

# Exploring Prime Numbers and Modular Functions I: On the Exponential of Prime Number via Dedekind Eta Function

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**ABSTRACT.** The main objective this paper is to develop asymptotic formulas for the exponential of prime number, using Dedekind eta function, and afterwards an elliptic modular function.

## 1. INTRODUCTION

As consequence of the prime number theorem, I get the asymptotic formula for the  $n$ th prime number, denoted by  $p_n$ :

$$p_n \sim n \ln n. \quad (1)$$

M. Pervouchine, in *Mémoires de la Société physic-mathématique de Kasan*, [1, page 848] deduced that

$$\frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{5}{12 \ln n} + \frac{1}{24 \ln \ln n}. \quad (2)$$

On the other hand, Ernest Cesáro, in *Sur une formule empirique de M. Pervouchine* [1, page 849], I encounter the formula

$$\frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2}, \quad (3)$$

in modern notation,

$$\frac{p_n}{n} = \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2} + O\left(\frac{1}{(\ln n)^2}\right). \quad (4)$$

On the other hand, the Dedekind eta function was introduced by Richard Dedekind, in 1877, and is defined in the half-plane  $\mathbb{H} = \{\tau: \Im(\tau) > 0\}$  by the equation [2, page 47]

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}). \quad (5)$$

The infinite product has the form  $\prod(1 - x^n)$ , where  $x = e^{2\pi i \tau}$ . If  $\tau \in \mathbb{H}$ , then  $|x| < 1$ , so the product converges absolutely and is nonzero.

In [2, page 48], I have

$$\frac{1}{2} \ln y = \ln \eta(i/y) - \ln \eta(iy), \quad (6)$$

hence,

$$\ln y = 2[\ln \eta(i/y) - \ln \eta(iy)] = 2 \ln \left[ \frac{\eta(i/y)}{\eta(iy)} \right], \quad (7)$$

for  $y > 0$ .

In this paper, I prove, among other things, that

$$\frac{p_n}{e^{2n}} \approx \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ i n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]},$$

$$e^{\frac{p_n}{2n} - \frac{\pi n}{12}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi n k(3k-1)} \sim e^{-\frac{\pi}{12n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi k(3k-1)}{n}},$$

$$e^{\frac{p_n}{2n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi n(6k-1)^2}{12}} \sim \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi(6k-1)^2}{12n}}$$

and

$$e^{\frac{p_n}{2n} - \frac{\pi n}{12}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left[ e^{-\pi n k(3k-1)} + e^{-\pi n k(3k+1)} \right] \right\} \sim e^{-\frac{\pi}{12n}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left[ e^{-\frac{\pi k(3k-1)}{n}} + e^{-\frac{\pi k(3k+1)}{n}} \right] \right\}.$$

## 2. THEOREMS

### PART 1

In this part, I develop one asymptotic connection between elliptic modular functions, more specifically, the Dedekind eta function, and the exponential function of a prime number.

**THEOREM 1.** *I have*

$$\frac{p_n}{e^{2n}} \approx \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ i n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]},$$

where  $p_n$  denotes the  $n$ th prime number and  $\eta(\tau)$  denotes the Dedekind eta function.

*Proof.* By (2), I have

$$\begin{aligned} \frac{p_n}{n} &\approx \ln n + \ln \ln n - 1 + \frac{5}{12 \ln n} + \frac{1}{24 \ln \ln n} \\ &= \ln n + \ln \ln n - \ln e + \ln \left( e^{\frac{5}{12 \ln n}} \right) + \ln \left( e^{\frac{1}{24 \ln \ln n}} \right) \\ &= \ln \left( n \ln n e^{\frac{5}{12 \ln n} + \frac{1}{24 \ln \ln n} - 1} \right) = \ln \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]. \end{aligned} \tag{8}$$

I substitute (7) in the right hand side of (8), and obtain

$$\frac{p_n}{n} \approx 2 \ln \left( \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ i n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]} \right). \tag{9}$$

The exponentiation of (9) give me

$$\begin{aligned}
 e^{\frac{p_n}{n}} &\approx \left( \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ in \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]} \right)^2 \Leftrightarrow \\
 e^{p_n \cdot \frac{1}{n}} &\approx \left( \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ in \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]} \right)^2 \Leftrightarrow \\
 (e^{p_n})^{\frac{1}{n}} &\approx \left( \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ in \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]} \right)^2 \Leftrightarrow \\
 e^{\frac{p_n}{2n}} &\approx \frac{\eta \left\{ i / \left[ n \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right] \right\}}{\eta \left[ in \ln n e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \right]}. \square
 \end{aligned}$$

COROLLARY 1. *I have*

$$e^{\frac{p_n}{2n}} \sim \frac{\eta \left\{ i / \left[ n \left( 10 + \frac{\ln n}{\ln \ln n} \right) \right] \right\}}{\eta \left[ in \left( 10 + \frac{\ln n}{\ln \ln n} \right) \right]},$$

where  $p_n$  denotes the  $n$ th prime number and  $\eta(\tau)$  denotes the Dedekind eta function.

