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Exploring Prime Numbers and Modular Functions I: On the Exponential of Prime Number via Dedekind Eta Function

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ABSTRACT. The main objective this paper is to develop asymptotic formulas for the exponential of prime number, using Dedekind eta function, and afterwards an elliptic modular function.

1. INTRODUCTION

As consequence of the prime number theorem, I get the asymptotic formula for the *n*th prime number, denoted by p_n:

$$p_n \sim n \ln n. \tag{1}$$

M. Pervouchine, in *Mémories de la Société physic-mathématique de Kasan*, [1, page 848] deduced that

$$\frac{p_n}{n} \approx \ln n + \ln\ln n - 1 + \frac{5}{12 \ln n} + \frac{1}{24 \ln\ln n}. \tag{2}$$

On the other hand, Ernest Cesáro, in *Sur une formule empirique de M. Pervouchine* [1, page 849], I encounter the formula

$$\frac{p_n}{n} \approx \ln n + \ln\ln n - 1 + \frac{\ln\ln n - 2}{\ln n} - \frac{(\ln\ln n)^2 - 6\ln\ln n + 11}{2(\ln n)^2}, \tag{3}$$

in modern notation,

$$\frac{p_n}{n} = \ln n + \ln\ln n - 1 + \frac{\ln\ln n - 2}{\ln n} - \frac{(\ln\ln n)^2 - 6\ln\ln n + 11}{2(\ln n)^2} + O\left(\frac{1}{(\ln n)^2}\right). \tag{4}$$

On the other hand, the Dedekind eta function was introduced by Richard Dedekind, in 1877, and is defined in the half-plane $\mathbb{H} = \{\tau : \Im(\tau) > 0\}$ by the equation [2, page 47]

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau}\right). \tag{5}$$

The infinite product has the form $\prod(1 - x^n)$, where $x = e^{2\pi i \tau}$. If $\tau \in \mathbb{H}$, then $|x| < 1$, so the product converges absolutely and is nonzero.

In [2, page 48], I have

$$\frac{1}{2}\ln y = \ln \eta(i/y) - \ln \eta(iy), \tag{6}$$

hence,

$$\ln y = 2[\ln \eta(i/y) - \ln \eta(iy)] = 2 \ln\left[\frac{\eta(i/y)}{\eta(iy)}\right], \tag{7}$$

for $y > 0$.

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In this paper, I prove, among other things, that

$$e^{\frac{p_n}{2n}} \approx \frac{\eta\left\{i/\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]},$$

$$e^{\frac{p_n}{2n}-\frac{\pi n}{12}}\sum_{k=-\infty}^{\infty}(-1)^k e^{-\pi nk(3k-1)} \sim e^{-\frac{\pi}{12n}}\sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi k(3k-1)}{n}},$$

$$e^{\frac{p_n}{2n}}\sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi n(6k-1)^2}{12}} \sim \sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi(6k-1)^2}{12n}}$$

and

$$e^{\frac{p_n}{2n}-\frac{\pi n}{12}}\left\{1+\sum_{k=1}^{\infty}(-1)^k\left[e^{-\pi nk(3k-1)}+e^{-\pi nk(3k+1)}\right]\right\} \sim e^{-\frac{\pi}{12n}}\left\{1\right.$$
$$\left.+\sum_{k=1}^{\infty}(-1)^k\left[e^{-\frac{\pi k(3k-1)}{n}}+e^{-\frac{\pi k(3k+1)}{n}}\right]\right\}.$$

2. THEOREMS

PART 1

In this part, I develop one asymptotic connection between elliptic modular functions, more specifically, the Dedekind eta function, and the exponential function of a prime number.

