

## An elementary proof of Fermat-Wiles and Catalan-Mihailescu theorems and generalization to Beal conjecture

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**Abstract:** We begin with Fermat, Catalan and Fermat-Catalan equations and solve them.

### The Fermat equation

Fermat equation is  $y^n = x^n \pm z^n = x^n + az^n$

Let  $x^{n-2}y^2 - y^{n-2}x^2 = Aa$

If  $A = 0 \Rightarrow x^{n-4} = y^{n-4}$  but  $GCD(x, y) = 1 \Rightarrow n = 4$  impossible, there is no solution and

If  $A^2 = z^{2n}; n \geq 3 \Rightarrow x^{n-3}y - y^{n-2} = \frac{Aaz^n}{x} \in \mathbb{Z}$  impossible  $\Rightarrow n = 2$

We have

$$(x^{n-2}y^2 - y^{n-2}x^2)z^c = Aaz^n = Ay^{n-2}y^2 - Ax^{n-2}x^2 \Rightarrow (x^{n-2}z^n - Ay^{n-2})y^2 - (x^{n-2}z^n - Ay^{n-2})x^2$$

We have then four cases :

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$

$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

or

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

or

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$

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or

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

With  $u, v \in \mathbb{Z}$

First case

or

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$

$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

We have

$$y^n = uv(-y^n z^{2n} + A^2 x^n - Az^n(x^2 y^{n-2} - y^2 x^{n-2})) = uv(-y^n z^{2n} + A^2 x^n + A^2 az^n) = uv(A^2 - z^{2n})y^n$$

$$uv(A^2 - z^{2n}) = 1$$

Impossible because  $A, u, v$  are integers

Second case

or

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

We have

$$uvy^n = -y^n z^{2n} + A^2 x^n - Az^n (x^2 y^{n-2} - y^2 x^{n-2}) = -y^n z^{2n} + A^2 x^n + A^2 az^n = (A^2 - z^{2n})y^n$$

$$uv = A^2 - z^{2n}$$

And

$$uv(y^2 x^{n-2} - x^2 y^{n-2}) = uvA = uz^{2n}(-y^{2n-4} + x^{2n-4})A = vz^{2n}(-y^4 + x^4)A$$

$$\Rightarrow u = z^{2n}(-y^4 + x^4); v = z^{2n}(-y^{2n-4} + x^{2n-4})$$

$$uv = A^2 - z^{2n} = (y^4 - x^4)(y^{2n-4} - x^{2n-4}) = uvz^{-4n} \Rightarrow uv = A^2 - z^{2n} = 0$$

It is impossible  $\Rightarrow n = 2$

Third case

or

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$

$$vx^{n-2} = x^2 z^n + Ay^2; vy^{n-2} = y^2 z^n + Ax^2$$

We have

$$vy^n = u(-y^n z^{2n} + A^2 x^n - Az^n (x^2 y^{n-2} - y^2 x^{n-2})) = u(-y^n z^{2n} + A^2 x^n + A^2 az^n) = u(A^2 - z^{2n})y^n$$

$$v = u(A^2 - z^{2n})$$

And

$$v(y^2 x^{n-2} - x^2 y^{n-2}) = vA = uvz^{2n}(-y^{2n-4} + x^{2n-4})A = z^{2n}(-y^4 + x^4)A$$

$$\Rightarrow uz^{2n}(-y^{2n-4} + x^{2n-4}) = 1 \Rightarrow u = \infty; n = 2$$

Impossible because u, A are integers

Fourth case

or

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$x^{n-2} = v(x^2 z^n + Ay^2); y^{n-2} = v(y^2 z^n + Ax^2)$$

We have

$$uy^n = v(-y^n z^{2n} + A^2 x^n - Az^n (x^2 y^{n-2} - y^2 x^{n-2})) = v(-y^n z^{2n} + A^2 x^n + A^2 az^n) = v(A^2 - z^{2n})y^n$$

$$u = v(A^2 - z^{2n})$$

And

$$u(y^2 x^{n-2} - x^2 y^{n-2}) = uvA = uz^{2n}(-y^{2n-4} + x^{2n-4})A = uvz^{2n}(-y^4 + x^4)A$$

$$\Rightarrow vz^{2n}(-y^{2n-4} + x^{2n-4}) = 1 \Rightarrow v = \infty, x^4 = y^4 + 1$$

Impossible! u, v, A are integers!

