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Three Integral Representations for the Trigamma Function and Some Special Identities

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Abstract. I proved three new integral representations for the trigamma function.

1. INTRODUCTION

Utilizing a classical integral representation and unorthodox method of substitution, I demonstrated that:

$$\psi_1(s) = \int_0^\infty \frac{(u+1)^{-s}\ln(u+1)}{u} du$$

and

$$\psi_1(s) = \int_0^\infty \frac{te^{(1-s)t}}{e^t - 1} dt;$$

as well as proved, using an integral representation of the natural logarithm function, that:

$$\psi_1(s) = \int_0^1 \int_0^1 \frac{x^{s-1}}{y(\ln y - 1)(\ln y - x)} dxdy.$$

2. THEOREM

Theorem 1. *For* $\Re(s) > 0$, *then*

$$\psi_1(s) = \int_0^\infty \frac{(u+1)^{-s}\ln(u+1)}{u} du, \tag{1}$$

where $\psi_1(s)$ *denotes the trigamma function and* $\ln s$ *denotes the natural logarithm.*

Proof. In [1], the trigamma function is defined by

$$\psi_1(s) := -\int_0^1 \frac{t^{s-1}}{1-t} \ln t \, dt. \tag{2}$$

Take $t = \frac{a+bu}{1+u}$ and $dt = \left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2}\right] du$ in (2), I obtain

$$\psi_1(s) = -\int_{-\frac{a}{b}}^{-\frac{a-1}{b-1}} \left[\frac{\left(\frac{a+bu}{1+u}\right)^{s-1}}{1-\frac{a+bu}{1+u}} \ln\left(\frac{a+bu}{1+u}\right)\right]\left[\frac{b}{1+u} - \frac{a+bu}{(1+u)^2}\right] du, \tag{3}$$

Let $-\frac{a-1}{b-1} = \tan(\pi\theta)$ and $-\frac{a}{b} = \cot(\pi\theta)$, therefore,

$$a = \frac{(\tan(\pi\theta)+1)\cot(\pi\theta)}{\cot(\pi\theta)-\tan(\pi\theta)}, \quad b = -\frac{\tan(\pi\theta)+1}{\cot(\pi\theta)-\tan(\pi\theta)}. \tag{4}$$

Put (4) in (3) and simplifying, I have

$$\psi_1(s) = -\int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u,\theta) du, \tag{5}$$

where

$$f(u,\theta)=\frac{2\cot(2\pi\theta)(\frac{\cos(\pi\theta)-u\sin(\pi\theta)}{(u+1)(\cos(\pi\theta)-\sin(\pi\theta))})^{s}\ln(\frac{\cos(\pi\theta)-u\sin(\pi\theta)}{(u+1)(\cos(\pi\theta)-\sin(\pi\theta))})}{(u-\tan(\pi\theta))(u-\cot(\pi\theta))}.$$

Consider the limit $\theta\to 1$ in (5)

$$\lim_{\theta\to1}\psi_1(s) = -\lim_{\theta\to1}\int_{\cot(\pi\theta)}^{\tan(\pi\theta)} f(u,\theta)du, \tag{6}$$

then

$$\lim_{\theta\to1}\psi_1(s)=\psi_1(s),$$

$$\lim_{\theta\to1} f(u,\theta)=-\frac{\left(\frac{1}{u+1}\right)^{s}\ln\left(\frac{1}{u+1}\right)}{u},$$

$$\lim_{\theta\to1}\tan(\pi\theta)=0,$$

$$\lim_{\theta\to1}\cot(\pi\theta)=\infty. \tag{7}$$

Hence, from (6) and (7), it follows that

$$\psi_1(s)=\int_0^\infty\frac{(u+1)^{-s}\ln(u+1)}{u}du.\square$$

Corollary 1. *For* $\Re(s)>0$, *then*

$$\psi_1(s)=\int_0^\infty\frac{te^{(1-s)t}}{e^t-1}dt.$$

Proof. I substitute $u+1=e^t$ and $du=e^t dt$ in Theorem 1. $\square$

3.1.11. THEOREM. *For* $\Re(s)>0$, *then*

$$\psi_1(s)=\int_0^1\int_0^1\frac{x^{s-1}}{y(\ln y-1)(\ln y-x)}dxdy, \tag{8}$$

where $\psi_1(z)$ *denotes the trigamma function and* $\ln s$ *denotes the natural logarithm.*

