

An Investigation on the Beta Function I: New Versions of the Euler Beta Function

Edigles Guedes¹ and Prof. Dr. K. Raja Rama Gandhi²

Number Theorist, Brazil¹

Resource perosn in Mathematics for Oxford University Press and Professor at BITS-Vizag²

Abstract. I developed new versions of the Euler beta function.

1. Introduction

In this paper, I discovered the formulas, among others:

$$\frac{1}{\pi} = -\frac{4\Gamma\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(-\frac{5}{6}\right)},$$

$$\frac{1}{\pi} = \frac{1}{\left[\frac{1}{8}(1 + \sqrt{5}) - \frac{1}{4}\sqrt{\frac{3}{2}(5 - \sqrt{5})}\right]\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)},$$

$$\frac{1}{\pi} = -\frac{2}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}$$

and

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{(1 - \sqrt{3})\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}.$$

2. THEOREM

Theorem 1. For $\Re(x) > 0$ and $\Re(y) > 0$, then

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y + \frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(x+k)}{4^k k!^2 \Gamma\left(x+y+k + \frac{1}{2}\right)}.$$

Proof. In [1, equation 7], I have

$$\frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} z^{2k}, \tag{1}$$

consequently,

$$\frac{1}{\sqrt{1-t}} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} t^k. \tag{2}$$

On the other hand, [2, page 898], I encounter the definition

$$B(x, y) \stackrel{\text{def}}{=} \int_0^1 t^{x-1}(1-t)^{y-1} dt. \tag{3}$$

I did the following algebraic manipulation in the above definition

$$B(x, y) = \int_0^1 \frac{t^{x-1}(1-t)^{y-\frac{1}{2}}}{\sqrt{1-t}} dt. \quad (4)$$

I substitute (2) into (4), and obtain

$$\begin{aligned} B(x, y) &= \int_0^1 t^{x-1}(1-t)^{y-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} t^k dt \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} \int_0^1 t^{x+k-1}(1-t)^{y-\frac{1}{2}} dt \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} \cdot \frac{\Gamma\left(y + \frac{1}{2}\right) \Gamma(x+k)}{\Gamma\left(x+y+k + \frac{1}{2}\right)} \\ &= \Gamma\left(y + \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(x+k)}{4^k k!^2 \Gamma\left(x+y+k + \frac{1}{2}\right)}. \end{aligned} \quad (5)$$

In [2, page 899], I meet

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (6)$$

From (5) and (6), I take

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y + \frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(x+k)}{4^k k!^2 \Gamma\left(x+y+k + \frac{1}{2}\right)}.$$

Corollary 1. For $\Re(y) > 0$, then

$$\frac{\Gamma(y)}{\Gamma(y+1)\Gamma\left(y + \frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k! \Gamma\left(y+k + \frac{3}{2}\right)}$$

Proof. Let $x = 1$ and this completes the proof. \square

Corollary 2. For $\Re(x) > 0$ and $\Re(y) > 0$, then

$$\begin{aligned} \frac{1}{\pi} &= \frac{1}{\cos(\pi(x+y))\Gamma\left(-x-y + \frac{1}{2}\right)\Gamma\left(x+y + \frac{1}{2}\right)}, \\ \frac{1}{\pi} &= \frac{-1}{\cos(\pi y)\Gamma\left(-y - \frac{1}{2}\right)\Gamma\left(y + \frac{3}{2}\right)} \end{aligned}$$

and

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)}.$$

Proof. In [2, page 887], I have

$$\Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos(\pi z)},$$

hence,

$$\frac{1}{\Gamma\left(z + \frac{1}{2}\right)} = \Gamma\left(\frac{1}{2} - z\right) \frac{\cos(\pi z)}{\pi}. \tag{7}$$

I substitute (8) into Theorem 1 and Corollary 1, separately.

$$\frac{\pi\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y + \frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \cos[\pi(x+y+k)] \Gamma(x+k)\Gamma\left(\frac{1}{2} - x - y - k\right)}{4^k k!^2},$$

so

$$\begin{aligned} \frac{\pi\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y + \frac{1}{2}\right)} &= \frac{\Gamma(x)\Gamma(y)\cos(\pi(x+y))\Gamma\left(-x-y+\frac{1}{2}\right)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y + \frac{1}{2}\right)} \Rightarrow \\ \frac{1}{\pi} &= \frac{1}{\cos(\pi(x+y))\Gamma\left(-x-y+\frac{1}{2}\right)\Gamma\left(x+y+\frac{1}{2}\right)} \end{aligned} \tag{8}$$