THEOREM 1. *I have*

$$e^{\frac{p_n}{2n}} \approx \frac{\eta\left\{i/\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]},$$

*where p_n denotes the n*th *prime number and $\eta(\tau)$ denotes the Dedekind eta function.*

Proof. By (2), I have

$$\frac{p_n}{n} \approx \ln n+\ln\ln n-1+\frac{5}{12\ln n}+\frac{1}{24\ln\ln n}$$
$$=\ln n+\ln\ln n-\ln e+\ln\left(e^{\frac{5}{12\ln n}}\right)+\ln\left(e^{\frac{1}{24\ln\ln n}}\right)$$
$$=\ln\left(n\ln n\, e^{\frac{5}{12\ln n}+\frac{1}{24\ln\ln n}-1}\right)=\ln\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]. \tag{8}$$

I substitute (7) in the right hand side of (8), and obtain

$$\frac{p_n}{n} \approx 2\ln\left(\frac{\eta\left\{i/\left[n\ln n e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]}\right). \tag{9}$$

The exponentiation of (9) give me

$$e^{\frac{p_n}{n}} \approx \left(\frac{\eta\left\{i/\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]}\right)^2 \Leftrightarrow$$

$$e^{p_n\cdot\frac{1}{n}} \approx \left(\frac{\eta\left\{i/\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]}\right)^2 \Leftrightarrow$$

$$(e^{p_n})^{\frac{1}{n}} \approx \left(\frac{\eta\left\{i/\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]}\right)^2 \Leftrightarrow$$

$$e^{\frac{p_n}{2n}} \approx \frac{\eta\left\{i/\left[n\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]\right\}}{\eta\left[in\ln n\, e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}\right]}. \square$$

COROLLARY 1. *I have*

$$e^{\frac{p_n}{2n}} \sim \frac{\eta\left\{i/\left[n\left(10+\frac{\ln n}{\ln\ln n}\right)\right]\right\}}{\eta\left[in\left(10+\frac{\ln n}{\ln\ln n}\right)\right]},$$

*where p_n denotes the n*th *prime number and $\eta(\tau)$ denotes the Dedekind eta function.*

Proof. The representation in series power of $e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}$ is

$$e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)} = 1+\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)+\frac{1}{2!}\left[\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)\right]^2+\cdots,$$

so,

$$e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)} \sim 1+\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)=\frac{1}{\ln\ln n}+\frac{10}{\ln n}.$$

I substitute the above result in Theorem 1, and this completes the proof. □

THEOREM 2. *I have*

$$e^{\frac{p_n}{2n}} \sim \frac{\eta(i/n)}{\eta(in)},$$

*where p_n denotes the n*th *prime number and $\eta(\tau)$ denotes the Dedekind eta function.*

Proof. By (1), I find

$$\frac{p_n}{n} \sim \ln n \tag{10}$$

I substitute (7) in the right hand side of (10), and obtain

$$\frac{p_n}{n} \sim 2 \ln \left[\frac{\eta(i/n)}{\eta(in)}\right]. \tag{11}$$

The exponentiation of (11) give me

$$e^{\frac{p_n}{n}} \sim \left[\frac{\eta(i/n)}{\eta(in)}\right]^2 \Leftrightarrow$$

$$e^{p_n \cdot \frac{1}{n}} \sim \left[\frac{\eta(i/n)}{\eta(in)}\right]^2 \Leftrightarrow$$

$$(e^{p_n})^{\frac{1}{n}} \sim \left[\frac{\eta(i/n)}{\eta(in)}\right]^2 \Leftrightarrow$$

$$e^{\frac{p_n}{2n}} \sim \frac{\eta(i/n)}{\eta(in)}. \square$$

THEOREM 3. *I have*

$$e^{\frac{p_n}{2n}} \approx \frac{\eta\left(i/\left\{n \ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2 - 2(\ln n+3)\ln\ln n + 4\ln n + 2(\ln n)^2 + 11]}\right\}\right)}{\eta\left\{in \ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2 - 2(\ln n+3)\ln\ln n + 4\ln n + 2(\ln n)^2 + 11]}\right\}},$$

where p_n *denotes the* nth *prime number and* $\eta(\tau)$ *denotes the Dedekind eta function.*

