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Exploring Prime Numbers and Modular Functions III: On the Exponential of Prime Number via Dedekind Eta Function

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ABSTRACT. The main objective this paper is to develop asymptotic formulas for the exponential of prime number, using Dedekind eta function.

1. INTRODUCTION

As consequence of the prime number theorem, I get the asymptotic formula for the *n*th prime number, denoted by p_n:

$$p_n \sim n \ln n. \tag{1}$$

M. Pervouchine, in *Mémories de la Société physic-mathématique de Kasan*, [1, page 848] deduced that

$$\frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{5}{12 \ln n} + \frac{1}{24 \ln \ln n}. \tag{2}$$

On the other hand, Ernest Cesáro, in *Sur une formule empirique de M. Pervouchine* [1, page 849], I encounter the formula

$$\frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2}, \tag{3}$$

in modern notation,

$$\frac{p_n}{n} = \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2} + O\left(\frac{1}{(\ln n)^2}\right). \tag{4}$$

On the other hand, the Dedekind eta function was introduced by Richard Dedekind, in 1877, and is defined in the half-plane $\mathbb{H} = \{\tau : \Im(\tau) > 0\}$ by the equation [2, page 47]

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau}\right). \tag{5}$$

The infinite product has the form $\prod(1 - x^n)$, where $x = e^{2\pi i \tau}$. If $\tau \in \mathbb{H}$, then $|x| < 1$, so the product converges absolutely and is nonzero.

In [2, page 48], I have

$$-\frac{1}{2} \ln y = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} - \frac{\pi}{12}\left(y - \frac{1}{y}\right). \tag{6}$$

Hence,

$$\ln y = 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} + \frac{\pi}{6}\left(y - \frac{1}{y}\right) \tag{7}$$

and

$$y = \frac{6 \ln y}{\pi} - \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} + \frac{1}{y}, \tag{8}$$

for $y > 0$.

In this paper, I prove, among other things, that

$$p_n \approx n\left(1-\frac{1}{\ln n}-\frac{2}{(\ln n)^2}\right)\left(\frac{6\ln\ln n}{\pi}+\frac{1}{\ln n}+\frac{12}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right)$$

$$+n\left(1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right)\left[\frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)-2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]-\frac{11n}{2(\ln n)^2}.$$

2. THEOREM

LEMMA 1. *I have*

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n(n^2-n-2)+(2n^2+2n+6-\ln n)\ln n-11,$$

where p_n denotes the nth prime number defined by $p_n\colon \mathbb{N}\to\mathbb{R}^+$.

Proof. Multiplying (3) by $2(\ln n)^2$, I obtain

$$\frac{2p_n(\ln n)^2}{n} \approx 2(\ln n)^3+2(\ln n)^2\ln\ln n-2(\ln n)^2+2\ln n\,(\ln\ln n-2)-(\ln\ln n)^2+6\ln\ln n-11,$$

thereupon, letting $n \to e^n$, I find

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n^3+2n^2\ln n-2n^2+2n(\ln n-2)-(\ln n)^2+6\ln n-11,$$

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n(n^2-n-2)+(2n^2+2n+6-\ln n)\ln n-11. \square$$

COROLLARY 1. *I have*

$$\frac{2n^2 p_{\lfloor e^n\rfloor}}{\lfloor e^n\rfloor} \approx 2n(n^2-n-2)+(2n^2+2n+6-\ln n)\ln n-11,$$

where p_n denotes the nth prime number defined by $p_n\colon \mathbb{N}\to\mathbb{R}^+$ and $\lfloor n\rfloor$ denotes the floor function.

Proof. I note that $\lfloor e^n\rfloor \approx e^n$, arising from the definition of the floor function. □

THEOREM 1. *For n sufficiently large, then*

$$p_n \approx \left[\frac{12\ln\ln n}{\pi}+\frac{2}{\ln n}-\frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{n^{2\pi k}-1}+\frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]\left[n\left(\frac{1}{2}-\frac{1}{2\ln n}-\frac{1}{(\ln n)^2}\right)\right]$$

$$+\left[\frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)+2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{n^{2\pi k}-1}-2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]\left\{n\left[1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right]\right\}$$

$$-\frac{11n}{2(\ln n)^2},$$

where p_n denotes the nth prime number.