Proof. In [2], I prove that

$$\frac{\ln x}{x-1}=\int_0^1\frac{1}{y(\ln y-x)(\ln y-1)}dy, \tag{9}$$

and I know that trigamma function is defined by

$$\psi_1(s)\overset{\text{def}}{=}-\int_0^1\frac{x^{s-1}\ln x}{1-x}dx. \tag{10}$$

Substituing (9) in (10), then,

$$\psi_1(s)=\int_0^1\int_0^1\frac{x^{s-1}}{y(\ln y-1)(\ln y-x)}dxdy.\square$$

Special Identities. Utilizing the Corollary 1, I calculate

$$\psi_1\left(\frac{1}{3}\right)=\frac{4\pi^2}{3}-\psi_1\left(\frac{2}{3}\right), \tag{11}$$

$$\psi_1\left(\frac{1}{4}\right)=\frac{G_{3,3}^{3,2}\left(1\middle|\begin{matrix}0,\frac{3}{4},1\\0,0,0\end{matrix}\right)}{\Gamma\left(\frac{1}{4}\right)}, \tag{12}$$

$$\psi_1\left(\frac{1}{5}\right)=\frac{8\pi^2}{5-\sqrt{5}}-\psi_1\left(\frac{4}{5}\right), \tag{13}$$

$$\psi_1\left(\frac{1}{6}\right)=4\pi^2-\psi_1\left(\frac{5}{6}\right), \tag{14}$$

$$\psi_1\left(\frac{1}{8}\right)=\frac{4\pi^2}{2-\sqrt{2}}-\psi_1\left(\frac{7}{8}\right), \tag{15}$$

where $G_{t,u}^{r,s}\left(x\middle|\begin{matrix}a,b,c\\d,e,f\end{matrix}\right)$ denotes the Meijer G- function and $\Gamma(x)$ denotes the gamma function.

Using the Theorem 1, I calculate

$$\psi_1\left(\frac{1}{4}\right)=8C+\pi^2, \tag{16}$$

$$\psi_1\left(\frac{1}{6}\right)=8\pi^2+\frac{9}{4}\psi_1\left(\frac{1}{3}\right)-\frac{3}{2}\psi_1\left(\frac{2}{3}\right)-\frac{1}{4}\psi_1\left(\frac{5}{6}\right), \tag{17}$$

$$\psi_1\left(\frac{1}{8}\right)=32C+4\pi^2-\psi_1\left(\frac{5}{8}\right), \tag{18}$$

where C denotes the Catalan constant.

From (12) and (16), it follows that

$$G_{3,3}^{3,2}\left(1\middle|\begin{matrix}0,\frac{3}{4},1\\0,0,0\end{matrix}\right)=\Gamma\left(\frac{1}{4}\right)(8C+\pi^2). \tag{19}$$

From (14) and (17), it follows that

$$4\pi^2-\frac{3}{4}\psi_1\left(\frac{5}{6}\right)=8\pi^2+\frac{9}{4}\psi_1\left(\frac{1}{3}\right)-\frac{3}{2}\psi_1\left(\frac{2}{3}\right). \tag{20}$$

From (11) and (20), it follows that

$$5\psi_1\left(\frac{2}{3}\right)-\psi_1\left(\frac{5}{6}\right)=\frac{28}{3}\pi^2. \tag{21}$$

From (15) and (18), it follows that

$$\psi_1\left(\frac{5}{8}\right)-\psi_1\left(\frac{7}{8}\right)=32C+4\pi^2-\frac{4\pi^2}{2-\sqrt{2}}. \tag{22}$$

