3s+2; 3s+3, 3s+3, 3s+3, 3s+3; 1)}{(3s+2)^4},$$

$$\left( \Re(s) > -\frac{2}{3} \right),$$

$$\int_0^\infty \left[ \frac{(u+1)^{-s} \ln(u+1)}{u} \right]^4 du = 24 \frac{{}_6F_5(4, 4s+3, 4s+3, 4s+3, 4s+3, 4s+3; 4s+4, 4s+4, 4s+4, 4s+4, 4s+4; 1)}{(4s+3)^5},$$

$$\left( \Re(s) > -\frac{3}{4} \right),$$

for  $n \in \mathbb{N}$

$$\int_0^\infty \left[ \frac{(u+1)^{-s-1} \ln(u+1)}{u} \right]^n du = n! \frac{{}_{n+2}F_{n+1} \left( n, \frac{ns+(n-1), \dots, ns+(n-1)}{(n+1)\text{times}}; \frac{ns+n, \dots, ns+n}{(n+1)\text{times}}; 1 \right)}{(ns+(n-1))^{n+1}},$$

$$\left( \Re(s) > -\frac{n-1}{n} \right),$$

$$\int_0^\infty \frac{(u+1)^{-s-1} \ln(u+1)}{u} du = \psi_1(s+1)$$

$$= \psi_1(s) - \frac{1}{s^2} = \frac{{}_3F_2(1, s+1, s+1; s+2, s+2; 1)}{(s+1)^2}, \quad \left( \Re(s) > -1 \right),$$

$$\int_0^\infty \left[ \frac{(u+1)^{-s-1} \ln(u+1)}{u} \right]^2 du = 2 \frac{{}_4F_3(2, 2s+3, 2s+3, 2s+3; 2s+4, 2s+4, 2s+4; 1)}{(2s+3)^3}, \quad \left( \Re(s) > -\frac{3}{2} \right),$$

$$\int_0^\infty \left[ \frac{(u+1)^{-s-1} \ln(u+1)}{u} \right]^3 du = 6 \frac{{}_5F_4(3, 3s+5, 3s+5, 3s+5, 3s+5; 3s+6, 3s+6, 3s+6, 3s+6; 1)}{(3s+5)^4},$$

$$\left( \Re(s) > -\frac{5}{3} \right),$$

$$\begin{aligned}
& \int_0^{\infty} \left[ \frac{(u+1)^{-s-1} \ln(u+1)}{u} \right]^4 du \\
&= 24 \frac{{}_6F_5(4, 4s+7, 4s+7, 4s+7, 4s+7, 4s+7; 4s+8, 4s+8, 4s+8, 4s+8, 4s+8; 1)}{(4s+7)^5}, \\
& \left( \Re(s) > -\frac{7}{4} \right),
\end{aligned}$$

for  $n \in \mathbb{N}$

$$\begin{aligned}
& \int_0^{\infty} \left[ \frac{(u+1)^{-s-1} \ln(u+1)}{u} \right]^n du \\
&= n! \frac{{}_{n+2}F_{n+1}(n, \frac{ns+(2n-1)}{(n+1)\text{times}}, \dots, \frac{ns+(2n-1)}{(n+1)\text{times}}; \frac{ns+(2n)}{(n+1)\text{times}}, \dots, \frac{ns+(2n)}{(n+1)\text{times}}; 1)}{(ns+(2n-1))^{n+1}}, \\
& \left( \Re(s) > -\frac{2n-1}{n} \right)
\end{aligned}$$

and for  $n \in \mathbb{N}$

$$\begin{aligned}
& \int_0^{\infty} \left[ \frac{(u+1)^{-s-j} \ln(u+1)}{u} \right]^n du \\
&= n! \frac{{}_{n+2}F_{n+1}(n, \frac{ns+(j+1)n-1}{(n+1)\text{times}}, \dots, \frac{ns+(j+1)n-1}{(n+1)\text{times}}; \frac{ns+(j+1)n}{(n+1)\text{times}}, \dots, \frac{ns+(j+1)n}{(n+1)\text{times}}; 1)}{(ns+(j+1)n-1)^{n+1}}, \\
& \left( \Re(s) > -\frac{(j+1)n-1}{n} \right).
\end{aligned}$$

## REFERENCES

- [1] [http://en.wikipedia.org/wiki/Trigamma\\_function](http://en.wikipedia.org/wiki/Trigamma_function), available in March 11, 2014.  
[2] Guedes, Edigles, *The Natural Logarithm Function and its Applications*, to appear.