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On the Complete Elliptic Integrals and Babylonian Identity V: The Complete Elliptic Integral of first kind and Approximation by Inverse Sine Function

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Abstract. I approximate the complete elliptic integral of first kind using inverse sine function.

1. INTRODUCTION

I developed the following approximation:

$$K(k) \cong \frac{\sqrt{\pi}}{k}\sin^{-1}(k) + \frac{\sqrt{\pi}}{3k}\sin^{-1}(k)$$

$$+\sqrt{\pi}\left[\frac{35\left(315\sqrt{\pi}-256\right)k^8}{294912} + \frac{5\left(35\sqrt{\pi}-32\right)k^6}{3584} + \frac{3}{640}\left(15\sqrt{\pi}-16\right)k^4 + \frac{1}{24}\left(3\sqrt{\pi}-4\right)k^2 + \frac{\sqrt{\pi}}{2} - 1\right]$$

$$+\frac{\sqrt{\pi}}{30240}\left[-1225k^8 - 2025k^6 - 4536k^4 + \frac{10080k^2}{\sqrt{1-k^2}} - 6720k^2 - \frac{10080}{\sqrt{1-k^2}} - 10080k^2\ln\left(\frac{1+\sqrt{1-k^2}}{2}\right)\right],$$

for $0 < k < 1/2$.

2. THEOREM

Theorem 1. *For* $0 < k < 1/2$, *then*

$$K(k) \cong \frac{\sqrt{\pi}}{k}\sin^{-1}(k) + \frac{\sqrt{\pi}}{3k}\sin^{-1}(k)$$

$$+\sqrt{\pi}\left[\frac{35\left(315\sqrt{\pi}-256\right)k^8}{294912} + \frac{5\left(35\sqrt{\pi}-32\right)k^6}{3584} + \frac{3}{640}\left(15\sqrt{\pi}-16\right)k^4 + \frac{1}{24}\left(3\sqrt{\pi}-4\right)k^2 + \frac{\sqrt{\pi}}{2} - 1\right]$$

$$+\frac{\sqrt{\pi}}{30240}\left[-1225k^8 - 2025k^6 - 4536k^4 + \frac{10080k^2}{\sqrt{1-k^2}} - 6720k^2 - \frac{10080}{\sqrt{1-k^2}} - 10080k^2\ln\left(\frac{1+\sqrt{1-k^2}}{2}\right)\right],$$

where $K(k)$ *denotes the complete elliptic integral of first kind,* $\sin^{-1}(k)$ *denotes the inverse sine function and* $\ln(k)$ *denotes the logarithm function.*

Proof. In previous paper [1], Corollary 3, I proved that

$$K(k) = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2\Gamma\left(n+\frac{3}{2}\right)}{(1)_n\left(\frac{3}{2}\right)_n n!}k^{2n}. \tag{1}$$

On the other hand, I calculated that

$$\frac{\sqrt{\pi}}{k}\sin^{-1}k = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2\Gamma(n+1)}{(1)_n\left(\frac{3}{2}\right)_n n!}k^{2n}. \tag{2}$$

I subtract (1) of (2), and obtain

$$K(k) - \frac{\sqrt{\pi}}{k}\sin^{-1}(k) = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2\Gamma\left(n+\frac{3}{2}\right)}{(1)_n\left(\frac{3}{2}\right)_n n!}k^{2n} - \sqrt{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2\Gamma(n+1)}{(1)_n\left(\frac{3}{2}\right)_n n!}k^{2n}$$

$$= \sqrt{\pi}\sum_{n=0}^{\infty}\left[\Gamma\left(n+\frac{3}{2}\right) - \Gamma(n+1)\right]\frac{\left(\frac{1}{2}\right)_n^2 k^{2n}}{(1)_n\left(\frac{3}{2}\right)_n n!}$$

$$= \sqrt{\pi}\sum_{n=0}^{4}\left[\Gamma\left(n+\frac{3}{2}\right) - \Gamma(n+1)\right]\frac{\left(\frac{1}{2}\right)_n^2 k^{2n}}{(1)_n\left(\frac{3}{2}\right)_n n!} + \sqrt{\pi}\sum_{n=5}^{\infty}\left[\Gamma\left(n+\frac{3}{2}\right) - \Gamma(n+1)\right]\frac{\left(\frac{1}{2}\right)_n^2 k^{2n}}{(1)_n\left(\frac{3}{2}\right)_n n!}$$

$$= \sqrt{\pi}\left[\frac{35(315\sqrt{\pi}-256)k^8}{294912} + \frac{5(35\sqrt{\pi}-32)k^6}{3584} + \frac{3}{640}(15\sqrt{\pi}-16)k^4 + \frac{1}{24}(3\sqrt{\pi}-4)k^2 + \frac{\sqrt{\pi}}{2} - 1\right] + \sqrt{\pi}\sum_{n=5}^{\infty}\left[\Gamma\left(n+\frac{3}{2}\right) - \Gamma(n+1)\right]\frac{\left(\frac{1}{2}\right)_n^2 k^{2n}}{(1)_n\left(\frac{3}{2}\right)_n n!}. \tag{3}$$