and

$$\frac{\pi\Gamma(y)}{\Gamma(y+1)\Gamma\left(y + \frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma\left(-\frac{1}{2} - y - k\right) \cos[\pi(y+k+1)]}{4^k k!},$$

so

$$\frac{\pi\Gamma(y)}{\Gamma(y+1)\Gamma\left(y + \frac{1}{2}\right)} = -\frac{\cos(\pi y)\Gamma\left(-y - \frac{1}{2}\right)\Gamma(y)\Gamma\left(y + \frac{3}{2}\right)}{\Gamma(y+1)\Gamma\left(y + \frac{1}{2}\right)} \Rightarrow \frac{1}{\pi} = \frac{-1}{\cos(\pi y)\Gamma\left(-y - \frac{1}{2}\right)\Gamma\left(y + \frac{3}{2}\right)}$$

I take $y = \frac{1}{2}$ in (8), and have

$$\Gamma(x+1)\Gamma(-x) = \frac{\pi}{\cos\left(\pi\left(x + \frac{1}{2}\right)\right)} = -\frac{\pi}{\sin(\pi x)}.$$

On the other hand, $\Gamma(x+1) = x\Gamma(x)$, that substituting in the previous equation gives the desired result. \square

Corollary 3. *I have*

$$\Gamma\left(\frac{1}{30}\right) = 30\Gamma\left(\frac{31}{30}\right),$$

$$\Gamma\left(\frac{1}{3}\right) = 3\Gamma\left(\frac{4}{3}\right),$$

$$\Gamma\left(\frac{1}{12}\right) = 12\Gamma\left(\frac{13}{12}\right),$$

$$\frac{1}{\pi} = -\frac{4\Gamma\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(-\frac{5}{6}\right)},$$

$$\frac{1}{\pi} = \frac{1}{\left[\frac{1}{8}(1+\sqrt{5}) - \frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})}\right]\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)},$$

$$\frac{1}{\pi} = -\frac{2}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}$$

and

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{(1-\sqrt{3})\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}.$$

Proof. Let $x = \frac{1}{3}$ and $y = \frac{1}{5}$ in Theorem 1, then

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{7}{10}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma\left(\frac{1}{3}+k\right)}{4^k k!^2 \Gamma\left(\frac{31}{30}+k\right)} = \frac{30\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{31}{30}\right)}{\Gamma\left(\frac{1}{30}\right)\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{7}{10}\right)} \Rightarrow \Gamma\left(\frac{1}{30}\right) = 30\Gamma\left(\frac{31}{30}\right).$$

Let $x = \frac{1}{3}$ and $y = \frac{1}{2}$ in Theorem 1, then

$$\frac{\Gamma\left(\frac{1}{3}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{6}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma\left(\frac{1}{3}+k\right)}{4^k k!^2 \Gamma\left(\frac{4}{3}+k\right)} = \frac{3\sqrt{\pi}\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} \Rightarrow \Gamma\left(\frac{1}{3}\right) = 3\Gamma\left(\frac{4}{3}\right).$$

Let $x = \frac{1}{3}$ and $y = \frac{1}{4}$ in Theorem 1, then

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{3}{4}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma\left(\frac{1}{3}+k\right)}{4^k k!^2 \Gamma\left(\frac{13}{12}+k\right)} = \frac{12\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{13}{12}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{3}{4}\right)} \Rightarrow \Gamma\left(\frac{1}{12}\right) = 12\Gamma\left(\frac{13}{12}\right).$$

Let $y = \frac{1}{3}$ in Corollary 2, then

$$\begin{aligned} \frac{\pi\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)} &= \sum_{k=0}^{\infty} \frac{(2k)!\Gamma\left(-\frac{1}{2}-y-k\right)\cos[\pi(y+k+1)]}{4^k k!} = -\frac{5}{4}\Gamma\left(-\frac{5}{6}\right) \\ &\Rightarrow \frac{1}{\pi} = -\frac{4\Gamma\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(-\frac{5}{6}\right)}. \end{aligned}$$

Let $x = \frac{1}{5}$ and $y = \frac{1}{3}$ in Corollary 2, then

$$\frac{1}{\pi} = \frac{1}{\cos\left(\frac{8\pi}{15}\right)\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)} = \frac{1}{\left[\frac{1}{8}(1+\sqrt{5}) - \frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})}\right]\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)}$$

and

$$\frac{1}{\pi} = \frac{-1}{\cos\left(\frac{\pi}{3}\right)\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)} = \frac{-2}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}.$$

