Another Proof that the Catalan's Constant is Irrational

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ABSTRACT. We use the contradiction method for prove, again, that the Catalan's constant is irrational.

1. INTRODUCTION

In Mathematics, the Catalan's constant [1] is defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \quad (1.1)$$

The Catalan's constant was named after Eugène Charles Catalan (30 May 1814 – 14 February 1894), a French and Belgian mathematician.

In previous paper [2], we prove that the constant G is irrational. In this paper, we damos outra prova de que the constant G is irrational.

2. THE PROOF

LEMMA. *The Catalan's constant have the following representation in series*

$$G = 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{(16n^2 + 16n + 3)^2}.$$

Proof. We developed the power series formula from the definition of Catalan's constant as follows

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\ &= \frac{(-1)^0}{(2 \cdot 0 + 1)^2} + \frac{(-1)^1}{(2 \cdot 1 + 1)^2} + \frac{(-1)^2}{(2 \cdot 2 + 1)^2} + \frac{(-1)^3}{(2 \cdot 3 + 1)^2} + \frac{(-1)^4}{(2 \cdot 4 + 1)^2} + \frac{(-1)^5}{(2 \cdot 5 + 1)^2} + \dots \\ &= \frac{1}{(2 \cdot 0 + 1)^2} - \frac{1}{(2 \cdot 1 + 1)^2} + \frac{1}{(2 \cdot 2 + 1)^2} - \frac{1}{(2 \cdot 3 + 1)^2} + \frac{1}{(2 \cdot 4 + 1)^2} - \frac{1}{(2 \cdot 5 + 1)^2} + \dots \\ &= \frac{1}{(2 \cdot 0 + 1)^2} + \frac{1}{(2 \cdot 2 + 1)^2} + \frac{1}{(2 \cdot 4 + 1)^2} + \dots - \left[\frac{1}{(2 \cdot 1 + 1)^2} + \frac{1}{(2 \cdot 3 + 1)^2} + \frac{1}{(2 \cdot 5 + 1)^2} + \dots \right] \\ &= \frac{1}{[2 \cdot (2 \cdot 0) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 1) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 2) + 1]^2} + \dots \\ &\quad - \left\{ \frac{1}{[2 \cdot (2 \cdot 0 + 1) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 2 + 1) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 2 + 1) + 1]^2} + \dots \right\} \\ &= \frac{1}{(4 \cdot 0 + 1)^2} + \frac{1}{(4 \cdot 1 + 1)^2} + \frac{1}{(4 \cdot 2 + 1)^2} + \dots - \left[\frac{1}{(4 \cdot 0 + 3)^2} + \frac{1}{(4 \cdot 1 + 3)^2} + \frac{1}{(4 \cdot 2 + 3)^2} + \dots \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\ &= \sum_{n=0}^{\infty} \frac{(4n+3)^2 - (4n+1)^2}{(4n+1)^2(4n+3)^2} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{16n+8}{(4n+1)^2(4n+3)^2} \\
 &= 8 \sum_{n=0}^{\infty} \frac{2n+1}{[(4n+1)(4n+3)]^2} \\
 &= 8 \sum_{n=0}^{\infty} \frac{2n+1}{(16n^2+16n+3)^2}. \square
 \end{aligned}$$

THEOREM. *The Catalan's constant is irrational.*

Proof. We will use the *reductio ad absurdum*.

By hypothesis, we suppose that G is a rational number. Of course, there exist two positive integers a and b , such that $G = a/b$, where, clearly, $b > 1$. Firstly, we define the number

$$x := \frac{(16b^2+16b+3)!^2}{4^{8b^2+8b+1}(8b^2+8b+1)!(8b^2+8b)!} \cdot \left(G - 8 \sum_{n=0}^b \frac{2n+1}{(16n^2+16n+3)^2} \right). \quad (2.1)$$

If G is rational, then x is an integer. We substitute $G = a/b$ into this definition to find

$$\begin{aligned}
 x &= \frac{(16b^2+16b+3)!^2}{4^{8b^2+8b+1}(8b^2+8b+1)!(8b^2+8b)!} \cdot \left(\frac{a}{b} - 8 \sum_{n=0}^b \frac{2n+1}{(16n^2+16n+3)^2} \right) \\
 &= \frac{(16b^2+16b+3)!^2 a}{4^{8b^2+8b+1}b(8b^2+8b+1)!(8b^2+8b)!} - 8 \sum_{n=0}^b \frac{(16b^2+16b+3)!^2(2n+1)}{4^{8b^2+8b+1}(8b^2+8b+1)!(8b^2+8b)!(16n^2+16n+3)^2}. \quad (2.2)
 \end{aligned}$$

It is obvious that the first term is an integer; because, $b > 1$, then $4^b(b!)^2 < (2b+1)!^2$. The second term is an integer; because, for $b > 1$, then $(2n+1)^2 4^b b((b-1)!)^2 < (2b+1)!^2$. Hence x is an integer.

We, now, demonstrate that $0 < x < 1$.

First, we demonstrate that x is strictly positive, we insert the series representation of G into the definition of x and we find

$$\begin{aligned}
 x &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^b \frac{(-1)^n}{(2n+1)^2} \right| = \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \right| = \\
 &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{\cos(\pi n)}{(2n+1)^2} \right| > \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \int_{b+1}^{\infty} \frac{\cos(\pi x)}{(2x+1)^2} dx \right| = \\
 &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| -\frac{1}{4} \cos \left(\left(\left(b + \frac{3}{2} \right) \pi \right) \right) - \frac{\cos(\pi b)}{4b+6} \right| > 0. \quad (2.3)
 \end{aligned}$$

On the other hand, for all terms with $2n+1 \geq 2b+2$, i.e., $2n \geq 2b+1$, we have the upper estimate

$$\frac{(2b+1)!}{(2n+1)!} \leq \frac{1}{(2b+2)^{2n-2b}}. \quad (2.4)$$

This inequality is strict for every $2n+1 \geq 2b+3$, i.e., $n \geq b+1$. Thereof, we substitute (1.1) and (2.4) in (2.1)

$$\begin{aligned}
 x &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^b \frac{(-1)^n}{(2n+1)^2} \right| \\
 &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \right| < \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2n+1)!^2} \right|
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4^b b ((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n (2b+1)!^2}{(2n+1)!^2} \right| < \frac{1}{4^b b ((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2b+2)^{2n-2b}} \right| \\
&= \frac{1}{4^b b ((b-1)!)^2} \left| -\frac{(-1)^b}{4b^2+8b+5} \right| < 1. \tag{2.5}
\end{aligned}$$

Since there is no integer strictly between 0 and 1, we have get in a contradiction, and so G must be irrational. $\square$

REFERENCES

- [1] http://en.wikipedia.org/wiki/Catalan's_constant, available in July 12, 2013.
- [2] Guedes, Edigles, *An Elegant Proof that the Catalan's Constant is Irrational*, July 12, 2013, vixra.

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