I note that: (i) for $n \geq 5$, then

$$\Gamma\left(n+\frac{3}{2}\right) - \Gamma(n+1) > \frac{n \cdot n!}{n-1},$$

From (i) and (3), I can write

$$K(k) - \frac{\sqrt{\pi}}{k}\sin^{-1}(k)$$

$$> \sqrt{\pi}\left[\frac{35(315\sqrt{\pi}-256)k^8}{294912} + \frac{5(35\sqrt{\pi}-32)k^6}{3584} + \frac{3}{640}(15\sqrt{\pi}-16)k^4 + \frac{1}{24}(3\sqrt{\pi}-4)k^2 + \frac{\sqrt{\pi}}{2} - 1\right] + \sqrt{\pi}\sum_{n=5}^{\infty}\left(\frac{n \cdot n!}{n-1}\right)\frac{\left(\frac{1}{2}\right)_n^2 k^{2n}}{(1)_n\left(\frac{3}{2}\right)_n n!}$$

$$= \sqrt{\pi}\left[\frac{35(315\sqrt{\pi}-256)k^8}{294912}+\frac{5(35\sqrt{\pi}-32)k^6}{3584}+\frac{3}{640}(15\sqrt{\pi}-16)k^4+\frac{1}{24}(3\sqrt{\pi}-4)k^2+\frac{\sqrt{\pi}}{2}-1\right]+\sqrt{\pi}\sum_{n=5}^{\infty}\left(\frac{n}{n-1}\right)\frac{\left(\frac{1}{2}\right)_n^2 k^{2n}}{\left(\frac{3}{2}\right)_n n!}$$

$$= \sqrt{\pi}\left[\frac{35(315\sqrt{\pi}-256)k^8}{294912}+\frac{5(35\sqrt{\pi}-32)k^6}{3584}+\frac{3}{640}(15\sqrt{\pi}-16)k^4+\frac{1}{24}(3\sqrt{\pi}-4)k^2+\frac{\sqrt{\pi}}{2}-1\right]$$

$$+\frac{\sqrt{\pi}}{30240}\left[-1225k^8-2025k^6-4536k^4+\frac{10080k^2}{\sqrt{1-k^2}}-6720k^2-\frac{10080}{\sqrt{1-k^2}}-10080k^2\ln\left(\frac{1+\sqrt{1-k^2}}{2}\right)+\frac{10080\sin^{-1}(k)}{k}\right],$$

accordingly, since $0 < k < 1/2$, I deduced easily that

$$K(k) \cong \frac{\sqrt{\pi}}{k}\sin^{-1}(k)+\frac{\sqrt{\pi}}{3k}\sin^{-1}(k)$$

$$+\sqrt{\pi}\left[\frac{35(315\sqrt{\pi}-256)k^8}{294912}+\frac{5(35\sqrt{\pi}-32)k^6}{3584}+\frac{3}{640}(15\sqrt{\pi}-16)k^4+\frac{1}{24}(3\sqrt{\pi}-4)k^2+\frac{\sqrt{\pi}}{2}-1\right]$$

$$+\frac{\sqrt{\pi}}{30240}\left[-1225k^8-2025k^6-4536k^4+\frac{10080k^2}{\sqrt{1-k^2}}-6720k^2-\frac{10080}{\sqrt{1-k^2}}-10080k^2\ln\left(\frac{1+\sqrt{1-k^2}}{2}\right)\right].\square$$

