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An Investigation on the Beta Function I: New Versions of the Euler Beta Function

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Abstract. I developed new versions of the Euler beta function.

1. Introduction

In this paper, I discovered the formulas, among others:

$$\frac{1}{\pi}=-\frac{4\Gamma\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(-\frac{5}{6}\right)},$$

$$\frac{1}{\pi}=\frac{1}{\left[\frac{1}{8}\left(1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5-\sqrt{5}\right)}\right]\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)},$$

$$\frac{1}{\pi}=-\frac{2}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}$$

and

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{\left(1-\sqrt{3}\right)\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}.$$

2. THEOREM

Theorem 1. *For* $\Re(x)>0$ *and* $\Re(y)>0$, *then*

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(x+k)}{4^k k!^2\,\Gamma\left(x+y+k+\frac{1}{2}\right)}.$$

Proof. In [1, equation 7], I have

$$\frac{1}{\sqrt{1-z^2}}=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}z^{2k}, \tag{1}$$

consequently,

$$\frac{1}{\sqrt{1-t}}=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}t^{k}. \tag{2}$$

On the other hand, [2, page 898], I encounter the definition

$$B(x,y)\stackrel{\text{def}}{=}\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t. \tag{3}$$

I did the following algebraic manipulation in the above definition

$$B(x,y)=\int_0^1 \frac{t^{x-1}(1-t)^{y-\frac{1}{2}}}{\sqrt{1-t}}\,\mathrm{d}t. \tag{4}$$

I substitute (2) into (4), and obtain

$$\begin{aligned}B(x,y)&=\int_0^1 t^{x-1}(1-t)^{y-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}t^k\,\mathrm{d}t\\&=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}\int_0^1 t^{x+k-1}(1-t)^{y-\frac{1}{2}}\,\mathrm{d}t\\&=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}\cdot\frac{\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+k)}{\Gamma\left(x+y+k+\frac{1}{2}\right)}\\&=\Gamma\left(y+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(2k)!\Gamma(x+k)}{4^k k!^2\Gamma\left(x+y+k+\frac{1}{2}\right)}.\end{aligned} \tag{5}$$

In [2, page 899], I meet

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{6}$$

From (5) and (6), I take

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(x+k)}{4^k k!^2\,\Gamma\left(x+y+k+\frac{1}{2}\right)}.$$

Corollary 1. *For* $\Re(y)>0$, *then*

$$\frac{\Gamma(y)}{\Gamma(y+1)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!\,\Gamma\left(y+k+\frac{3}{2}\right)}$$

Proof. Let $x=1$ and this completes the proof. □

Corollary 2. *For* $\Re(x)>0$ *and* $\Re(y)>0$, *then*

$$\frac{1}{\pi}=\frac{1}{\cos(\pi(x+y))\Gamma\left(-x-y+\frac{1}{2}\right)\Gamma\left(x+y+\frac{1}{2}\right)},$$

$$\frac{1}{\pi}=\frac{-1}{\cos(\pi y)\Gamma\left(-y-\frac{1}{2}\right)\Gamma\left(y+\frac{3}{2}\right)}$$

and

$$\Gamma(x)\Gamma(-x)=-\frac{\pi}{x\sin(\pi x)}.$$

Proof. In [2, page 887], I have

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos(\pi z)},$$

hence,

$$\frac{1}{\Gamma\left(z+\frac{1}{2}\right)}=\Gamma\left(\frac{1}{2}-z\right)\frac{\cos(\pi z)}{\pi}. \tag{7}$$

I substitute (8) into Theorem 1 and Corollary 1, separately.

$$\frac{\pi\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\cos[\pi(x+y+k)]\,\Gamma(x+k)\Gamma\left(\frac{1}{2}-x-y-k\right)}{4^k k!^2},$$

so

$$\frac{\pi\Gamma(x)\Gamma(y)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\frac{\Gamma(x)\Gamma(y)\cos(\pi(x+y))\Gamma\left(-x-y+\frac{1}{2}\right)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}\Rightarrow$$

$$\frac{1}{\pi}=\frac{1}{\cos(\pi(x+y))\Gamma\left(-x-y+\frac{1}{2}\right)\Gamma\left(x+y+\frac{1}{2}\right)} \tag{8}$$

and

$$\frac{\pi\Gamma(y)}{\Gamma(y+1)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(-\frac{1}{2}-y-k\right)\cos[\pi(y+k+1)]}{4^k k!},$$

so

$$\frac{\pi\Gamma(y)}{\Gamma(y+1)\Gamma\left(y+\frac{1}{2}\right)}=-\frac{\cos(\pi y)\Gamma\left(-y-\frac{1}{2}\right)\Gamma(y)\Gamma\left(y+\frac{3}{2}\right)}{\Gamma(y+1)\Gamma\left(y+\frac{1}{2}\right)}\Rightarrow\frac{1}{\pi}=\frac{-1}{\cos(\pi y)\Gamma\left(-y-\frac{1}{2}\right)\Gamma\left(y+\frac{3}{2}\right)}$$

