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A New Hilbert-type Inequality with the Homogeneous Kernel of Degree -2 and with the Integral

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Abstract. By using the weight functions and by means of Hadamard's inequality, we present a new Hilbert-type inequality with the integral in whole plane, a best constant factor and a homogeneous kernel of degree-2 .

1 Introduction

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^2(x)\mathrm{d}x < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi \left\{\int_0^\infty f(x)dx \int_0^\infty g(x)\mathrm{d}x\right\}^{1/2} \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which has been extended by Hardy-Riesz as [2]
if $p > 1, 1/p + 1/q = 1, f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^q(x)\mathrm{d}x < \infty$
then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/q}. \tag{1.2}$$

with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$.

In recent years, inequalities (1.1) and (1.2) have many generalizations and variants. (1.1) has been strengthened by Yang and others. (including double series inequalities)[3-14].

In this paper, by using the way of weight function, we give a new Hilbert-type inequality with the integral in whole plane, a best constant factor and a homogeneous kernel of degree-2. We also consider its equivalent forms and the reverses.

In the following, we always suppose that: $1/p + 1/q = 1, p \neq 0, \theta \in (0, \pi)$.

2 Some lemmas

Lemma 2.1 Define the weight functions as follow:

$$w(x) := \int_{-\infty}^\infty \frac{|x|dy}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}},\quad w(y) := \int_{-\infty}^\infty \frac{|y|dx}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}.$$

Then, we have

$$w(x) = w(y) = K := \frac{1}{2\sin\frac{\theta}{2}}\ln\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} + \frac{1}{2\cos\frac{\theta}{2}}\ln\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}} \tag{2.1}$$

Proof. We only prove that $w(x)=K$ for $x\in(-\infty,0)$.

Setting $y=tx$, and $t=(1-u)/u$ then[3]

$$w(x)=\int_{-\infty}^{0}\frac{|x|\,dy}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}+\int_{0}^{\infty}\frac{|x|\,dy}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}:=w_1+w_2,$$

$$w_1=\int_{-\infty}^{0}\frac{(-x)\,dy}{(-x-y)\sqrt{y^2+2xy\cos\theta+x^2}}=\int_0^{\infty}\frac{dt}{(1+t)\sqrt{t^2+2t\cos\theta+1}}$$

$$=\int_0^1\frac{du}{\sqrt{(2-2\cos\theta)u^2-(2-2\cos\theta)u+1}}$$

$$=\frac{1}{2\sin\frac{\theta}{2}}\ln\left|\sqrt{(2-2\cos\theta)u^2-(2-2\cos\theta)u+1}+\sqrt{2-2\cos\theta}\,u-\frac{2-2\cos\theta}{2\sqrt{2-2\cos\theta}}\right|_0^1$$

$$=\frac{1}{2\sin\frac{\theta}{2}}\ln\frac{2\sin\frac{\theta}{2}+1-\cos\theta}{2\sin\frac{\theta}{2}+\cos\theta-1}$$

$$=\frac{1}{2\sin\frac{\theta}{2}}\ln\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}.$$

Setting $y=-tx$, then

$$w_2=\int_0^{\infty}\frac{(-x)\,dy}{(-x+y)\sqrt{y^2+2xy\cos\theta+x^2}}=\int_0^{\infty}\frac{dt}{(1+t)\sqrt{t^2-2t\cos\theta+1}}$$

$$=\int_0^{\infty}\frac{dt}{(1+t)\sqrt{t^2+2t\cos(\pi-\theta)+1}}$$

$$=\frac{1}{2\cos\frac{\theta}{2}}\ln\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}},$$

$$w(x)=w_1+w_2=\frac{1}{2\sin\frac{\theta}{2}}\ln\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}+\frac{1}{2\cos\frac{\theta}{2}}\ln\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}=K.$$

and then

Easily if $x\in(0,\infty)$, then we have $w(x)=K$ and

$$w(y)=K.$$

The lemma is proved.