Remark: to the delight of the reader, I left the Table 1. In this table, I used

$$G\left(\frac{1}{m}\right)^* \cong m\sqrt{\pi}\sin^{-1}\left(\frac{1}{m}\right)+\frac{m\sqrt{\pi}}{3}\sin^{-1}\left(\frac{1}{m}\right)$$

$$+\sqrt{\pi}\left[\frac{35(315\sqrt{\pi}-256)}{294912m^8}+\frac{5(35\sqrt{\pi}-32)}{3584m^6}+\frac{3(15\sqrt{\pi}-16)}{640m^4}+\frac{3\sqrt{\pi}-4}{24m^2}+\frac{\sqrt{\pi}}{2}-1\right]$$

$$+\frac{\sqrt{\pi}}{30240}\left[-\frac{1225}{m^8}-\frac{2025}{m^6}-\frac{4536}{m^4}+\frac{10080}{m^2\sqrt{1-\left(\frac{1}{m}\right)^2}}-\frac{6720}{m^2}-\frac{10080}{\sqrt{1-\left(\frac{1}{m}\right)^2}}-\frac{10080}{m^2}\ln\left(\frac{1+\sqrt{1-\left(\frac{1}{m}\right)^2}}{2}\right)\right].\square$$

Table 1

In this table, I have: first column: m; second column: $k = 1/m$; third column: $K\left(\frac{1}{m}\right)$; fourth column: $G\left(\frac{1}{m}\right)^*$; fifth column: $\frac{K\left(\frac{1}{m}\right)}{G\left(\frac{1}{m}\right)^*}$.

2	$\frac{1}{2}$	1.6857503548125960428	1.6857403110718361953	1.0000059580593130987
3	$\frac{1}{3}$	1.6173867356247324265	1.6173866088101303537	1.0000000784071052536
4	$\frac{1}{4}$	1.5962422221317835101	1.5962422157029001064	1.0000000040275112012
5	$\frac{1}{5}$	1.5868678474541662373	1.5868678467960866801	1.00000000041470344145
6	$\frac{1}{6}$	1.5818784363176980222	1.5818784362141167553	1.00000000006547991582
7	$\frac{1}{7}$	1.5789039188597221945	1.5789039188378899243	1.00000000001382748502
8	$\frac{1}{8}$	1.5769867712158131421	1.5769867712101267978	1.00000000000360582877
9	$\frac{1}{9}$	1.575678422624177276	1.5756784226224381409	1.00000000000110373739
10	$\frac{1}{10}$	1.5747455615173559526	1.5747455615167525137	1.00000000000038319772
11	$\frac{1}{11}$	1.5740569481969505752	1.5740569481967187635	1.00000000000014727026
12	$\frac{1}{12}$	1.5735341080428901972	1.573534108042793357	1.00000000000006154314
13	$\frac{1}{13}$	1.5731277560114288442	1.5731277560113854425	1.00000000000002758938
14	$\frac{1}{14}$	1.5728056640231232325	1.5728056640231025824	1.00000000000001312948
15	$\frac{1}{15}$	1.5725460328830585891	1.5725460328830482449	1.00000000000000657798
16	$\frac{1}{16}$	1.5723336873167065823	1.5723336873167011632	1.0000000000000034465
17	$\frac{1}{17}$	1.5721577982816330998	1.572157798281630147	1.00000000000000187817
18	$\frac{1}{18}$	1.5720104697444878613	1.5720104697444861953	1.00000000000000105975
19	$\frac{1}{19}$	1.5718858341464005877	1.5718858341463996181	1.0000000000000006168
20	$\frac{1}{20}$	1.5717794574832945285	1.5717794574832939483	1.00000000000000036912
21	$\frac{1}{21}$	1.5716879385035419064	1.5716879385035415504	1.00000000000000022651
22	$\frac{1}{22}$	1.5716086328456104228	1.5716086328456101994	1.0000000000000001422
23	$\frac{1}{23}$	1.5715394594947149985	1.5715394594947148552	1.00000000000000009114
24	$\frac{1}{24}$	1.5714787626279964161	1.5714787626279963226	1.00000000000000005953
25	$\frac{1}{25}$	1.5714252114413814438	1.5714252114413813816	1.00000000000000003957
26	$\frac{1}{26}$	1.5713777264737404576	1.5713777264737404156	1.00000000000000002673
27	$\frac{1}{27}$	1.5713354247087490886	1.5713354247087490598	1.00000000000000001832
28	$\frac{1}{28}$	1.571297578176380478	1.5712975781763804579	1.00000000000000001273
29	$\frac{1}{29}$	1.571263582388512046	1.5712635823885120319	1.00000000000000000896
30	$\frac{1}{30}$	1.5712329320261925969	1.5712329320261925869	1.00000000000000000639
31	$\frac{1}{31}$	1.5712052020348712371	1.5712052020348712299	1.0000000000000000046
32	$\frac{1}{32}$	1.5711800327950414382	1.5711800327950414329	1.00000000000000000335

REFERENCES

[1] Guedes, Edigles, *On the Complete Elliptic Integrals and Babylonian Identity I: The* $1/\pi$ *formulaes Involving Gamma functions and Summations,* 2013.