The only solution is A = 1 and n = 2

### Resolution of Catalan equation

Catalan equation is  $y^p = x^q + 1$

Let  $x^{q-3}y^2 - y^{p-2}x^3 = 1$

If A = 0  $\Rightarrow x^{q-6} = y^{p-4}$

$GCD(x, y) = 1 \Rightarrow p = 4$  impossible, there is no solution, Ko Chao proved it and

If  $A^2 = 1; p \geq 3 \Rightarrow x^{q-3}y - y^{p-3}x^3 = 1/y \in \mathbb{Z}$  impossible thus  $p=2$

We have

$$x^{q-3}y^2 - y^{p-2}x^3 = 1 = Ay^{p-2}y^2 - Ax^{q-3}x^3$$

$$\begin{aligned} \Rightarrow (x^{q-3} - Ay^{p-2})y^2 &= (y^{p-2} - Ax^{q-3})x^3 \\ (y^2 + Ax^3)x^{q-3} &= (x^3 + Ay^2)y^{p-2} \\ \text{GCD}(x, y) &= 1 \end{aligned}$$

We have then four cases :

$$\begin{aligned} x^3 &= u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3}) \\ x^{q-3} &= v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3) \end{aligned}$$

or

$$\begin{aligned} ux^3 &= -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3} \\ vx^{q-3} &= x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3 \end{aligned}$$

or

$$\begin{aligned} x^3 &= u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3}) \\ vx^{q-3} &= x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3 \end{aligned}$$

or

$$\begin{aligned} ux^3 &= -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3} \\ x^{q-3} &= v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3) \end{aligned}$$

$u, v \in \mathbb{Z}$

First case

$$\begin{aligned} x^3 &= u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3}) \\ x^{q-3} &= v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3) \end{aligned}$$

We have

$$\begin{aligned} y^p &= uv(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = uv(-y^p + A^2x^q + A^2) = uv(A^2 - 1)y^p \\ uv(A^2 - 1) &= 1 \end{aligned}$$

Impossible because A,u,v are integers

Second case

$$\begin{aligned} ux^3 &= -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3} \\ vx^{q-3} &= x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3 \end{aligned}$$

We have

$$\begin{aligned} uvv^p &= -y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3}) = -y^p + A^2x^q + A^2 = (A^2 - 1)y^p \\ uv &= A^2 - 1 \end{aligned}$$

And

$$\begin{aligned} uv(y^2x^{q-3} - x^3y^{p-2}) &= uvA = u(-y^{2p-4} + x^{2q-6})A = v(-y^4 + x^6)A \\ \Rightarrow u &= -y^4 + x^6; v = -y^{2p-4} + x^{2q-6} \\ uv &= A^2 - 1 = (-y^4 + x^6)(-y^{2p-4} + x^{2q-6}) \\ \Rightarrow (y^2x^{q-3} - x^3y^{p-2})^2 - 1 &= (y^4 - x^6)(y^{2p-4} - x^{2q-6}) \\ \Rightarrow x^{2q} + y^{2p} + 2x^qy^p &= 2y^4x^{2q-6} + 2x^6y^{2p-4} - 1 \\ = (x^q + y^p)^2 &= (2y^p - 1)^2 = 4y^{2p} - 4y^p + 1 \end{aligned}$$

$$2p > 4 \Rightarrow 2y^3x^{2q-6} + 2x^6y^{2p-5} - 4y^{2p-1} + 4y^{p-1} = 2/y \in \mathbb{Z}$$

Impossible :  $\Rightarrow p = 2$

Third case

$$\begin{aligned} x^3 &= u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3}) \\ vx^{q-3} &= x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3 \end{aligned}$$

We have

$$vy^p = u(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = u(-y^p + A^2x^q + A^2) = u(A^2 - 1)y^p$$

$$v = u(A^2 - 1)$$

And

$$v(y^2x^{q-3} - x^3y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-6})A = (-y^4 + x^6)A \Rightarrow u(-y^{2p-4} + x^{2q-6}) = 1 \Rightarrow p - 2 = q - 3 = 0$$

Impossible because u, A are integers

Fourth case

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}x^{q-3} = v(x^3 + Ay^2);$$

$$y^{p-2} = v(y^2 + Ax^3)$$

We have

$$uy^p = v(-y^p + A^2x^q - A(x^2y^{p-2} - y^2x^{q-3})) = v(-y^p + A^2x^q + A^2) = v(A^2 - 1)y^p$$

$$u = v(A^2 - 1)$$

And

$$u(y^2x^{q-3} - x^3y^{p-2}) = uA = (-y^{2p-4} + x^{2q-6})A = uv(-y^4 + x^6)A \Rightarrow v(-y^4 + x^6) = 1$$

Impossible, because v, A are integers !

The only solution is A=1 and p=2 this case has been studied by Ko Cha

### The Fermat-Catalan equation

The equation now is  $y^p = x^q \pm z^c = x^q + az^c$

$$\text{Let } x^{q-w}y^2 - y^{p-2}x^w = A$$

If

$$A = 0 \Rightarrow x^{q-2w} = y^{p-4}$$

$$\text{GCD}(x, y) = 1 \Rightarrow p = 4$$

It means that p=2 is the prime solution !

