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# An elementary proof of Fermat-Wiles and Catalan-Mihailescu theorems and generalization to Beal conjecture

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**Abstract:** We begin with Fermat, Catlan and Fermat-Catalan equations and solve them.



## The Fermat equation

Fermat equation is $y^n = x^n \pm z^n = x^n + az^n$
Let $x^{n-2}y^2 - y^{n-2}x^2 = Aa$
If $A = 0 \Rightarrow x^{n-4} = y^{n-4}$ but $GCD(x, y) = 1 \Rightarrow n = 4$ impossible, there is no solution and
If $A^2 = z^{2n}; n \geq 3 \Rightarrow x^{n-3}y - y^{n-2} = \frac{Aaz^n}{x} \in Z$ impossible $\Rightarrow n = 2$
We have
$(x^{n-2}y^2 - y^{n-2}x^2)z^c = Aaz^n = Ay^{n-2}y^2 - Ax^{n-2}x^2 \Rightarrow (x^{n-2}z^n - Ay^{n-2})y^2 = (y^{n-2}z^n - Ax^{n-2})x^2$

We have then four cases :

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

*or*

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$
$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

*or*

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

*or*

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

With $u, v \in Z$
First case

*or*

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

We have

$$y^n = uv(-y^nz^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2})) = uv(-y^nz^{2n} + A^2x^n + A^2az^n) = uv(A^2 - z^{2n})y^n$$
$$uv(A^2 - z^{2n}) = 1$$

Impossible because A,u,v are integers
Second case

*or*

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$
$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

We have

$$uvy^n = -y^n z^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2}) = -y^n z^{2n} + A^2x^n + A^2az^n = (A^2 - z^{2n})y^n$$

$$uv = A^2 - z^{2n}$$

And

$$uv(y^2x^{n-2} - x^2y^{n-2}) = uvA = uz^{2n}(-y^{2n-4} + x^{2n-4})A = vz^{2n}(-y^4 + x^4)A$$

$$\Rightarrow u = z^{2n}(-y^4 + x^4); v = z^{2n}(-y^{2n-4} + x^{2n-4})$$

$$uv = A^2 - z^{2n} = (y^4 - x^4)(y^{2n-4} - x^{2n-4}) = uvz^{-4n} \Rightarrow uv = A^2 - z^{2n} = 0$$

It is impossible $\Rightarrow n = 2$

Third case

*or*

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$

$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

We have

$$vy^n = u(-y^n z^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2})) = u(-y^n z^{2n} + A^2x^n + A^2az^n) = u(A^2 - z^{2n})y^n$$

$$v = u(A^2 - z^{2n})$$

And

$$v(y^2x^{n-2} - x^2y^{n-2}) = vA = uvz^{2n}(-y^{2n-4} + x^{2n-4})A = z^{2n}(-y^4 + x^4)A$$

$$\Rightarrow uz^{2n}(-y^{2n-4} + x^{2n-4}) = 1 \Rightarrow u = \infty; n = 2$$

Impossible because u, A are integers

Fourth case

*or*

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

We have

$$uy^n = v(-y^n z^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2})) = v(-y^n z^{2n} + A^2x^n + A^2az^n) = v(A^2 - z^{2n})y^n$$

$$u = v(A^2 - z^{2n})$$

And

$$u(y^2x^{n-2} - x^2y^{n-2}) = uA = z^{2n}(-y^{2n-4} + x^{2n-4})A = uvz^{2n}(-y^4 + x^4)A$$

$$\Rightarrow vz^{2n}(x^4 - y^4) = 1 \Rightarrow v = 1, x^4 = y^4 + 1$$

Impossible ! Because v,A are integers !