Proof. I put (7) and (8) into Lemma 1, as following

$$\frac{2n^2 p_{e^n}}{e^n} \approx \left[\frac{12\ln n}{\pi} - \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/n}} + \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi kn}} + \frac{2}{n}\right](n^2-n-2)$$
$$+ (2n^2+2n+6-\ln n)\left[2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/n}} - 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi kn}} + \frac{\pi}{6}\left(n-\frac{1}{n}\right)\right] - 11.$$

Dividing both members of above equation by $2n^2+2n+6-\ln n$

$$\frac{2n^2 p_{e^n}}{e^n(2n^2+2n+6-\ln n)}$$
$$\approx \left[\frac{12\ln n}{\pi} - \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/n}} + \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi kn}} + \frac{2}{n}\right]\left(\frac{n^2-n-2}{2n^2+2n+6-\ln n}\right)$$
$$+\left[2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/n}} - 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi kn}} + \frac{\pi}{6}\left(n-\frac{1}{n}\right)\right] - \frac{11}{2n^2+2n+6-\ln n}.$$

I return to replace n by $\ln n$ in the previous equation

$$\frac{2(\ln n)^2 p_n}{n[2(\ln n)^2+2\ln n+6-\ln\ln n]}$$
$$\approx \left[\frac{12\ln\ln n}{\pi} - \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/\ln n}} + \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-n^{2\pi k}} + \frac{2}{\ln n}\right]\left[\frac{(\ln n)^2-\ln n-2}{2(\ln n)^2+2\ln n+6-\ln\ln n}\right] +$$
$$\left[2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/\ln n}} - 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-n^{2\pi k}} + \frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)\right] - \frac{11}{2(\ln n)^2+2\ln n+6-\ln\ln n}. \quad (9)$$

Multiplying both members of (9) by $\frac{n[2(\ln n)^2+2\ln n+6-\ln\ln n]}{2(\ln n)^2}$, I encounter

$$p_n \approx \left[\frac{12\ln\ln n}{\pi} - \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/\ln n}} + \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-n^{2\pi k}} + \frac{2}{\ln n}\right]\left[n\left(\frac{1}{2}-\frac{1}{2\ln n}-\frac{1}{(\ln n)^2}\right)\right]$$
$$+\left[2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-e^{2\pi k/\ln n}} - 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-n^{2\pi k}} + \frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)\right]\left\{n\left[1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right]\right\}$$
$$-\frac{11n}{2(\ln n)^2},$$

consequently,

$$p_n \approx \left[\frac{12\ln\ln n}{\pi} + \frac{2}{\ln n} - \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{n^{2\pi k}-1} + \frac{24}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]\left[n\left(\frac{1}{2}-\frac{1}{2\ln n}-\frac{1}{(\ln n)^2}\right)\right]$$
$$+\left[\frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right) + 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{n^{2\pi k}-1} - 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]\left\{n\left[1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right]\right\}$$
$$-\frac{11n}{2(\ln n)^2}.$$

This completes the Theorem. □

COROLLARY 2. *For n sufficiently large, then*

$$p_n \approx n\left(1-\frac{1}{\ln n}-\frac{2}{(\ln n)^2}\right)\left(\frac{6\ln\ln n}{\pi}+\frac{1}{\ln n}+\frac{12}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right)$$

$$+n\left(1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right)\left[\frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)-2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]-\frac{11n}{2(\ln n)^2},$$

*where p_n denotes the n*th *prime number.*

Proof. I note that, for $n \in \mathbb{N}_{>1}$,

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{n^{2\pi k}-1} \downarrow 0. \tag{11}$$

From Theorem 1 and (11), I obtain

$$p_n \approx n\left(1-\frac{1}{\ln n}-\frac{2}{(\ln n)^2}\right)\left(\frac{6\ln\ln n}{\pi}+\frac{1}{\ln n}+\frac{12}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right)$$

$$+n\left(1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right)\left[\frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)-2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]-\frac{11n}{2(\ln n)^2}.$$

This completes the proof. □

CONJECTURE: I observe that, numerically, p_n is better approximate if I add the term $\frac{5n}{12\ln n}+\frac{n}{24\ln\ln n}$, so

$$p_n \approx n\left(1-\frac{1}{\ln n}-\frac{2}{(\ln n)^2}\right)\left(\frac{6\ln\ln n}{\pi}+\frac{1}{\ln n}+\frac{12}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right)$$

$$+n\left(1+\frac{1}{\ln n}+\frac{3}{(\ln n)^2}-\frac{\ln\ln n}{2(\ln n)^2}\right)\left[\frac{\pi}{6}\left(\ln n-\frac{1}{\ln n}\right)-2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi k/\ln n}-1}\right]-\frac{11n}{2(\ln n)^2}$$

$$+\frac{5n}{12\ln n}+\frac{n}{24\ln\ln n}.$$

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