And

$$A^2 = z^{2c}; p \geq 3 \Rightarrow x^{q-w}y - y^{p-2}x^w = 1/y \in \mathbb{Z}$$

impossible thus p = 2

We have

$$(x^{q-w}y^2 - y^{p-2}x^w) - Az^c = (y^{p-2}y^2 - Ax^{q-w}x^w)$$

$$\Rightarrow (x^{q-w}z^c + y^{p-2})y^2 - (y^{p-2}z^c - Ax^{q-w})x^2$$

$$x^{q-w}y^2 - y^{p-2}x^w = aAz^c - Ay^{p-2}y^2 - Ax^{q-w}x^w$$

$$\Rightarrow (x^{q-w} - Ay^{p-2} - Ax^{q-w})x^w$$

$$(y^2 + Ax^w) - (x^w + Ay^2)y^{p-2}$$

$$\text{GCD}(x, y) = 1$$

We have then four cases :

First case

$$x^w = u(-x^{q-w}z^c + Ay^{p-2}); y^2 = u(-y^{p-2}z^c + Ax^{q-w})$$

$$x^{q-w} = v(x^wz^c + Ay^2); y^{p-2} = v(y^2z^c + Ax^w)$$

We have

$$y^p = uv(-y^p z^{2c} + A^2x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w})) = uv(-y^p z^{2c} + A^2x^q + A^2az^c) = uv(A^2 - z^{2c})y^p$$

$$uv(A^2 - z^{2c}) = 1$$

Impossible because A,u,v are integers

Second case

$$ux^w = -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w}$$

$$vx^{q-w} = x^wz^c + Ay^2; vy^{p-2} = y^2z^c + Ax^w$$

We have

$$uvy^p = -y^p z^{2c} + A^2 x^q - Az^c (x^w y^{p-2} - y^2 x^{q-w}) = -y^p z^{2c} + A^2 x^q + A^2 az^c = (A^2 - z^{2c})y^p$$

$$uv = A^2 - z^{2c}$$

And

$$uv(y^2 x^{q-w} - x^w y^{p-2}) = uvA = u(-y^{2p-4} + x^{2q-2w})A = v(-y^4 + x^{2w})A$$

$$\Rightarrow u = -y^4 + x^{2w}; v = -y^{2p-4} + x^{2q-2w}$$

$$uv = A^2 - z^{2c} = (y^4 - x^{2w})(y^{2p-4} - x^{2q-2w}) = uvz^{4c} \Rightarrow uv = A^2 - z^{2c} = 0$$

Impossible :  $\Rightarrow p = 2$

Third case

$$x^w = u(-x^{q-w}z^c + Ay^{p-2}); y^2 = u(-y^{p-2}z^c + Ax^{q-w})$$

$$vx^{q-w} = x^wz^c + Ay^2; vy^{p-2} = y^2z^c + Ax^w$$

We have

$$vy^p = u(-y^p z^{2c} + A^2 x^q - Az^c (x^w y^{p-2} - y^2 x^{q-w})) = u(-y^p z^{2c} + A^2 x^q + A^2 az^c) = (A^2 - z^{2c})y^p$$

$$v = u(A^2 - z^{2c})$$

And

$$v(y^2 x^{q-w} - x^w y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-2w})A = (-y^4 + x^{2w})A$$

$$\Rightarrow u(-y^{2p-4} + x^{2q-2w}) = 1 \Rightarrow p = 2$$

Impossible because u, A are integers

Fourth case

$$ux^w = -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w}$$

$$x^{q-w} = v(x^wz^c + Ay^2); y^{p-2} = v(y^2z^c + Ax^w)$$

We have

$$uy^p = v(-y^p z^{2c} + A^2 x^q - Az^c (x^w y^{p-2} - y^2 x^{q-w})) = v(-y^p z^{2c} + A^2 x^q + A^2 az^c) = v(A^2 - z^{2c})y^p$$

$$u = v(A^2 - z^{2c})$$

And

$$u(y^2 x^{q-w} - x^w y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-2w})A = uv(-y^4 + x^{2w})A$$

$$\Rightarrow v(-y^4 + x^{2w}) = 1$$

Impossible because u, A are integers ! In the Fermat-Catalan equation, one of the exponents must be equal to 1 ! The Beal conjecture has been proved !

In fact, in the three precedent equations studied here, one of the exponent greater or equal to 2 must be minimum, which means that it must be 2 !

### Conclusion

We have solved both three equations by the same method and proved two theorems and one conjecture.

### References

- [1] Paolo Ribenboim, The Catalan's conjecture, Academic Press, 1994.
- [2] Robert Tijdeman, On the equation of Catalan, Acta Arith, 1976.