The only solution is A = 1 and n = 2

**Resolution of Catalan equation**

Catalan equation is $y^p = x^q + 1$

Let $x^{q-3}y^2 - y^{p-2}x^3 = 1$

If $A = 0 \Rightarrow x^{q-6} = y^{p-4}$

$GCD(x, y) = 1 \Rightarrow p = 4$ impossible, there is no solution, Ko Chao proved it and

If $A^2 = 1$; $p \geq 3 \Rightarrow x^{q-3}y - y^{p-3}x^3 = 1/y \in Z$ impossible thus p=2

We have

$$x^{q-3}y^2 - y^{p-2}x^3 = 1 = Ay^{p-2}y^2 - Ax^{q-3}x^3$$

$$\Rightarrow (x^{q-3} - Ay^{p-2})y^2 = (y^{p-2} - Ax^{q-3})x^3$$
$$(y^2 + Ax^3)x^{q-3} = (x^3 + Ay^2)y^{p-2}$$
$$GCD(x, y) = 1$$

We have then four cases :

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$
$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

*or*

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$
$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

*or*

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$
$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

or

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$
$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$
$$u, v \in Z$$

First case

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$
$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

We have

$$y^p = uv(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = uv(-y^p + A^2x^q + A^2) = uv(A^2 - 1)y^n$$
$$uv(A^2 - 1) = 1$$

Impossible because A,u,v are integers

Second case

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$
$$vx^{q-3} = x^2 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

We have

$$uvy^p = -y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3}) = -y^p + A^2x^q + A^2 = (A^2 - 1)y^p$$
$$uv = A^2 - 1$$

And

$$uv(y^2x^{q-3} - x^3y^{p-2}) = uvA = u(-y^{2p-4} + x^{2q-6})A = v(-y^4 + x^6)A$$
$$\Rightarrow u = -y^4 + x^6; v = -y^{2p-4} + x^{2q-6}$$
$$uv = A^2 - 1 = (y^4 - x^4)(y^{2p-4} - x^{2q-6})$$
$$\Rightarrow (y^2x^{q-3} - x^3y^{p-2})^2 - 1 = (y^4 - x^6)(y^{2p-4} - x^{2q-6})$$
$$\Rightarrow x^{2q} + y^{2p} + 2x^qy^p = 2y^4x^{2q-6} + 2x^6y^{2p-4} - 1$$
$$= (x^q + y^p)^2 = (2y^p - 1)^2 = 4y^{2p} - 4y^p + 1$$

$$2p > 4 \Rightarrow 2y^3x^{2q-6} + 2x^6y^{2p-5} - 4y^{2p-1} + 4y^{p-1} = 2/y \in Z$$

Impossible : $\Rightarrow p = 2$

Third case

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$
$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

We have
$vy^p = u(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = u(-y^p + A^2x^q + A^2) = u(A^2-1)y^p$
$v = u(A^2-1)$
And
$v(y^2x^{q-3} - x^3y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-6})A = (-y^4 + x^6)A \Rightarrow u(-y^{2p-4} + x^{2q-6}) = 1 \Rightarrow p-2 = q-3 = 0$

Impossible because u, A are integers
Fourth case

$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}x^{q-3} = v(x^3 + Ay^2);$
$y^{p-2} = v(y^2 + Ax^3)$
We have
$uy^p = v(-y^p + A^2x^q - A(x^2y^{p-2} - y^2x^{q-3})) = v(-y^p + A^2x^q + A^2) = v(A^2-1)y^p$
$u = v(A^2-1)$

And

$u(y^2x^{q-3} - x^3y^{p-2}) = uA = (-y^{2p-4} + x^{2q-6})A = uv(-y^4 + x^6)A \Rightarrow v(-y^4 + x^6) = 1$

Impossible, because v,A are integers !
The only solution is A=1 and p=2 this case has been studied by Ko Chao

**The Fermat-Catalan equation**

The equation now is $y^p = x^q \pm z^c = x^q + az^c$
Let $x^{q-w}y^2 - y^{p-2}x^w = A$
If
$A = 0 \Rightarrow x^{q-2w} = y^{p-4}$
$GCD(x,y) = 1 \Rightarrow p = 4$

It means that p=2 is the prime solution !

