

An Investigation on the Beta Function II: The Summations of $1/\sqrt{\pi}$

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Abstract. I developed new summations of $1/\sqrt{\pi}$.

1. Introduction

In this paper, I demonstrated the formulas, among others:

$$\frac{1}{\sqrt{\pi}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{4^n n! \Gamma\left(n + \frac{5}{2}\right)},$$

$$\frac{1}{\sqrt{\pi}} = \frac{3}{16} \sum_{n=0}^{\infty} \frac{(2n+7)(2n)!}{4^n n! \Gamma\left(n + \frac{7}{2}\right)}$$

and

$$\frac{1}{\sqrt{\pi}} = \frac{45}{512} \sum_{n=0}^{\infty} \frac{(4n^2 + 28n + 53)(2n)!}{4^n n! \Gamma\left(n + \frac{9}{2}\right)}.$$

2. THEOREM

Theorem 1. For an integer positive, $M = 1, 2, 3, \dots$, then

$$\sum_{n=1}^M \frac{1}{n \Gamma\left(n + \frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!} \sum_{n=1}^M \frac{1}{\Gamma\left(n + k + \frac{3}{2}\right)}.$$

Proof. In previous paper [1], Theorem 1, I demonstrated that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma(x+k)}{4^k k!^2 \Gamma\left(x+y+k+\frac{1}{2}\right)}, \quad (1)$$

for $\Re(x) > 0$ and $\Re(y) > 0$. Let $x = 1$ and $y = n$ in (1)

$$\frac{\Gamma(n)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k! \Gamma\left(n+k+\frac{3}{2}\right)}$$

$$\frac{1}{n \Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k! \Gamma\left(n+k+\frac{3}{2}\right)} \quad (2)$$

Let the sum from 1 at M in n , then

$$\sum_{n=1}^M \frac{1}{n \Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!} \sum_{n=1}^M \frac{1}{\Gamma\left(n+k+\frac{3}{2}\right)}. \quad \square$$

Explicit Evaluation. Let $M = 1, 2, 3, 4, 5$ and 6 in Theorem 1, then:

$$\frac{1}{\sqrt{\pi}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{4^n n! \Gamma\left(n + \frac{5}{2}\right)},$$

$$\frac{1}{\sqrt{\pi}} = \frac{3}{16} \sum_{n=0}^{\infty} \frac{(2n+7)(2n)!}{4^n n! \Gamma\left(n + \frac{7}{2}\right)},$$

$$\frac{1}{\sqrt{\pi}} = \frac{45}{512} \sum_{n=0}^{\infty} \frac{(4n^2 + 28n + 53)(2n)!}{4^n n! \Gamma\left(n + \frac{9}{2}\right)},$$

$$\frac{1}{\sqrt{\pi}} = \frac{315}{7264} \sum_{n=0}^{\infty} \frac{(8n^3 + 92n^2 + 358n + 485)(2n)!}{4^n n! \Gamma\left(n + \frac{11}{2}\right)},$$

$$\frac{1}{\sqrt{\pi}} = \frac{4725}{218432} \sum_{n=0}^{\infty} \frac{(16n^4 + 272n^3 + 1728n^2 + 4908n + 5351)(2n)!}{4^n n! \Gamma\left(n + \frac{13}{2}\right)},$$

$$\frac{1}{\sqrt{\pi}} = \frac{155925}{14421632} \sum_{n=0}^{\infty} \frac{(32n^5 + 752n^4 + 6992n^3 + 32280n^2 + 74506n + 69595)}{4^n n! \Gamma\left(n + \frac{15}{2}\right)}.$$

REFERENCES

- [1] Guedes, Edigles, *An Investigation on the Beta Function I: New Versions of the Euler Beta Function*, 2013.