Lemma 2.2 For $\frac{q}{4} > \varepsilon > 0, q > 1$ define both functions, f and g as follow:

$$f(x) = \begin{cases} x^{-2\varepsilon/p}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{-2\varepsilon/p}, & \text{if } x \in (-\infty,1), \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^{-2\varepsilon/q}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{-2\varepsilon/q}, & \text{if } x \in (-\infty,1). \end{cases}$$

Then

$$I(\varepsilon) := \varepsilon \left\{ \int_{-\infty}^{\infty} |x|^{-1} f^p(x) dx \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} |x|^{-1} g^q(x) dx \right\}^{1/q} = 1; \tag{2.2}$$

$$\tilde{I}(\varepsilon) := 2\varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(|x|+|y|)\sqrt{y^2 + 2xy\cos\theta + x^2}} dxdy = K + o(1) \;\; (\text{for } \varepsilon \to 0^+), \tag{2.3}$$

Proof Easily, we have

$$I(\varepsilon) := 2\varepsilon \left\{ \int_1^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} \left\{ \int_1^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/q} = 1;$$

Let $y = -Y$, using $f(-x) = f(x)$, $g(-x) = g(x)$, and

$$f(-x) \int_{-\infty}^{\infty} \frac{g(y)dy}{(|x|+|y|)\sqrt{y^2 - 2xy\cos\theta + x^2}} = f(x) \int_{-\infty}^{\infty} g(Y) \frac{g(Y)dY}{(|x|+|Y|)\sqrt{Y^2 + 2xY\cos\theta + x^2}}$$

we find that $F(x) = f(x) \int_{-\infty}^{\infty} \frac{g(y)dy}{(|x|+|y|)\sqrt{y^2 + 2xy\cos\theta + x^2}}$ is an even function, and

$$\tilde{I}(\varepsilon) = 2\varepsilon \int_0^{\infty} f(x) \left(\int_{-\infty}^{\infty} \frac{g(y)dy}{(|x|+|y|)\sqrt{y^2 + 2xy\cos\theta + x^2}} \right) dx := I_1 + I_2,$$

$$I_1 = 2\varepsilon \int_1^{\infty} x^{-2\varepsilon/p} \left(\int_{-\infty}^{-1} \frac{(-y)^{-\frac{2\varepsilon}{q}} dy}{(x-y)\sqrt{y^2 + 2xy\cos\theta + x^2}} \right) dx,$$

$$I_2 = 2\varepsilon \int_1^{\infty} x^{-2\varepsilon/p} \left(\int_1^{\infty} \frac{y^{-\frac{2\varepsilon}{q}} dy}{(x+y)\sqrt{y^2 + 2xy\cos\theta + x^2}} \right) dx.$$

Setting $y = xu$, then

$$I_1 = 2\varepsilon \int_1^{\infty} x^{-2\varepsilon/p} \left(\int_1^{\infty} \frac{y^{-\frac{2\varepsilon}{q}}}{(x+y)\sqrt{y^2 - 2xy\cos\theta + x^2}} dy \right) dx$$

$$= 2\varepsilon \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{\frac{1}{x}}^{\infty} \frac{u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2 - 2u\cos\theta + 1}} du \right) dx$$

$$= 2\varepsilon\left[\int_1^\infty x^{-1-2\varepsilon}\left(\int_1^\infty \frac{u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du\right)dx\right.$$

$$\left.+\int_1^\infty x^{-1-2\varepsilon}\left(\int_{\frac{1}{x}}^1 \frac{u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du\right)dx\right]$$

$$=\int_1^\infty \frac{u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du+2\varepsilon\int_0^1 \frac{u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}\left(\int_{1/u}^\infty x^{-1-2\varepsilon}dx\right)du$$

$$=\int_1^\infty \frac{u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du+\int_0^1 \frac{u^{\frac{2\varepsilon}{p}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du$$

$$=\int_0^\infty \frac{du}{(u+1)\sqrt{u^2-2u\cos\theta+1}}$$

$$+\int_1^\infty \frac{u^{-\frac{2\varepsilon}{q}}-1}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du+\int_0^1 \frac{u^{\frac{2\varepsilon}{p}}-1}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du$$

$$=\frac{1}{2\cos\frac{\theta}{2}}\ln\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}+o(1)\ (\text{for } \varepsilon\to 0^+).$$

If fact we find

$$\left|\frac{1-u^{-\frac{2\varepsilon}{q}}}{(u+1)\sqrt{u^2-2u\cos\theta+1}}\right|\le\frac{1}{(u+1)\sqrt{u^2-2u\cos\theta+1}},\ u\in(1,\infty),$$

$$\int_1^\infty \frac{1}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du\le w_2<\infty,$$

$$\left|\frac{u^{\frac{2\varepsilon}{p}}-1}{(u+1)\sqrt{u^2-2u\cos\theta+1}}\right|\le\frac{1}{(u+1)\sqrt{u^2-2u\cos\theta+1}},\ u\in(0,1),$$

$$\int_0^1 \frac{1}{(u+1)\sqrt{u^2-2u\cos\theta+1}}du\le w_2<\infty$$

And by Lebesgue convergence theorem, we have the last expression.

then we have the conclusion as follows:

$$I_1 \to w_2 \quad (\varepsilon \to 0^+),$$

Similarly $I_2 \to w_1 \,(\text{for } \varepsilon \to 0^+)$, and we have

$$\tilde{I}(\varepsilon) = I_1 + I_2 = K + o(1) \ (\text{for } \varepsilon \to 0^+).$$

The lemma is proved.

