

## Another Proof that the Catalan's Constant is Irrational

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**ABSTRACT.** We use the contradiction method for prove, again, that the Catalan's constant is irrational.

### 1. INTRODUCTION

In Mathematics, the Catalan's constant [1] is defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \quad (1.1)$$

The Catalan's constant was named after Eugène Charles Catalan (30 May 1814 – 14 February 1894), a French and Belgian mathematician.

In previous paper [2], we prove that the constant  $G$  is irrational. In this paper, we damos outra prova de que the constant  $G$  is irrational.

### 2. THE PROOF

LEMMA. The Catalan's constant have the following representation in series

$$G = 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{(16n^2 + 16n + 3)^2}.$$

*Proof.* We developed the power series formula from the definition of Catalan's constant as follows

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\ &= \frac{(-1)^0}{(2 \cdot 0 + 1)^2} + \frac{(-1)^1}{(2 \cdot 1 + 1)^2} + \frac{(-1)^2}{(2 \cdot 2 + 1)^2} + \frac{(-1)^3}{(2 \cdot 3 + 1)^2} + \frac{(-1)^4}{(2 \cdot 4 + 1)^2} + \frac{(-1)^5}{(2 \cdot 5 + 1)^2} + \dots \\ &= \frac{1}{(2 \cdot 0 + 1)^2} - \frac{1}{(2 \cdot 1 + 1)^2} + \frac{1}{(2 \cdot 2 + 1)^2} - \frac{1}{(2 \cdot 3 + 1)^2} + \frac{1}{(2 \cdot 4 + 1)^2} - \frac{1}{(2 \cdot 5 + 1)^2} + \dots \\ &= \frac{1}{(2 \cdot 0 + 1)^2} + \frac{1}{(2 \cdot 2 + 1)^2} + \frac{1}{(2 \cdot 4 + 1)^2} + \dots - \left[ \frac{1}{(2 \cdot 1 + 1)^2} + \frac{1}{(2 \cdot 3 + 1)^2} + \frac{1}{(2 \cdot 5 + 1)^2} + \dots \right] \\ &= \frac{1}{[2 \cdot (2 \cdot 0) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 1) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 2) + 1]^2} + \dots \\ &\quad - \left\{ \frac{1}{[2 \cdot (2 \cdot 0 + 1) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 2 + 1) + 1]^2} + \frac{1}{[2 \cdot (2 \cdot 2 + 1) + 1]^2} + \dots \right\} \\ &= \frac{1}{(4 \cdot 0 + 1)^2} + \frac{1}{(4 \cdot 1 + 1)^2} + \frac{1}{(4 \cdot 2 + 1)^2} + \dots - \left[ \frac{1}{(4 \cdot 0 + 3)^2} + \frac{1}{(4 \cdot 1 + 3)^2} + \frac{1}{(4 \cdot 2 + 3)^2} + \dots \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\ &= \sum_{n=0}^{\infty} \frac{(4n+3)^2 - (4n+1)^2}{(4n+1)^2(4n+3)^2} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{16n + 8}{(4n + 1)^2(4n + 3)^2} \\
 &= 8 \sum_{n=0}^{\infty} \frac{2n + 1}{[(4n + 1)(4n + 3)]^2} \\
 &= 8 \sum_{n=0}^{\infty} \frac{2n + 1}{(16n^2 + 16n + 3)^2}. \square
 \end{aligned}$$

**THEOREM.** *The Catalan’s constant is irrational.*

*Proof.* We will use the *reductio ad absurdum*.