Proof. By (2), I meet

$$\frac{p_n}{n} \approx \ln n + \ln\ln n - 1 + \frac{\ln\ln n - 2}{\ln n} - \frac{(\ln\ln n)^2 - 6\ln\ln n + 11}{2(\ln n)^2}$$

$$= \ln n + \ln\ln n - \ln e + \ln\left(e^{\frac{\ln\ln n - 2}{\ln n}}\right) - \ln\left(e^{\frac{(\ln\ln n)^2 - 6\ln\ln n + 11}{2(\ln n)^2}}\right)$$

$$= \ln\left(n \ln n\, e^{\frac{\ln\ln n - 2}{\ln n} - \frac{(\ln\ln n)^2 - 6\ln\ln n + 11}{2(\ln n)^2} - 1}\right)$$

$$= \ln\left\{n \ln n\, e^{\frac{1}{2(\ln n)^2}[2\ln n(\ln\ln n - 2) - (\ln\ln n)^2 + 6\ln\ln n - 11 - 2(\ln n)^2]}\right\}$$

$$= \ln\left\{n \ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2 - 2(\ln n+3)\ln\ln n + 4\ln n + 2(\ln n)^2 + 11]}\right\}. \tag{12}$$

I take (12) in (7), and obtain

$$\frac{p_n}{n} \approx 2 \ln \left[\frac{\eta\left(i/\left\{n \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2 - 2(\ln n+3)\ln\ln n + 4\ln n + 2(\ln n)^2 + 11]}\right\}\right)}{\eta\left\{in \ln n e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2 - 2(\ln n+3)\ln\ln n + 4\ln n + 2(\ln n)^2 + 11]}\right\}}\right]. \tag{13}$$

The exponentiation of (13) give me

$$e^{\frac{p_n}{n}} \approx \left[\frac{\eta\left(i/\left\{n\ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}\right\}\right)}{\eta\left\{in\ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}\right\}}\right]^2 \Leftrightarrow$$

$$e^{\frac{p_n}{2n}} \approx \frac{\eta\left(i/\left\{n\ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}\right\}\right)}{\eta\left\{in\ln n\, e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}\right\}}. \square$$

COROLLARY 2. *I have*

$$e^{\frac{p_n}{2n}}\sim\frac{\eta(-2i\ln n/\{n[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+11]\})}{\eta\left(-\frac{in}{2\ln n}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+11]\right)},$$

where p_n *denotes the* nth *prime number and* $\eta(\tau)$ *denotes the Dedekind eta function.*

Proof. The representation in series power of $e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}$ is

$$e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}$$
$$=1-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]$$
$$+\frac{1}{2!}\left\{\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]\right\}^2+\cdots,$$

so,

$$e^{-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11]}$$
$$\sim 1-\frac{1}{2(\ln n)^2}[(\ln\ln n)^2-2(\ln n+3)\ln\ln n+4\ln n+2(\ln n)^2+11].$$

I substitute the previous result in Theorem 3, and this completes the proof. □

THEOREM 4. *I have*

$$e^{\frac{p_n}{2n}-\frac{\pi n}{12}}\sum_{k=-\infty}^{\infty}(-1)^k e^{-\pi nk(3k-1)}\sim e^{-\frac{\pi}{12n}}\sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi k(3k-1)}{n}},$$
$$e^{\frac{p_n}{2n}}\sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi n(6k-1)^2}{12}}\sim\sum_{k=-\infty}^{\infty}(-1)^k e^{-\frac{\pi(6k-1)^2}{12n}}$$

and

$$e^{\frac{p_n}{2n}-\frac{\pi n}{12}}\left\{1+\sum_{k=1}^{\infty}(-1)^k\left[e^{-\pi nk(3k-1)}+e^{-\pi nk(3k+1)}\right]\right\}\sim e^{-\frac{\pi}{12n}}\left\{1\right.$$
$$\left.+\sum_{k=1}^{\infty}(-1)^k\left[e^{-\frac{\pi k(3k-1)}{n}}+e^{-\frac{\pi k(3k+1)}{n}}\right]\right\},$$

where p_n *denotes the* nth *prime number.*

Proof. In [3], the Dedekind eta function have the following series sum representation