*Proof.* The representation in series power of  $e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)}$  is

$$e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} = 1 + \frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right) + \frac{1}{2!} \left[ \frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right) \right]^2 + \dots,$$

so,

$$e^{\frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right)} \sim 1 + \frac{1}{24} \left( \frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24 \right) = \frac{1}{\ln \ln n} + \frac{10}{\ln n}.$$

I substitute the above result in Theorem 1, and this completes the proof.  $\square$

THEOREM 2. *I have*

$$e^{\frac{p_n}{2n}} \sim \frac{\eta(i/n)}{\eta(in)},$$

where  $p_n$  denotes the  $n$ th prime number and  $\eta(\tau)$  denotes the Dedekind eta function.

*Proof.* By (1), I find

$$\frac{p_n}{n} \sim \ln n \tag{10}$$

I substitute (7) in the right hand side of (10), and obtain

$$\frac{p_n}{n} \sim 2 \ln \left[ \frac{\eta(i/n)}{\eta(in)} \right]. \tag{11}$$

The exponentiation of (11) give me

$$\begin{aligned} e^{\frac{p_n}{n}} &\sim \left[ \frac{\eta(i/n)}{\eta(in)} \right]^2 \Leftrightarrow \\ e^{p_n \cdot \frac{1}{n}} &\sim \left[ \frac{\eta(i/n)}{\eta(in)} \right]^2 \Leftrightarrow \\ (e^{p_n})^{\frac{1}{n}} &\sim \left[ \frac{\eta(i/n)}{\eta(in)} \right]^2 \Leftrightarrow \\ e^{\frac{p_n}{2n}} &\sim \frac{\eta(i/n)}{\eta(in)}. \quad \square \end{aligned}$$

THEOREM 3. I have

$$e^{\frac{p_n}{2n}} \approx \frac{\eta \left( i / \left\{ n \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\} \right)}{\eta \left\{ in \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\}},$$

where  $p_n$  denotes the  $n$ th prime number and  $\eta(\tau)$  denotes the Dedekind eta function.

Proof. By (2), I meet

$$\begin{aligned} \frac{p_n}{n} &\approx \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2} \\ &= \ln n + \ln \ln n - \ln e + \ln \left( e^{\frac{\ln \ln n - 2}{\ln n}} \right) - \ln \left( e^{\frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2}} \right) \\ &= \ln \left( n \ln n e^{\frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2} - 1} \right) \\ &= \ln \left\{ n \ln n e^{\frac{1}{2(\ln n)^2} [2 \ln n (\ln \ln n - 2) - (\ln \ln n)^2 + 6 \ln \ln n - 11 - 2(\ln n)^2]} \right\} \\ &= \ln \left\{ n \ln n e^{-\frac{1}{2(\ln n)^2} [(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\}. \tag{12} \end{aligned}$$

I take (12) in (7), and obtain

$$\frac{p_n}{n} \approx 2 \ln \left[ \frac{\eta \left( i / \left\{ n \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\} \right)}{\eta \left\{ in \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\}} \right]. \tag{13}$$

The exponentiation of (13) give me

$$e^{\frac{p_n}{n}} \approx \left[ \frac{\eta \left( i / \left\{ n \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\} \right)}{\eta \left\{ in \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\}} \right]^2 \Leftrightarrow$$

$$e^{\frac{p_n}{2n}} \approx \frac{\eta \left( i / \left\{ n \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\} \right)}{\eta \left\{ in \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]} \right\}} . \square$$

COROLLARY 2. I have

$$e^{\frac{p_n}{2n}} \sim \frac{\eta(-2i \ln n / \{n[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 11]\})}{\eta\left(-\frac{in}{2 \ln n} [(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 11]\right)},$$

where  $p_n$  denotes the  $n$ th prime number and  $\eta(\tau)$  denotes the Dedekind eta function.