And

$A^2 = z^{2c}; p \geq 3 \Rightarrow x^{q-w}y - y^{p-3}x^w = 1/y \in Z$

impossible thus p = 2

We have
$(x^{q-w}y^2 - y^{p-2}x^w)z^c = Aaz^c = Ay^{p-2}y^2 - Ax^{q-w}x^w$
$\Rightarrow (x^{q-w}z^c + Ay^{p-2})y^2 = (y^{p-2}z^c - Ax^{q-w})x^2$
$x^{q-w}y^2 - y^{p-2}x^w = aAz^c = Ay^{p-2}y^2 - Ax^{q-w}x^w$
$\Rightarrow (x^{q-w} + Ay^{p-2})y^2 = (y^{p-2} - Ax^{q-w})x^w$
$(y^2 + Ax^w)x^{q-w} = (x^w + Ay^2)y^{p-2}$
$GCD(x,y) = 1$

We have then four cases :
First case

$x^w = u(-x^{q-w}z^c + Ay^{p-2}); y^2 = u(-y^{p-2}z^c + Ax^{q-w})$
$x^{q-w} = v(x^wz^c + Ay^2); y^{p-2} = v(y^2z^c + Ax^w)$

We have

$y^p = uv(-y^pz^{2c} + A^2x^q - Az^c(x^wy^{p-2} - y^2x^{q-w})) = uv(-y^pz^{2c} + A^2x^q + A^2az^c) = uv(A^2 - z^{2c})y^p$
$uv(A^2 - z^{2c}) = 1$

Impossible because A,u,v are integers
Second case

$$ux^w = -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w}$$
$$vx^{q-w} = x^w z^c + Ay^2; vy^{p-2} = y^2 z^c + Ax^w$$

We have
$$uvy^p = -y^p z^{2c} + A^2 x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w}) = -y^p z^{2c} + A^2 x^q + A^2 az^c) = (A^2 - z^{2c})y^p$$
$$uv = A^2 - z^{2c}$$

And
$$uv(y^2 x^{q-w} - x^w y^{p-2}) = uvA = u(-y^{2p-4} + x^{2q-2w})A = v(-y^4 + x^{2w})A$$
$$\Rightarrow u = -y^4 + x^{2w}; v = -y^{2p-4} + x^{2q-2w}$$
$$uv = A^2 - z^{2c} = (y^4 - x^{2w})(y^{2p-4} - x^{2q-2w}) = uvz^{4c} \Rightarrow uv = A^2 - z^{2c} = 0$$

Impossible : $\Rightarrow p = 2$
Third case

$$x^w = u(-x^{q-w}z^c + Ay^{p-2}); y^2 = u(-y^{p-2}z^c + Ax^{q-w})$$
$$vx^{q-w} = x^w z^c + Ay^2; vy^{p-2} = y^2 z^c + Ax^w$$

We have

$$vy^p = u(-y^p z^{2c} + A^2 x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w})) = u(-y^p z^{2c} + A^2 x^q + A^2 az^c) = u(A^2 - z^{2c})y^p$$
$$v = u(A^2 - z^{2c})$$

And
$$v(y^2 x^{q-w} - x^w y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-2w})A = (-y^4 + x^{2w})A$$
$$\Rightarrow u(-y^{2p-4} + x^{2q-2w}) = 1 \Rightarrow p = 2$$

Impossible because u, A are integers
Fourth case

$$ux^w = -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w}$$
$$x^{q-w} = v(x^w z^c + Ay^2); y^{p-2} = v(y^2 z^c + Ax^w)$$

We have

$$uy^p = v(-y^p z^{2c} + A^2 x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w})) = v(-y^p z^{2c} + A^2 x^q + A^2 az^c) = v(A^2 - z^{2c})y^p$$
$$u = v(A^2 - z^{2c})$$

And

$$u(y^2 x^{q-w} - x^w y^{p-2}) = uA = -y^{2p-4} + x^{2q-2w}A = uv(-y^4 + x^{2w})A$$
$$\Rightarrow v(-y^4 + x^{2w}) = 1$$

Impossible because [illegible],A are integers ! In the Fermat-Catalan equation, one of the exponents must be equal to 2 ! The Beal conjecture has been proved !
In fact, in the three precedent equations studied here, one of the exponent greater or equal to 2 must be minimum, which means that it must be 2 !

## Conclusion

We have solved both three equations by the same method and proved two theorems and one conjecture.

## References


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