Lemma 2.3. If $p>1,\ f(x)\ge 0,\ 0<\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\,\mathrm{d}x<\infty$, then we have

$$J := \int_{-\infty}^{\infty}|y|^{p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right)^p dy \le K^p\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\,\mathrm{d}x. \tag{2.4}$$

Proof By lemma 2.2,we find

$$\left(\int_{-\infty}^{\infty} f(x)\left(\frac{1}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}\right)^{\frac{1}{p}}\left(\frac{1}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}\right)^{\frac{1}{q}}dx\right)^p$$

$$\le \int_{-\infty}^{\infty}\frac{1}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}} f^p(x)dx\left(\int_{-\infty}^{\infty}\frac{1}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right)^{p-1}$$

$$= K^{p-1}|y|^{-p+1}\int_{-\infty}^{\infty}\frac{f^p(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx,$$

$$J \le K^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{f^p(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right]dy$$

$$= K^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{dy}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}\right]f^p(x)dx$$

$$= K^p\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\,\mathrm{d}x \tag{2.5}$$

The lemma is proved.

3 Main results

Theorem 3.1 If both functions $f(x)$ and $g(x)$ are nonnegative measurable functions, and satisfy $0<\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\,\mathrm{d}x<\infty$,and $0<\int_{-\infty}^{\infty}|x|^{-1} g^q(x)\,\mathrm{d}x<\infty$, then

$$I^* := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dxdy$$

$$< K\left(\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\,\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1} g^q(x)\,\mathrm{d}x\right)^{1/q}, \tag{3.1}$$

And

$$J=\int_{-\infty}^{\infty}|y|^{p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right)^p dy<K^p\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\,\mathrm{d}x \tag{3.2}$$

Inequalities (3.1)and (3.2) are equivalent, and the constant factors in the two forms are all the best possible.

Proof If (2.5) takes the form of equality for some $y\in(-\infty,0)\cup(0,\infty)$, then there exist constants M and N, such that they are not all zero, and

$$Mf^p(x)=N \quad \text{a.e.} \quad \text{in} \quad (-\infty,\infty).$$

We claim that $M\neq 0$, otherwise $N=0$, then $|x|^{-1}f^p(x)=N/(M|x|)$ a.e. in $(-\infty,\infty)$ which contradicts the fact that $0<\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\,\mathrm{d}x<\infty$. Hence by (2.4), we have (3.2).

By Holder's inequality with weight and (3.2), we have,

$$I^*=\int_{-\infty}^{\infty}\left[|y|^{1/q}\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right]\left[|y|^{-1/q}g(y)\right]dy$$
$$\le (J)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{-1}g^q(y)\,\mathrm{d}y\right)^{1/q}. \tag{3.3}$$

Using (3.2),we have (3.1).

Setting $g(y)=|y|^{p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right)^{p-1}$, then $J=\int_{-\infty}^{\infty}|y|^{-1}g^q(y)\,\mathrm{d}y$, by (2.5) we have $J<\infty$. If $J=0$ then (3.2) is proved; If $0<J<\infty$, Then by (3.1),we obtain

$$0<\int_{-\infty}^{\infty}|y|^{-1}g^q(y)\,\mathrm{d}y=J=I^*<K\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\,\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}g^q(x)\,\mathrm{d}x\right)^{1/q},$$

$$\left(\int_{-\infty}^{\infty}|x|^{-1}g^q(x)\,\mathrm{d}x\right)^{1/p}=J^{1/p}<K\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\,\mathrm{d}x\right)^{1/p},$$

namely (3.1)and (3.2) are equivalent.