*Proof.* The representation in series power of  $e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]}$  is

$$e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]}$$

$$= 1 - \frac{1}{2(\ln n)^2} [(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]$$

$$+ \frac{1}{2!} \left\{ \frac{1}{2(\ln n)^2} [(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11] \right\}^2 + \dots,$$

so,

$$e^{-\frac{1}{2(\ln n)^2}[(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11]}$$

$$\sim 1 - \frac{1}{2(\ln n)^2} [(\ln \ln n)^2 - 2(\ln n + 3) \ln \ln n + 4 \ln n + 2(\ln n)^2 + 11].$$

I substitute the previous result in Theorem 3, and this completes the proof.  $\square$

THEOREM 4. I have

$$e^{\frac{p_n}{2n} - \frac{\pi n}{12}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi n k(3k-1)} \sim e^{-\frac{\pi}{12n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi k(3k-1)}{n}},$$

$$e^{\frac{p_n}{2n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi n(6k-1)^2}{12}} \sim \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi(6k-1)^2}{12n}}$$

and

$$e^{\frac{p_n}{2n} - \frac{\pi n}{12}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k [e^{-\pi n k(3k-1)} + e^{-\pi n k(3k+1)}] \right\} \sim e^{-\frac{\pi}{12n}} \left\{ 1 \right.$$

$$\left. + \sum_{k=1}^{\infty} (-1)^k \left[ e^{-\frac{\pi k(3k-1)}{n}} + e^{-\frac{\pi k(3k+1)}{n}} \right] \right\},$$

where  $p_n$  denotes the  $n$ th prime number.

*Proof.* In [3], the Dedekind eta function have the following series sum representation

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \sum_{k=-\infty}^{\infty} (-1)^k e^{\pi i \tau k(3k-1)} \tag{14}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k e^{\frac{\pi i \tau (6k-1)^2}{12}} \tag{15}$$

$$= e^{\frac{\pi i \tau}{12}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left[ e^{\pi i \tau k(3k-1)} + e^{\pi i \tau k(3k+1)} \right] \right\} \tag{16}$$

Substituting (14), (15) and (16) in Theorem 2, I obtain

$$e^{\frac{p_n}{2n}} \eta(in) \sim \eta(i/n),$$

$$e^{\frac{p_n}{2n}} \frac{\pi n}{12} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi n k(3k-1)} \sim e^{-\frac{\pi}{12n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi k(3k-1)}{n}},$$

$$e^{\frac{p_n}{2n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi n (6k-1)^2}{12}} \sim \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi (6k-1)^2}{12n}},$$

$$e^{\frac{p_n}{2n}} \frac{\pi n}{12} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left[ e^{-\pi n k(3k-1)} + e^{-\pi n k(3k+1)} \right] \right\} \sim e^{-\frac{\pi}{12n}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left[ e^{-\frac{\pi k(3k-1)}{n}} + e^{-\frac{\pi k(3k+1)}{n}} \right] \right\}.$$

This completes the proof. □

**PART 2**

In this part, I extend the previous work with the elliptical modular functions, demonstrating other asymptotic formula for the exponential function of prime number.

**THEOREM 5.** *I have*

$$e^{\frac{p_n}{2n}} \sim e^{\frac{\pi i}{12}} \frac{\eta^2 \left( \frac{1}{2} \left[ \frac{i}{\frac{n}{6} \left( 5 + \frac{\ln n}{2 \ln \ln n} \right) + 1} \right] \right) \eta^2 \left( \frac{i}{n} \right)}{\eta \left( \frac{i}{\frac{n}{6} \left( 5 + \frac{\ln n}{2 \ln \ln n} \right) + 1} \right) \eta(in) \eta^2 \left( \frac{i+n}{2n} \right)}$$

and

$$e^{\frac{p_n}{2n}} \sim \frac{\eta^2 \left( \frac{1}{2} \left[ \frac{i}{\frac{n}{6} \left( 5 + \frac{\ln n}{2 \ln \ln n} \right) + 1} \right] \right) \eta^2 \left( \frac{i}{n} \right)}{2\eta \left( \frac{i}{\frac{n}{6} \left( 5 + \frac{\ln n}{2 \ln \ln n} \right) + 1} \right) \eta(in) \eta^2 \left( \frac{2i}{n} \right)}$$

where  $p_n$  denotes the  $n$ th prime number and  $\eta(\tau)$  denotes the Dedekind eta function.

