On the Inverse Sine Function, π and Babylonian Identity

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Abstract: We evaluate the constant π using the Babylonian identity and the inverse sine function.

1. INTRODUCTION

By means of the inverse sine function and Babylonian identity, we demonstrated the identities following, among others:

$$\begin{aligned}\pi &= \frac{2\sqrt{2}}{3} + \frac{16\sqrt{2+\sqrt{2}}}{3} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2}}}{2} \right)^{2n}, \\ \pi &= \frac{5}{8} \sqrt{\frac{5}{2} - \frac{\sqrt{5}}{2}} + 5 \sqrt{\frac{5}{2} + \frac{\sqrt{5}}{2}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \right)^{2n}, \\ \pi &= \frac{4}{7} \sqrt{2-\sqrt{2}} + \frac{32}{7} \sqrt{2+\sqrt{2+\sqrt{2}}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \right)^{2n}, \\ \pi &= \frac{3}{5} + \frac{24(1+\sqrt{3})}{5\sqrt{2}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{1+\sqrt{3}}{2\sqrt{2}} \right)^{2n}, \\ \pi &= \frac{5(\sqrt{5}-1)}{18} + \frac{20(1+\sqrt{5}+\sqrt{10-2\sqrt{5}})}{9\sqrt{2}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{1+\sqrt{5}}{4\sqrt{2}} + \frac{\sqrt{5-\sqrt{5}}}{4} \right)^{2n}, \\ \pi &= \frac{5(1+\sqrt{5})}{14} + \frac{10}{7} \left(2\sqrt{5+\sqrt{5}} + \sqrt{10} - \sqrt{2} \right) \sum_{k=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{5+\sqrt{5}}}{4} - \frac{1-\sqrt{5}}{4\sqrt{2}} \right)^{2n} \\ \text{and} \\ \pi &= \frac{3\sqrt{3}}{4} + 6\sqrt{3} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{3}}{2} \right)^{2n}.\end{aligned}$$

2. LEMMAS

Lemma 1. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = \frac{1}{2} \sqrt{1 - \left(\frac{a-b}{a+b} \right)^2} \quad (1)$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b} \right)^2}} \quad (2)$$

Proof. we knew the Babylonian identity [1, page 119]

$$ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]. \quad (3)$$

Make the following algebraic manipulation in (3)

$$ab = \left(\frac{a+b}{2}\right)^2 \left[1 - \left(\frac{a-b}{a+b}\right)^2\right],$$

hence,

$$a^{\frac{1}{2}}b^{\frac{1}{2}} = \frac{a+b}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2} \Leftrightarrow \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = \frac{1}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2},$$

and inverting both members, we have

$$a^{-\frac{1}{2}}b^{-\frac{1}{2}} = \frac{2}{a+b} \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}} \Leftrightarrow \frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}}. \square$$

Lemma 2. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 (2k-1)} \left(\frac{a-b}{a+b}\right)^{2k} \quad (4)$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} \left(\frac{a-b}{a+b}\right)^{2k}. \quad (5)$$

Proof. we calculate

$$\sqrt{1-z^2} = -\sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 (2k-1)} z^{2k} \quad (6)$$

and

$$\frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} z^{2k} \quad (7)$$

Take $z = \frac{a-b}{a+b}$ in (6) and (7), then replace in (1) and (2) respectively, completing the proof. $\square$

3. THEOREMS

Theorem 1. For $x \in [0,1]$, then

$$\sin^{-1}(x) = x \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2 (2n+1)} x^{2n}.$$

where $\sin^{-1}(x)$ denotes the inverse sine function.

Proof. Put $\frac{a-b}{a+b} = xt$ in (5), we encounter

$$\frac{1}{\sqrt{1-x^2t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} x^{2n} t^{2n}. \quad (8)$$

we integrate (8) from 0 at 1 in t then

$$\int_0^1 \frac{dt}{\sqrt{1-x^2t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} x^{2n} \int_0^1 t^{2n} dt. \quad (9)$$

we calculate

$$\int_0^1 \frac{dt}{\sqrt{1-x^2t^2}} = \frac{\sin^{-1}(x)}{x} \quad (10)$$

and

$$\int_0^1 t^{2n} dt = \frac{1}{2n+1}, \quad (11)$$

$$n > -\frac{1}{2}.$$

for

From (9), (10) and (11), it follows

$$\frac{\sin^{-1}(x)}{x} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n}. \quad (12)$$

Multiply (12) by x and the desired result follows. $\square$

Explicit Calculus. We obtain easily

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3},$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

and

$$\sin^{-1}(1) = \frac{\pi}{2}.$$

Theorem 2. For $x \in [0,1]$, then

$$\sin^{-1}(x) = x\sqrt{1-x^2} + 4x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n},$$

where $\sin^{-1}(x)$ denotes the inverse sine function.