If the constant factor *K* in (3.1) is not the best possible, then there exists a positive number *h* (with *h*<*K*),such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dxdy<h\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\,\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}g^q(x)\,\mathrm{d}x\right)^{1/q}. \tag{3.4}$$

For ε>0,by (3.4),using lemma 2.2,we have

$$\tilde{I}(\varepsilon)=k+o(1)<\varepsilon h\left(\int_{-\infty}^{\infty}|x|^{-1}\tilde{f}^p(x)\,\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}\tilde{g}^q(x)\,\mathrm{d}x\right)^{1/q}=h.$$

Hence we find $K+o(1)<h$. For $\varepsilon\to 0^+$ it follows that $K\le h$, which contradicts the fact that *h*<*K*. Hence the constant *h* in (3.1) is the best possible. Since (3.1) and (3.2) are equivalent, if the constant factor in (3.2) is not the best possible, then by using (3.3), we can get a contradiction that the constant factor in (3.1) is not the best possible.

Thus we complete the proof of the theorem.

Theorem 3.2 If $1 > p > 0$, both functions $f(x)$ and $g(x)$ are nonnegative measurable functions, satisfying

$0 < \int_{-\infty}^{\infty} |x|^{-1} f^p(x)\mathrm{d}x < \infty$ and $0 < \int_{-\infty}^{\infty} |x|^{-1} g^q(x)\mathrm{d}x < \infty$, then

$$I^* := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dxdy$$

$$> K\left(\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1} g^q(x)\mathrm{d}x\right)^{1/q} \tag{3.5}$$

$$J = \int_{-\infty}^{\infty}|y|^{p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dx\right)^p dy > K^p\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\mathrm{d}x \tag{3.6}$$

and

$$L := \int_{-\infty}^{\infty}|x|^{q-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}g(y)dy\right)^q dx < K^q\int_{-\infty}^{\infty}|y|^{-1} g^q(y)dy. \tag{3.7}$$

Inequalities (3.5),(3.6)and (3.7) are equivalent, and the constant factors K, K^p and K^q are the best possible.

Proof By the reverse Holder's inequality and the same way, we can obtain the reverse forms of (2.5)and (3.3).And then we deduce (3.6),by the some way, we obtain (3.5).

Setting $g(y)$ as Theorem3.1 ,we obtain $J > 0$. If $J = \infty$, then we have (3.6) if $0 < J < \infty$, then by (3.5)

$$\int_{-\infty}^{\infty}|y|^{-1} g^q(y)dy = J = I^* > K\left(\int_{-\infty}^{\infty}|x|^{-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1} g^q(x)dx\right)^{1/q}.$$

Dividing $J^{\frac{1}{q}}$ in the above, and we have (3.6),Hence inequalities (3.5)and (3.6) are equivalent.

Setting

$$f(x) = |x|^{q-1}\int_{-\infty}^{\infty}\frac{g(y)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dy$$

By the same way, we find

$$\infty > \int_{-\infty}^{\infty}|x|^{-1} f^p(x)\mathrm{d}x = L = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dxdy$$

$$> K\left(\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1} g^q(x)\mathrm{d}x\right)^{1/q} \tag{3.8}$$

$$\int_{-\infty}^{\infty}|x|^{-1} f^p(x)\mathrm{d}x = L < K^q\int_{-\infty}^{\infty}|x|^{-1} g^q(x)\mathrm{d}x, \tag{3.9}$$

and we have (3.7). By the reverse Holder's inequality, we have

$$I^* = \int_{-\infty}^{\infty}\left[|x|^{-\frac{1}{p}} f(x)\right]\left[\int_{-\infty}^{\infty}|x|^{\frac{1}{p}} g(y)\frac{f(x)}{(|x|+|y|)\sqrt{y^2+2xy\cos\theta+x^2}}dy\right]dx$$

$$\geq \left(\int_{-\infty}^{\infty}|x|^{-1} f^p(x)dx\right)^{1/p} L^{1/q}. \tag{3.10}$$

Then by (3.7),we have (3.5),which is equivalent to (3.7).Therefore (3.5)-(3.7)are equivalent.

If there exist a constant $h^* > K$, such that (3.5) still valid as we replace K by h^*, the by the reverse of (3.4) we have: $K \geq h^* (\varepsilon \to 0^+)$,

Hence $h^* = K$ is the best value of (3.5). We conform that the constant factor of (3.6)(3.7) is the best possible, otherwise by the reverse of (3.3)(3.10), we can get a contradiction that the constant factor in (3.5)is not the best possible.

Remark. For $\theta = \frac{\pi}{4}$ in (3.1),we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)\sqrt{y^2+\sqrt{2}xy+x^2}}dxdy$$

$$<2\sqrt{2}\ln\left(\sqrt{2}+1\right)\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}g^q(x)\mathrm{d}x\right)^{1/q}$$

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