*Proof.* In [4, page 114], Weber defined the following functions

$$f(\tau) = \frac{e^{-\frac{\pi i}{24}\eta\left(\frac{\tau+1}{2}\right)}}{\eta(\tau)}, \quad (17)$$

$$f_2(\tau) = \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)}. \quad (18)$$

and conclude that

$$\vartheta_{00} = \eta(\tau)f(\tau)^2, \quad (19)$$

$$\vartheta_{10} = \eta(\tau)f_2(\tau)^2, \quad (20)$$

where  $\vartheta_{00}$ ,  $\vartheta_{01}$  and  $\vartheta_{10}$  are Jacobi theta functions.

The Theorem 2 assures me that

$$e^{\frac{pn}{2n}\eta(in)} \sim \eta(i/n), \quad (21)$$

Multiplying both members of (21) by  $f(i/n)^2$  and  $f_2(i/n)^2$ , respectively, I encounter

$$e^{\frac{pn}{2n}\eta(in)} f(i/n)^2 \sim \eta(i/n) f(i/n)^2, \quad (22)$$

$$e^{\frac{pn}{2n}\eta(in)} f_2(i/n)^2 \sim \eta(i/n) f_2(i/n)^2, \quad (23)$$

from (17) and (18), I set

$$e^{\frac{pn}{2n} \frac{e^{-\frac{\pi i}{12}\eta(in)} \eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)}} \sim \vartheta_{00}, \quad (24)$$

$$2e^{\frac{pn}{2n} \frac{\eta(in) \eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)}} \sim \vartheta_{10}. \quad (25)$$

In [5, page 173] the Ramanujan's theta function  $f(a, b)$  is defined by

$$f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1, \quad (26)$$

which is a generalization of Jacobi theta functions. Roughly speaking, I can suppose that

$$e^{\frac{pn}{2n} \frac{e^{-\frac{\pi i}{12}\eta(in)} \eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)}} = f_{00}(a, b), \quad (27)$$

$$2e^{\frac{pn}{2n} \frac{\eta(in) \eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)}} = f_{10}(a, b). \quad (28)$$

videlicet,

$$e^{\frac{pn}{2n} \frac{e^{-\frac{\pi i}{12}\eta(in)} \eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)}} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\sigma(n)k(k+1)/2} e^{-\pi\sigma(n)k(k-1)/2} = \vartheta_3(0, e^{-2\pi\sigma(n)}), \quad (29)$$

$$2e^{\frac{pn}{2n}} \frac{\eta(in)\eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\omega(n)k(k+1)/2} e^{-\pi\omega(n)k(k-1)/2} = \vartheta_3\left(0, e^{-2\pi\omega(n)}\right). \quad (30)$$

How calculate the functions  $\sigma(n)$  and  $\omega(n)$ ? Approximately, I have

$$e^{\frac{pn}{2n}} \frac{e^{-\frac{\pi i}{12}} \eta(in)\eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\sigma(n)k(k+1)/2} e^{-\pi\sigma(n)k(k-1)/2} \sim \int_{-\infty}^{\infty} e^{-\pi\sigma(n)x(x+1)/2} e^{-\pi\sigma(n)x(x-1)/2} dx = \frac{1}{\sqrt{2\sigma(n)}}, \quad (31)$$

$$2e^{\frac{pn}{2n}} \frac{\eta(in)\eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\omega(n)k(k+1)/2} e^{-\pi\omega(n)k(k-1)/2} \sim \int_{-\infty}^{\infty} e^{-\pi\omega(n)x(x+1)/2} e^{-\pi\omega(n)x(x-1)/2} dx = \frac{1}{\sqrt{2\omega(n)}}, \quad (32)$$

pursuant to,

$$\sigma(n) \sim \frac{\eta^4\left(\frac{i}{n}\right)}{2e^{\frac{pn}{n}} e^{-\frac{\pi i}{6}} \eta^2(in)\eta^4\left(\frac{i+n}{2n}\right)}, \quad (33)$$

$$\omega(n) \sim \frac{\eta^4\left(\frac{i}{n}\right)}{8e^{\frac{pn}{n}} \eta^2(in)\eta^4\left(\frac{2i}{n}\right)}. \quad (34)$$