By hypothesis, we suppose that  $G$  is a rational number. Of course, there exist two positive integers  $a$  and  $b$ , such that  $G = a/b$ , where, clearly,  $b > 1$ . Firstly, we define the number

$$x := \frac{(16b^2+16b+3)!^2}{4^{8b^2+8b+1}(8b^2+8b+1)!(8b^2+8b)!} \cdot \left( G - 8 \sum_{n=0}^b \frac{2n+1}{(16n^2+16n+3)^2} \right). \tag{2.1}$$

If  $G$  is rational, then  $x$  is an integer. We substitute  $G = a/b$  into this definition to find

$$\begin{aligned}
 x &= \frac{(16b^2 + 16b + 3)!^2}{4^{8b^2+8b+1}(8b^2 + 8b + 1)!(8b^2 + 8b)!} \cdot \left( \frac{a}{b} - 8 \sum_{n=0}^b \frac{2n + 1}{(16n^2 + 16n + 3)^2} \right) \\
 &= \frac{(16b^2+16b+3)!^2 a}{4^{8b^2+8b+1}b(8b^2+8b+1)!(8b^2+8b)!} - 8 \sum_{n=0}^b \frac{(16b^2+16b+3)!^2(2n+1)}{4^{8b^2+8b+1}(8b^2+8b+1)!(8b^2+8b)!(16n^2+16n+3)^2}. \tag{2.2}
 \end{aligned}$$

It is obvious that the first term is an integer; because, for  $b > 1$ , then  $4^b(b!)^2 < (2b + 1)!^2$ . The second term is an integer; because, for  $b > 1$ , then  $(2n + 1)^2 4^b b((b - 1)!)^2 < (2b + 1)!^2$ . Hence  $x$  is an integer.

We, now, demonstrate that  $0 < x < 1$ .

First, we demonstrate that  $x$  is strictly positive, we insert the series representation of  $G$  into the definition of  $x$  and we find

$$\begin{aligned}
 x &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} - \sum_{n=0}^b \frac{(-1)^n}{(2n+1)^2} \right| = \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \right| = \\
 &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{\cos(\pi n)}{(2n+1)^2} \right| > \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| \int_{b+1}^{\infty} \frac{\cos(\pi x)}{(2x+1)^2} dx \right| = \\
 &= \frac{(2b+1)!^2}{4^b b((b-1)!)^2} \left| -\frac{1}{4} \cos\left(\left(b + \frac{3}{2}\right)\pi\right) - \frac{\cos(\pi b)}{4b+6} \right| > 0. \tag{2.3}
 \end{aligned}$$

On the other hand, for all terms with  $2n + 1 \geq 2b + 2$ , i.e.,  $2n \geq 2b + 1$ , we have the upper estimate

$$\frac{(2b+1)!}{(2n+1)!} \leq \frac{1}{(2b+2)^{2n-2b}}. \tag{2.4}$$

This inequality is strict for every  $2n + 1 \geq 2b + 3$ , i.e.,  $n \geq b + 1$ . Thereof, we substitute (1.1) and (2.4) in (2.1)

$$\begin{aligned}
 x &= \frac{(2b + 1)!^2}{4^b b((b - 1)!)^2} \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} - \sum_{n=0}^b \frac{(-1)^n}{(2n + 1)^2} \right| \\
 &= \frac{(2b + 1)!^2}{4^b b((b - 1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2n + 1)^2} \right| < \frac{(2b + 1)!^2}{4^b b((b - 1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2n + 1)!^2} \right|
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4^b b ((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n (2b+1)!^2}{(2n+1)!^2} \right| < \frac{1}{4^b b ((b-1)!)^2} \left| \sum_{n=b+1}^{\infty} \frac{(-1)^n}{(2b+2)^{2n-2b}} \right| \\
&= \frac{1}{4^b b ((b-1)!)^2} \left| -\frac{(-1)^b}{4b^2+8b+5} \right| < 1. \tag{2.5}
\end{aligned}$$

Since there is no integer strictly between 0 and 1, we have get in a contradiction, and so  $G$  must be irrational.  $\square$

## REFERENCES

- [1] [http://en.wikipedia.org/wiki/Catalan's\\_constant](http://en.wikipedia.org/wiki/Catalan's_constant), available in July 12, 2013.
- [2] Guedes, Edigles, *An Elegant Proof that the Catalan's Constant is Irrational*, July 12, 2013, vixra.

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