

Exploring Prime Numbers and Modular Functions III: On the Exponential of Prime Number via Dedekind Eta Function

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ABSTRACT. The main objective this paper is to develop asymptotic formulas for the exponential of prime number, using Dedekind eta function.

1. INTRODUCTION

As consequence of the prime number theorem, I get the asymptotic formula for the n th prime number, denoted by p_n :

$$p_n \sim n \ln n. \quad (1)$$

M. Pervouchine, in *Mémoires de la Société physic-mathématique de Kasan*, [1, page 848] deduced that

$$\frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{5}{12 \ln n} + \frac{1}{24 \ln \ln n}. \quad (2)$$

On the other hand, Ernest Cesáro, in *Sur une formule empirique de M. Pervouchine* [1, page 849], I encounter the formula

$$\frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2}, \quad (3)$$

in modern notation,

$$\frac{p_n}{n} = \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2} + O\left(\frac{1}{(\ln n)^2}\right). \quad (4)$$

On the other hand, the Dedekind eta function was introduced by Richard Dedekind, in 1877, and is defined in the half-plane $\mathbb{H} = \{\tau: \Im(\tau) > 0\}$ by the equation [2, page 47]

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}). \quad (5)$$

The infinite product has the form $\prod(1 - x^n)$, where $x = e^{2\pi i \tau}$. If $\tau \in \mathbb{H}$, then $|x| < 1$, so the product converges absolutely and is nonzero.

In [2, page 48], I have

$$-\frac{1}{2} \ln y = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / y}} - \frac{\pi}{12} \left(y - \frac{1}{y}\right). \quad (6)$$

Hence,

$$\ln y = 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / y}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} + \frac{\pi}{6} \left(y - \frac{1}{y}\right) \quad (7)$$

and

$$y = \frac{6 \ln y}{\pi} - \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / y}} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} + \frac{1}{y}, \quad (8)$$

for $y > 0$.

In this paper, I prove, among other things, that

$$p_n \approx n \left(1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left(\frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k e^{2\pi k / \ln n} - 1} \right) \\ + n \left(1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[\frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2}.$$

2. THEOREM

LEMMA 1. *I have*

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n(n^2 - n - 2) + (2n^2 + 2n + 6 - \ln n) \ln n - 11,$$

where p_n denotes the n th prime number defined by $p_n: \mathbb{N} \rightarrow \mathbb{R}^+$.

Proof. Multiplying (3) by $2(\ln n)^2$, I obtain

$$\frac{2p_n(\ln n)^2}{n} \approx 2(\ln n)^3 + 2(\ln n)^2 \ln \ln n - 2(\ln n)^2 + 2 \ln n (\ln \ln n - 2) - (\ln \ln n)^2 + 6 \ln \ln n - 11,$$

thereupon, letting $n \rightarrow e^n$, I find

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n^3 + 2n^2 \ln n - 2n^2 + 2n(\ln n - 2) - (\ln n)^2 + 6 \ln n - 11,$$

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n(n^2 - n - 2) + (2n^2 + 2n + 6 - \ln n) \ln n - 11. \square$$

COROLLARY 1. *I have*

$$\frac{2n^2 p_{\lfloor e^n \rfloor}}{\lfloor e^n \rfloor} \approx 2n(n^2 - n - 2) + (2n^2 + 2n + 6 - \ln n) \ln n - 11,$$

where p_n denotes the n th prime number defined by $p_n: \mathbb{N} \rightarrow \mathbb{R}^+$ and $\lfloor n \rfloor$ denotes the floor function.