From (8), (33) and (34), I obtain

$$\sigma(n) \sim \frac{e^{\frac{\pi i}{6}} \eta^4\left(\frac{i}{n}\right)}{2n \ln n e^{\frac{1}{12}\left(\frac{5}{\ln n} + \frac{1}{2 \ln \ln n} - 12\right)} \eta^2(in)\eta^4\left(\frac{i+n}{2n}\right)}, \quad (35)$$

$$\omega(n) \sim \frac{\eta^4\left(\frac{i}{n}\right)}{8n \ln n e^{\frac{1}{12}\left(\frac{5}{\ln n} + \frac{1}{2 \ln \ln n} - 12\right)} \eta^2(in)\eta^4\left(\frac{2i}{n}\right)}. \quad (36)$$

On the other hand, I notice that

$$\frac{\eta^4\left(\frac{i}{n}\right)}{\eta^2(in)\eta^4\left(\frac{i+n}{2n}\right)} \searrow \frac{\sqrt{3}}{2} - \frac{1}{2}i, \quad (37)$$

$$\frac{\eta^4\left(\frac{i}{n}\right)}{\eta^2(in)\eta^4\left(\frac{2i}{n}\right)} \searrow 4. \quad (38)$$

I get (37) and (38) into (35) and (36)

$$\sigma(n) \sim \frac{e^{\frac{\pi i}{6}} \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)}{2n \ln n e^{\frac{1}{12}\left(\frac{5}{\ln n} + \frac{1}{2 \ln \ln n} - 12\right)}}, \quad (39)$$

$$\omega(n) \sim \frac{1}{2n \ln n e^{\frac{1}{12}\left(\frac{5}{\ln n} + \frac{1}{2 \ln \ln n} - 12\right)}}. \quad (40)$$



On the one hand, I readily saw that

$$e^{\frac{\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad (41)$$

and the representation in series power of  $e^{\frac{1}{24}\left(\frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24\right)}$  is

$$e^{\frac{1}{24}\left(\frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24\right)} = 1 + \frac{1}{24}\left(\frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24\right) + \frac{1}{2!}\left[\frac{1}{24}\left(\frac{1}{\ln \ln n} + \frac{10}{\ln n} - 24\right)\right]^2 + \dots, \quad (42)$$

I set (41) and (42) in (39) and (40)

$$\sigma(n) \sim \frac{1}{2n \ln n \left[1 + \frac{1}{12}\left(\frac{5}{\ln n} + \frac{1}{2 \ln \ln n} - 12\right)\right]} = \frac{1}{n \left[\frac{1}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)\right]}, \quad (43)$$

$$\omega(n) \sim \frac{1}{2n \ln n \left[1 + \frac{1}{12}\left(\frac{5}{\ln n} + \frac{1}{2 \ln \ln n} - 12\right)\right]} = \frac{1}{n \left[\frac{1}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)\right]}. \quad (44)$$

From (29), (30), (43) and (44), I conclude that

$$e^{\frac{p_n}{2n}} \sim \frac{e^{\frac{\pi i}{12}} \vartheta_3\left(0, e^{-\frac{\pi}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)}\right) \eta^2\left(\frac{i}{n}\right)}{\eta(in) \eta^2\left(\frac{i+n}{2n}\right)}, \quad (45)$$

$$e^{\frac{p_n}{2n}} \sim \frac{\vartheta_3\left(0, e^{-\frac{\pi}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)}\right) \eta^2\left(\frac{i}{n}\right)}{2\eta(in) \eta^2\left(\frac{2i}{n}\right)}. \quad (46)$$

In [3], I encounter

$$\vartheta_3(0, e^{\pi i \tau}) = \frac{\eta^2\left(\frac{\tau+1}{2}\right)}{\eta(\tau+1)}. \quad (47)$$

I substitute (47) into (45) and (46)

$$e^{\frac{p_n}{2n}} \sim e^{\frac{\pi i}{12}} \frac{\eta^2\left(\frac{1}{2}\left[\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)} + 1\right]\right) \eta^2\left(\frac{i}{n}\right)}{\eta\left(\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)} + 1\right) \eta(in) \eta^2\left(\frac{i+n}{2n}\right)},$$

$$e^{\frac{p_n}{2n}} \sim \frac{\eta^2\left(\frac{1}{2}\left[\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)} + 1\right]\right) \eta^2\left(\frac{i}{n}\right)}{2\eta\left(\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2 \ln \ln n}\right)} + 1\right) \eta(in) \eta^2\left(\frac{2i}{n}\right)}. \quad \square$$

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