Let $x = \frac{1}{4}$ and $y = \frac{1}{3}$ in Corollary 2, then

$$\frac{1}{\pi} = \frac{1}{\cos\left(\frac{7\pi}{12}\right)\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)} = \frac{1}{\frac{1-\sqrt{3}}{2\sqrt{2}}\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)} = \frac{2\sqrt{2}}{(1-\sqrt{3})\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}.$$

Corollary 4. For $\Re(x) > 0$ and $\Re(y) > 0$, then

$$\frac{\sqrt{\pi}\Gamma(x)\Gamma(y)}{2^{2x+2y-1}\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma(x+k)\Gamma(x+y+k)}{k!^2\Gamma[2(x+y+k)]}.$$

and

$$\frac{\sqrt{\pi}\Gamma(y)}{2^{2y+1}\Gamma(y+1)\Gamma\left(y+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma(y+k+1)}{k!\Gamma[2(y+k+1)]}$$

Proof. In [2, page 887], I have

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \Rightarrow \frac{1}{\Gamma\left(x+\frac{1}{2}\right)} = \frac{2^{2x-1}}{\sqrt{\pi}}\frac{\Gamma(x)}{\Gamma(2x)}. \tag{9}$$

I substitute (9) into Theorem 1 and Corollary 1, this completes the proof. \square

Corollary 6. I have

$$\sqrt{\pi} = 2\Gamma\left(\frac{3}{2}\right)$$

and

$$\Gamma\left(\frac{1}{4}\right) = 4\Gamma\left(\frac{5}{4}\right).$$

Proof. Let $y = \frac{1}{2}$ in Corollary 4, then

$$\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{3}{2}\right)\Gamma(1)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma\left(\frac{3}{2}+k\right)}{k!\Gamma(2k+3)} = \frac{\sqrt{\pi}}{2} \Rightarrow \sqrt{\pi} = 2\Gamma\left(\frac{3}{2}\right).$$

Let $y = \frac{1}{4}$ in Corollary 4, then

$$\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2^{\frac{3}{2}}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!\Gamma\left(\frac{5}{4}+k\right)}{k!\Gamma\left(\frac{5}{2}+2k\right)} = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{3}{4}\right)} \Rightarrow \Gamma\left(\frac{1}{4}\right) = 4\Gamma\left(\frac{5}{4}\right).$$

Theorem 2. For $\Re(z) > 0$, then

$$\Gamma(z) \sum_{n=1}^M \frac{\Gamma(n)}{\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(z+k)}{4^k k!^2} \sum_{n=1}^M \frac{1}{\Gamma\left(z+n+k+\frac{1}{2}\right)}$$

and

$$\frac{\sqrt{\pi}\Gamma(z)}{2^{2z-1}} \cdot \sum_{n=1}^M \frac{\Gamma(n)}{2^{2n}\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(z+k)}{k!^2} \sum_{n=1}^M \frac{\Gamma(z+n+k)}{\Gamma[2(z+n+k)]}$$

Proof. Let $x = z$ and $y = n$ in Theorem 1 and Corollary 4

$$\Gamma(z) \frac{\Gamma(n)}{\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(z+k)}{4^k k!^2 \Gamma\left(z+n+k+\frac{1}{2}\right)} \quad (10)$$

and

$$\frac{\sqrt{\pi}\Gamma(z)}{2^{2z-1}} \cdot \frac{\Gamma(n)}{2^{2n}\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(z+k) \Gamma(x+n+k)}{k!^2 \Gamma[2(z+n+k)]} \quad (11)$$

I let the summation from 1 at M to n in (10) and (11)

$$\Gamma(z) \sum_{n=1}^M \frac{\Gamma(n)}{\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(z+k)}{4^k k!^2} \sum_{n=1}^M \frac{1}{\Gamma\left(z+n+k+\frac{1}{2}\right)}$$

and

$$\frac{\sqrt{\pi}\Gamma(z)}{2^{2z-1}} \cdot \sum_{n=1}^M \frac{\Gamma(n)}{2^{2n}\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma(z+k)}{k!^2} \sum_{n=1}^M \frac{\Gamma(z+n+k)}{\Gamma[2(z+n+k)]} \cdot \square$$

REFERENCES

- [1] Guedes, Edigles, *On the Complete Elliptic Integrals and Babylonian Identity I: The $1/\pi$ formulae Involving Gamma functions and Summations*, 2013.
- [2] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, 2000.