NOTE. I leave the reader to prove the following identities

$$\psi_1(s)=\int_0^\infty\frac{(u+1)^{-s}\ln(u+1)}{u}du=\frac{{}_3F_2(1,s,s;s+1,s+1;1)}{s^2},\qquad(\Re(s)>0)\,,$$

$$\int_0^\infty \left[\frac{(u+1)^{-s}\ln(u+1)}{u}\right]^2 du$$
$$= 2\,\frac{{}_4F_3(2,2s+1,2s+1,2s+1;2s+2,2s+2,2s+2;1)}{(2s+1)^3},\left(\Re(s)>-\frac{1}{2}\right),$$

$$\int_0^\infty \left[\frac{(u+1)^{-s}\ln(u+1)}{u}\right]^3 du$$
$$= 6\,\frac{{}_5F_4(3,3s+2,3s+2,3s+2,3s+2;3s+3,3s+3,3s+3,3s+3;1)}{(3s+2)^4},$$
$$\left(\Re(s)>-\frac{2}{3}\right),$$

$$\int_0^\infty \left[\frac{(u+1)^{-s}\ln(u+1)}{u}\right]^4 du$$
$$= 24\,\frac{{}_6F_5(4,4s+3,4s+3,4s+3,4s+3,4s+3;4s+4,4s+4,4s+4,4s+4,4s+4;1)}{(4s+3)^5},$$
$$\left(\Re(s)>-\frac{3}{4}\right),$$

for $n\in\mathbb{N}$

$$\int_0^\infty \left[\frac{(u+1)^{-s-1}\ln(u+1)}{u}\right]^n du$$
$$= n!\,\frac{{}_{n+2}F_{n+1}\left(n,\underbrace{ns+(n-1),\dots,ns+(n-1)}_{(n+1)\text{times}};\underbrace{ns+n,\dots,ns+n}_{(n+1)\text{times}};1\right)}{\left(ns+(n-1)\right)^{n+1}},$$
$$\left(\Re(s)>-\frac{n-1}{n}\right),$$

$$\int_0^\infty \frac{(u+1)^{-s-1}\ln(u+1)}{u}\,du=\psi_1(s+1)$$
$$=\psi_1(s)-\frac{1}{s^2}=\frac{{}_3F_2(1,s+1,s+1;s+2,s+2;1)}{(s+1)^2},\qquad(\Re(s)>-1),$$

$$\int_0^\infty \left[\frac{(u+1)^{-s-1}\ln(u+1)}{u}\right]^2 du$$
$$= 2\,\frac{{}_4F_3(2,2s+3,2s+3,2s+3;2s+4,2s+4,2s+4;1)}{(2s+3)^3},\qquad\left(\Re(s)>-\frac{3}{2}\right),$$

$$\int_0^\infty \left[\frac{(u+1)^{-s-1}\ln(u+1)}{u}\right]^3 du$$
$$= 6\,\frac{{}_5F_4(3,3s+5,3s+5,3s+5,3s+5;3s+6,3s+6,3s+6,3s+6;1)}{(3s+5)^4},$$
$$\left(\Re(s)>-\frac{5}{3}\right),$$

$$\int_0^\infty \left[\frac{(u+1)^{-s-1}\ln(u+1)}{u}\right]^4 du$$
$$= 24\,\frac{{}_6F_5(4,4s+7,4s+7,4s+7,4s+7,4s+7;4s+8,4s+8,4s+8,4s+8,4s+8;1)}{(4s+7)^5},$$
$$\left(\Re(s) > -\frac{7}{4}\right),$$

for $n \in \mathbb{N}$

$$\int_0^\infty \left[\frac{(u+1)^{-s-1}\ln(u+1)}{u}\right]^n du$$
$$= n!\,\frac{{}_{n+2}F_{n+1}(n,\underbrace{ns+(2n-1),\ldots,ns+(2n-1)}_{(n+1)\text{times}};\underbrace{ns+(2n),\ldots,ns+(2n)}_{(n+1)\text{times}};1)}{(ns+(2n-1))^{n+1}},$$
$$\left(\Re(s) > -\frac{2n-1}{n}\right)$$

and for $n \in \mathbb{N}$

$$\int_0^\infty \left[\frac{(u+1)^{-s-j}\ln(u+1)}{u}\right]^n du$$
$$= n!\,\frac{{}_{n+2}F_{n+1}(n,\underbrace{ns+\big((j+1)n-1\big),\ldots,ns+\big((j+1)n-1\big)}_{(n+1)\text{times}};\underbrace{ns+\big((j+1)n\big),\ldots,ns+\big((j+1)n\big)}_{(n+1)\text{times}};1)}{(ns+\big((j+1)n-1\big))^{n+1}},$$
$$\left(\Re(s) > -\frac{(j+1)n-1}{n}\right).$$

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