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}} \sum_{k=-\infty}^{\infty} (-1)^k e^{\pi i\tau k(3k-1)} \tag{14}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k e^{\frac{\pi i\tau(6k-1)^2}{12}} \tag{15}$$

$$= e^{\frac{\pi i\tau}{12}} \left\{1 + \sum_{k=1}^{\infty} (-1)^k \left[e^{\pi i\tau k(3k-1)} + e^{\pi i\tau k(3k+1)}\right]\right\} \tag{16}$$

Substituting (14), (15) and (16) in Theorem 2, I obtain

$$e^{\frac{p_n}{2n}} \eta(in) \sim \eta(i/n),$$

$$e^{\frac{p_n}{2n}-\frac{\pi n}{12}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi nk(3k-1)} \sim e^{-\frac{\pi}{12n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi k(3k-1)}{n}},$$

$$e^{\frac{p_n}{2n}} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi n(6k-1)^2}{12}} \sim \sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{\pi(6k-1)^2}{12n}},$$

$$e^{\frac{p_n}{2n}-\frac{\pi n}{12}} \left\{1 + \sum_{k=1}^{\infty} (-1)^k \left[e^{-\pi nk(3k-1)} + e^{-\pi nk(3k+1)}\right]\right\} \sim e^{-\frac{\pi}{12n}} \left\{1 + \sum_{k=1}^{\infty} (-1)^k \left[e^{-\frac{\pi k(3k-1)}{n}} + e^{-\frac{\pi k(3k+1)}{n}}\right]\right\}.$$

This completes the proof. □

PART 2

In this part, I extend the previous work with the elliptical modular functions, demonstrating other asymptotic formula for the exponential function of prime number.

THEOREM 5. *I have*

$$e^{\frac{p_n}{2n}} \sim e^{\frac{\pi i}{12}} \frac{\eta^2\left(\frac{1}{2}\left[\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2\ln\ln n}\right)} + 1\right]\right)\eta^2\left(\frac{i}{n}\right)}{\eta\left(\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2\ln\ln n}\right)} + 1\right)\eta(in)\eta^2\left(\frac{i+n}{2n}\right)}$$

and

$$e^{\frac{p_n}{2n}} \sim \frac{\eta^2\left(\frac{1}{2}\left[\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2\ln\ln n}\right)} + 1\right]\right)\eta^2\left(\frac{i}{n}\right)}{2\eta\left(\frac{i}{\frac{n}{6}\left(5 + \frac{\ln n}{2\ln\ln n}\right)} + 1\right)\eta(in)\eta^2\left(\frac{2i}{n}\right)},$$

where p_n *denotes the* nth *prime number and* $\eta(\tau)$ *denotes the Dedekind eta function.*

Proof. In [4, page 114], Weber defined the following functions

$$f(\tau) = \frac{e^{-\frac{\pi i}{24}}\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}, \tag{17}$$

$$f_2(\tau) = \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)}. \tag{18}$$

and conclude that

$$\vartheta_{00} = \eta(\tau)f(\tau)^2, \tag{19}$$

$$\vartheta_{10} = \eta(\tau)f_2(\tau)^2, \tag{20}$$

where $\vartheta_{00}, \vartheta_{01}$ and ϑ_{10} are Jacobi theta functions.