Proof. Using the identity

$$\frac{1}{2n+1} = \frac{4n}{4n^2-1} - \frac{1}{2n-1} \quad (13)$$

in the right-hand side of Theorem 1, we have

$$x \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n} = 4x \sum_{n=0}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} - x \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n},$$

accordingly,

$$\sin^{-1}(x) = 4x \sum_{n=0}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} - x \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n}. \quad (14)$$

we calculate

$$\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n-1)} x^{2n} = -\sqrt{1-x^2}. \quad (15)$$

we substitute (15) into (14), and obtain

$$\begin{aligned} \sin^{-1}(x) &= 4x \sum_{n=0}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} + x\sqrt{1-x^2} \Rightarrow \\ 4x \sum_{n=0}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} &= \sin^{-1}(x) - x\sqrt{1-x^2} \Rightarrow \\ 4x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} x^{2n} &= \sin^{-1}(x) - x\sqrt{1-x^2}. \square \end{aligned}$$

Theorem 3. For $x \in [0, \pi]$, then

$$\sin^{-1}(\cos x) = \frac{\sin(2x)}{2} + 4 \cos x \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (\cos x)^{2n},$$

where $\sin^{-1}(x)$ denotes the inverse sine function, $\sin(x)$ denotes the sine function and $\cos(x)$ denotes the cosine function.

Proof. Let $x = \cos x$ in Theorem 2

$$4 \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} (\cos x)^{2n+1} = \sin^{-1}(\cos x) - \frac{\sin(2x)}{2} \quad (16)$$

We sum $\frac{\sin(2x)}{2}$ in both members of (16) and the proof is complete. $\square$

Explicit Calculations. we leave for the reader:

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2}}}{2} \right)^{2n} = \frac{3\pi - 2\sqrt{2}}{16\sqrt{2+\sqrt{2}}},$$

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \right)^{2n} = \frac{8\sqrt{2}\pi - 5\sqrt{5-\sqrt{5}}}{40\sqrt{5+\sqrt{5}}},$$

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \right)^{2n} = \frac{7\pi - 4\sqrt{2-\sqrt{2}}}{32\sqrt{2+\sqrt{2+\sqrt{2}}}},$$

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{1+\sqrt{3}}{2\sqrt{2}} \right)^{2n} = \frac{\sqrt{2}(5\pi-3)}{24(1+\sqrt{3})},$$

$$\sum_{k=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{1+\sqrt{5}}{4\sqrt{2}} + \frac{\sqrt{5-\sqrt{5}}}{4} \right)^{2n} = \frac{18\sqrt{2}\pi - 5(\sqrt{10}-\sqrt{2})}{40(1+\sqrt{5}+\sqrt{10-2\sqrt{5}})},$$

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{5+\sqrt{5}}}{4} - \frac{1-\sqrt{5}}{4\sqrt{2}} \right)^{2n} = \frac{14\pi - 5(1+\sqrt{5})}{20(2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2})}$$

and

$$\sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{3}}{2} \right)^{2n} = \frac{4\pi - 3\sqrt{3}}{24\sqrt{3}},$$

consequently,

$$\pi = \frac{2\sqrt{2}}{3} + \frac{16\sqrt{2+\sqrt{2}}}{3} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2}}}{2} \right)^{2n},$$

$$\pi = \frac{5}{8} \sqrt{\frac{5}{2} - \frac{\sqrt{5}}{2}} + 5 \sqrt{\frac{5}{2} + \frac{\sqrt{5}}{2}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \right)^{2n},$$

$$\pi = \frac{4}{7} \sqrt{2-\sqrt{2}} + \frac{32}{7} \sqrt{2+\sqrt{2+\sqrt{2}}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \right)^{2n},$$

$$\pi = \frac{3}{5} + \frac{24(1+\sqrt{3})}{5\sqrt{2}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{1+\sqrt{3}}{2\sqrt{2}} \right)^{2n},$$

$$\pi = \frac{5(\sqrt{5}-1)}{18} + \frac{20(1+\sqrt{5}+\sqrt{10-2\sqrt{5}})}{9\sqrt{2}} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{1+\sqrt{5}}{4\sqrt{2}} + \frac{\sqrt{5-\sqrt{5}}}{4} \right)^{2n},$$

$$\pi = \frac{5(1+\sqrt{5})}{14} + \frac{10}{7} (2\sqrt{5+\sqrt{5}}+\sqrt{10}-\sqrt{2}) \sum_{k=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{5+\sqrt{5}}}{4} - \frac{1-\sqrt{5}}{4\sqrt{2}} \right)^{2n}$$

and

$$\pi = \frac{3\sqrt{3}}{4} + 6\sqrt{3} \sum_{n=1}^{\infty} \frac{(2n)!n}{2^{2n}(n!)^2(4n^2-1)} \left(\frac{\sqrt{3}}{2} \right)^{2n}.$$

Corollary 1.

$$\frac{\pi^2}{8} = \frac{1}{2} + 2\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k-1)!(4k^2-1)\Gamma\left(k+\frac{3}{2}\right)}, \frac{\pi}{2} \ln x$$

Proof. we integrate the Theorem 3 from 0 at $\frac{\pi}{2} \ln x$

$$\int_0^{\frac{\pi}{2}} \sin^{-1}(\cos x) dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2x)}{2} dx + 4 \sum_{k=1}^{\infty} \frac{(2k)! k}{2^{2k}(k!)^2(4k^2-1)} \int_0^{\frac{\pi}{2}} (\cos x)^{2k+1} dx. \quad (17)$$

we calculate

$$\int_0^{\frac{\pi}{2}} \sin^{-1}(\cos x) dx = \frac{\pi^2}{8}, \quad (18)$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2x)}{2} dx = \frac{1}{2}, \quad (19)$$

and

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2k+1} dx = \frac{\sqrt{\pi}\Gamma(k+1)}{2\Gamma\left(k+\frac{3}{2}\right)}, \quad (20)$$

For $k > -1$

From (17), (18), (19) and (20), the proof follows. $\square$