Proof. I note that $\lfloor e^n \rfloor \approx e^n$, arising from the definition of the floor function. \square

THEOREM 1. *For n sufficiently large, then*

$$p_n \approx \left[\frac{12 \ln \ln n}{\pi} + \frac{2}{\ln n} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k n^{2\pi k} - 1} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k e^{2\pi k / \ln n} - 1} \right] \left[n \left(\frac{1}{2} - \frac{1}{2 \ln n} - \frac{1}{(\ln n)^2} \right) \right] \\ + \left[\frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) + 2 \sum_{k=1}^{\infty} \frac{1}{k n^{2\pi k} - 1} - 2 \sum_{k=1}^{\infty} \frac{1}{k e^{2\pi k / \ln n} - 1} \right] \left\{ n \left[1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right] \right\} \\ - \frac{11n}{2(\ln n)^2},$$

where p_n denotes the n th prime number.

Proof. I put (7) and (8) into Lemma 1, as following

$$\begin{aligned} \frac{2n^2 p_{e^n}}{e^n} &\approx \left[\frac{12 \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{2}{n} \right] (n^2 - n - 2) \\ &\quad + (2n^2 + 2n + 6 - \ln n) \left[2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{\pi}{6} \left(n - \frac{1}{n} \right) \right] - 11. \end{aligned}$$

Dividing both members of above equation by $2n^2 + 2n + 6 - \ln n$

$$\begin{aligned} &\frac{2n^2 p_{e^n}}{e^n (2n^2 + 2n + 6 - \ln n)} \\ &\approx \left[\frac{12 \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{2}{n} \right] \left(\frac{n^2 - n - 2}{2n^2 + 2n + 6 - \ln n} \right) \\ &\quad + \left[2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{\pi}{6} \left(n - \frac{1}{n} \right) \right] - \frac{11}{2n^2 + 2n + 6 - \ln n}. \end{aligned}$$

I return to replace n by $\ln n$ in the previous equation

$$\begin{aligned} &\frac{2(\ln n)^2 p_n}{n[2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n]} \\ &\approx \left[\frac{12 \ln \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/\ln n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{2}{\ln n} \right] \left[\frac{(\ln n)^2 - \ln n - 2}{2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n} \right] + \\ &\left[2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/\ln n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) \right] - \frac{11}{2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n}. \quad (9) \end{aligned}$$

Multiplying both members of (9) by $\frac{n[2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n]}{2(\ln n)^2}$, I encounter

$$\begin{aligned} p_n &\approx \left[\frac{12 \ln \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/\ln n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{2}{\ln n} \right] \left[n \left(\frac{1}{2} - \frac{1}{2 \ln n} - \frac{1}{(\ln n)^2} \right) \right] \\ &\quad + \left[2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/\ln n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) \right] \left\{ n \left[1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right] \right\} \\ &\quad - \frac{11n}{2(\ln n)^2}, \end{aligned}$$

consequently,

$$\begin{aligned} p_n &\approx \left[\frac{12 \ln \ln n}{\pi} + \frac{2}{\ln n} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k/\ln n} - 1} \right] \left[n \left(\frac{1}{2} - \frac{1}{2 \ln n} - \frac{1}{(\ln n)^2} \right) \right] \\ &\quad + \left[\frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) + 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k/\ln n} - 1} \right] \left\{ n \left[1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right] \right\} \\ &\quad - \frac{11n}{2(\ln n)^2}. \end{aligned}$$

This completes the Theorem. \square

COROLLARY 2. For n sufficiently large, then

$$p_n \approx n \left(1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left(\frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right) \\ + n \left(1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[\frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2},$$

where p_n denotes the n th prime number.

Proof. I note that, for $n \in \mathbb{N}_{>1}$,

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k - 1}} \downarrow 0. \tag{11}$$

From Theorem 1 and (11), I obtain

$$p_n \approx n \left(1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left(\frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right) \\ + n \left(1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[\frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2}.$$

This completes the proof. \square

CONJECTURE: I observe that, numerically, p_n is better approximate if I add the term $\frac{5n}{12 \ln n} + \frac{n}{24 \ln \ln n}$, so

$$p_n \approx n \left(1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left(\frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right) \\ + n \left(1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[\frac{\pi}{6} \left(\ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2} \\ + \frac{5n}{12 \ln n} + \frac{n}{24 \ln \ln n}.$$

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