I take $y=\frac{1}{2}$ in (8), and have

$$\Gamma(x+1)\Gamma(-x)=\frac{\pi}{\cos\left(\pi\left(x+\frac{1}{2}\right)\right)}=-\frac{\pi}{\sin(\pi x)}.$$

On the other hand, $\Gamma(x+1)=x\Gamma(x)$, that subsitituting in the previous equation gives the desired result. □

Corollary 3. *I have*

$$\Gamma\left(\frac{1}{30}\right)=30\Gamma\left(\frac{31}{30}\right),$$

$$\Gamma\left(\frac{1}{3}\right)=3\Gamma\left(\frac{4}{3}\right),$$

$$\Gamma\left(\frac{1}{12}\right)=12\Gamma\left(\frac{13}{12}\right),$$

$$\frac{1}{\pi}=-\frac{4\Gamma\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(-\frac{5}{6}\right)},$$

$$\frac{1}{\pi}=\frac{1}{\left[\frac{1}{8}(1+\sqrt{5})-\frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})}\right]\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)},$$

$$\frac{1}{\pi}=-\frac{2}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}$$

and

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{(1-\sqrt{3})\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}.$$

Proof. Let $x=\frac{1}{3}$ and $y=\frac{1}{5}$ in Theorem 1, then

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{7}{10}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(\frac{1}{3}+k\right)}{4^k k!^2\,\Gamma\left(\frac{31}{30}+k\right)}=\frac{30\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{31}{30}\right)}{\Gamma\left(\frac{1}{30}\right)\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{7}{10}\right)}\Rightarrow\Gamma\left(\frac{1}{30}\right)=30\Gamma\left(\frac{31}{30}\right).$$

Let $x=\frac{1}{3}$ and $y=\frac{1}{2}$ in Theorem 1, then

$$\frac{\Gamma\left(\frac{1}{3}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{6}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(\frac{1}{3}+k\right)}{4^k k!^2\,\Gamma\left(\frac{4}{3}+k\right)}=\frac{3\sqrt{\pi}\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}\Rightarrow\Gamma\left(\frac{1}{3}\right)=3\Gamma\left(\frac{4}{3}\right).$$

Let $x=\frac{1}{3}$ and $y=\frac{1}{4}$ in Theorem 1, then

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{3}{4}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(\frac{1}{3}+k\right)}{4^k k!^2\,\Gamma\left(\frac{13}{12}+k\right)}=\frac{12\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{13}{12}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{3}{4}\right)}\Rightarrow\Gamma\left(\frac{1}{12}\right)=12\Gamma\left(\frac{13}{12}\right).$$

Let $y=\frac{1}{3}$ in Corollary 2, then

$$\frac{\pi\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(-\frac{1}{2}-y-k\right)\cos[\pi(y+k+1)]}{4^k k!}=-\frac{5}{4}\Gamma\left(-\frac{5}{6}\right)$$

$$\Rightarrow\frac{1}{\pi}=-\frac{4\Gamma\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(-\frac{5}{6}\right)}.$$

Let $x=\frac{1}{5}$ and $y=\frac{1}{3}$ in Corollary 2, then

$$\frac{1}{\pi}=\frac{1}{\cos\left(\frac{8\pi}{15}\right)\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)}=\frac{1}{\left[\frac{1}{8}(1+\sqrt{5})-\frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})}\right]\Gamma\left(-\frac{1}{30}\right)\Gamma\left(\frac{31}{30}\right)}$$

and

$$\frac{1}{\pi}=\frac{-1}{\cos\left(\frac{\pi}{3}\right)\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}=\frac{-2}{\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)}.$$