The Theorem 2 assures me that

$$e^{\frac{pn}{2n}}\eta(in) \sim \eta(i/n), \tag{21}$$

Multiplying both members of (21) by $f(i/n)^2$ and $f_2(i/n)^2$, respectively, I encounter

$$e^{\frac{pn}{2n}}\eta(in)\, f(i/n)^2 \sim \eta(i/n)\, f(i/n)^2, \tag{22}$$

$$e^{\frac{pn}{2n}}\eta(in)\, f_2(i/n)^2 \sim \eta(i/n)\, f_2(i/n)^2, \tag{23}$$

from (17) and (18), I set

$$e^{\frac{pn}{2n}}\frac{e^{-\frac{\pi i}{12}}\eta(in)\eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)} \sim \vartheta_{00}, \tag{24}$$

$$2e^{\frac{pn}{2n}}\frac{\eta(in)\eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)} \sim \vartheta_{10}. \tag{25}$$

In [5, page 173] the Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \qquad |ab| < 1, \tag{26}$$

which is a generalization of Jacobi theta functions. Roughly speaking, I can suppose that

$$e^{\frac{pn}{2n}}\frac{e^{-\frac{\pi i}{12}}\eta(in)\eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)} = f_{00}(a,b), \tag{27}$$

$$2e^{\frac{pn}{2n}}\frac{\eta(in)\eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)} = f_{10}(a,b). \tag{28}$$

videlicet,

$$e^{\frac{pn}{2n}}\frac{e^{-\frac{\pi i}{12}}\eta(in)\eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\sigma(n)k(k+1)/2} e^{-\pi\sigma(n)k(k-1)/2} = \vartheta_3\left(0, e^{-2\pi\sigma(n)}\right), \tag{29}$$

$$2e^{\frac{p_n}{2n}}\frac{\eta(in)\eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\omega(n)k(k+1)/2}e^{-\pi\omega(n)k(k-1)/2} = \vartheta_3\left(0, e^{-2\pi\omega(n)}\right). \tag{30}$$

How calculate the functions $\sigma(n)$ and $\omega(n)$? Approximately, I have

$$e^{\frac{p_n}{2n}}\frac{e^{-\frac{\pi i}{12}}\eta(in)\eta^2\left(\frac{i+n}{2n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\sigma(n)k(k+1)/2}e^{-\pi\sigma(n)k(k-1)/2} \sim \int_{-\infty}^{\infty} e^{-\pi\sigma(n)x(x+1)/2}e^{-\pi\sigma(n)x(x-1)/2}dx = \frac{1}{\sqrt{2\sigma(n)}}, \tag{31}$$

$$2e^{\frac{p_n}{2n}}\frac{\eta(in)\eta^2\left(\frac{2i}{n}\right)}{\eta^2\left(\frac{i}{n}\right)} \approx \sum_{k=-\infty}^{\infty} e^{-\pi\omega(n)k(k+1)/2}e^{-\pi\omega(n)k(k-1)/2} \sim \int_{-\infty}^{\infty} e^{-\pi\omega(n)x(x+1)/2}e^{-\pi\omega(n)x(x-1)/2}\,dx = \frac{1}{\sqrt{2\omega(n)}}, \tag{32}$$

pursuant to,

$$\sigma(n) \sim \frac{\eta^4\left(\frac{i}{n}\right)}{2e^{\frac{p_n}{n}}e^{-\frac{\pi i}{6}}\eta^2(in)\eta^4\left(\frac{i+n}{2n}\right)}, \tag{33}$$

$$\omega(n) \sim \frac{\eta^4\left(\frac{i}{n}\right)}{8e^{\frac{p_n}{n}}\eta^2(in)\eta^4\left(\frac{2i}{n}\right)}. \tag{34}$$

From (8), (33) and (34), I obtain

$$\sigma(n) \sim \frac{e^{\frac{\pi i}{6}}\eta^4\left(\frac{i}{n}\right)}{2n\ln n e^{\frac{1}{12}\left(\frac{5}{\ln n}+\frac{1}{2\ln\ln n}-12\right)}\eta^2(in)\eta^4\left(\frac{i+n}{2n}\right)}, \tag{35}$$

$$\omega(n) \sim \frac{\eta^4\left(\frac{i}{n}\right)}{8n\ln n e^{\frac{1}{12}\left(\frac{5}{\ln n}+\frac{1}{2\ln\ln n}-12\right)}\eta^2(in)\eta^4\left(\frac{2i}{n}\right)}. \tag{36}$$