Let $x=\frac{1}{4}$ and $y=\frac{1}{3}$ in Corollary 2, then

$$\frac{1}{\pi}=\frac{1}{\cos\left(\frac{7\pi}{12}\right)\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}=\frac{1}{\frac{1-\sqrt{3}}{2\sqrt{2}}\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}=\frac{2\sqrt{2}}{(1-\sqrt{3})\Gamma\left(-\frac{1}{12}\right)\Gamma\left(\frac{13}{12}\right)}.$$

Corollary 4. *For* $\Re(x)>0$ *and* $\Re(y)>0$*, then*

$$\frac{\sqrt{\pi}\Gamma(x)\Gamma(y)}{2^{2x+2y-1}\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(x+k)\Gamma(x+y+k)}{k!^2\,\Gamma[2(x+y+k)]}.$$

and

$$\frac{\sqrt{\pi}\Gamma(y)}{2^{2y+1}\Gamma(y+1)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(y+k+1)}{k!\,\Gamma[2(y+k+1)]}$$

Proof. In [2, page 887], I have

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)\Rightarrow\frac{1}{\Gamma\left(x+\frac{1}{2}\right)}=\frac{2^{2x-1}}{\sqrt{\pi}}\frac{\Gamma(x)}{\Gamma(2x)}. \tag{9}$$

I substitute (9) into Theorem 1 and Corollary 1, this completes the proof. □

Corollary 6. *I have*

$$\sqrt{\pi}=2\Gamma\left(\frac{3}{2}\right)$$

and

$$\Gamma\left(\frac{1}{4}\right)=4\Gamma\left(\frac{5}{4}\right).$$

Proof. Let $y=\frac{1}{2}$ in Corollary 4, then

$$\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{3}{2}\right)\Gamma(1)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(\frac{3}{2}+k\right)}{k!\,\Gamma(2k+3)}=\frac{\sqrt{\pi}}{2}\Rightarrow\sqrt{\pi}=2\Gamma\left(\frac{3}{2}\right).$$

Let $y=\frac{1}{4}$ in Corollary 4, then

$$\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2^{\frac{3}{2}}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma\left(\frac{5}{4}+k\right)}{k!\,\Gamma\left(\frac{5}{2}+2k\right)}=\frac{\sqrt{2\pi}}{\Gamma\left(\frac{3}{4}\right)}\Rightarrow\Gamma\left(\frac{1}{4}\right)=4\Gamma\left(\frac{5}{4}\right).$$

Theorem 2. *For* $\Re(z) > 0$*, then*

$$\Gamma(z)\sum_{n=1}^{M}\frac{\Gamma(n)}{\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(z+k)}{4^k k!^2}\sum_{n=1}^{M}\frac{1}{\Gamma\left(z+n+k+\frac{1}{2}\right)}$$

and

$$\frac{\sqrt{\pi}\Gamma(z)}{2^{2z-1}}\cdot\sum_{n=1}^{M}\frac{\Gamma(n)}{2^{2n}\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(z+k)}{k!^2}\sum_{n=1}^{M}\frac{\Gamma(z+n+k)}{\Gamma[2(z+n+k)]}.$$

Proof. Let $x = z$ and $y = n$ in Theorem 1 and Corollary 4

$$\Gamma(z)\frac{\Gamma(n)}{\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\Gamma(z+k)}{4^k k!^2\Gamma\left(z+n+k+\frac{1}{2}\right)} \quad (10)$$

and

$$\frac{\sqrt{\pi}\Gamma(z)}{2^{2z-1}}\cdot\frac{\Gamma(n)}{2^{2n}\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\Gamma(z+k)\Gamma(x+n+k)}{k!^2\Gamma[2(z+n+k)]}. \quad (11)$$

I let the summation from 1 at M to n in (10) and (11)

$$\Gamma(z)\sum_{n=1}^{M}\frac{\Gamma(n)}{\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(z+k)}{4^k k!^2}\sum_{n=1}^{M}\frac{1}{\Gamma\left(z+n+k+\frac{1}{2}\right)}$$

and

$$\frac{\sqrt{\pi}\Gamma(z)}{2^{2z-1}}\cdot\sum_{n=1}^{M}\frac{\Gamma(n)}{2^{2n}\Gamma(z+n)\Gamma\left(n+\frac{1}{2}\right)}=\sum_{k=0}^{\infty}\frac{(2k)!\,\Gamma(z+k)}{k!^2}\sum_{n=1}^{M}\frac{\Gamma(z+n+k)}{\Gamma[2(z+n+k)]}.\ \square$$

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