On the other hand, I notice that

$$\frac{\eta^4\left(\frac{i}{n}\right)}{\eta^2(in)\eta^4\left(\frac{i+n}{2n}\right)} \searrow \frac{\sqrt{3}}{2} - \frac{1}{2}i, \tag{37}$$

$$\frac{\eta^4\left(\frac{i}{n}\right)}{\eta^2(in)\eta^4\left(\frac{2i}{n}\right)} \searrow 4. \tag{38}$$

I get (37) and (38) into (35) and (36)

$$\sigma(n) \sim \frac{e^{\frac{\pi i}{6}}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)}{2n\ln n e^{\frac{1}{12}\left(\frac{5}{\ln n}+\frac{1}{2\ln\ln n}-12\right)}}, \tag{39}$$

$$\omega(n) \sim \frac{1}{2n\ln n e^{\frac{1}{12}\left(\frac{5}{\ln n}+\frac{1}{2\ln\ln n}-12\right)}}. \tag{40}$$

On the one hand, I readily saw that

$$e^{\frac{\pi i}{6}}=\frac{\sqrt{3}}{2}+\frac{1}{2}i \tag{41}$$

and the representation in series power of $e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}$ is

$$e^{\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)}=1+\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)+\frac{1}{2!}\left[\frac{1}{24}\left(\frac{1}{\ln\ln n}+\frac{10}{\ln n}-24\right)\right]^2+\cdots, \tag{42}$$

I set (41) and (42) in (39) and (40)

$$\sigma(n)\sim\frac{1}{2n\ln n\left[1+\frac{1}{12}\left(\frac{5}{\ln n}+\frac{1}{2\ln\ln n}-12\right)\right]}=\frac{1}{n\left[\frac{1}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)\right]}, \tag{43}$$

$$\omega(n)\sim\frac{1}{2n\ln n\left[1+\frac{1}{12}\left(\frac{5}{\ln n}+\frac{1}{2\ln\ln n}-12\right)\right]}=\frac{1}{n\left[\frac{1}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)\right]}. \tag{44}$$

From (29), (30), (43) and (44), I conclude that

$$e^{\frac{p_n}{2n}}\sim\frac{e^{\frac{\pi i}{12}}\vartheta_3\left(0,e^{-\frac{\pi}{\frac{n}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)}}\right)\eta^2\left(\frac{i}{n}\right)}{\eta(in)\eta^2\left(\frac{i+n}{2n}\right)}, \tag{45}$$

$$e^{\frac{p_n}{2n}}\sim\frac{\vartheta_3\left(0,e^{-\frac{\pi}{\frac{n}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)}}\right)\eta^2\left(\frac{i}{n}\right)}{2\eta(in)\eta^2\left(\frac{2i}{n}\right)}. \tag{46}$$

In [3], I encounter

$$\vartheta_3\left(0,e^{\pi i\tau}\right)=\frac{\eta^2\left(\frac{\tau+1}{2}\right)}{\eta(\tau+1)}. \tag{47}$$

I substitute (47) into (45) and (46)

$$e^{\frac{p_n}{2n}}\sim e^{\frac{\pi i}{12}}\frac{\eta^2\left(\frac{1}{2}\left[\frac{i}{\frac{n}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)}+1\right]\right)\eta^2\left(\frac{i}{n}\right)}{\eta\left(\frac{i}{\frac{n}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)}+1\right)\eta(in)\eta^2\left(\frac{i+n}{2n}\right)},$$

$$e^{\frac{p_n}{2n}}\sim\frac{\eta^2\left(\frac{1}{2}\left[\frac{i}{\frac{n}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)}+1\right]\right)\eta^2\left(\frac{i}{n}\right)}{2\eta\left(\frac{i}{\frac{n}{6}\left(5+\frac{\ln n}{2\ln\ln n}\right)}+1\right)\eta(in)\eta^2\left(\frac{2i}{n}\right)}.\ \square$$

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