

On the Complete Elliptic Integrals and Babylonian Identity IV: The Complete Elliptic Integral of first kind as sum of two Gauss hypergeometric functions

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Abstract. I evaluate the complete elliptic integral of first kind as the sum of two Gauss hypergeometric functions.

1. INTRODUCTION

In this paper, I evaluate the complete elliptic integral of first kind as the sum of two Gauss hypergeometric functions, as follows:

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; k^2\right) + \frac{\pi k^2}{16} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; k^2\right).$$

2. THEOREM

Theorem 1. For $0 < k < 1$, then

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; k^2\right) + \frac{\pi k^2}{16} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; k^2\right),$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. In previous paper [1], Theorem 1, I proved that

$$\frac{K(k)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right) k^{2n}}{\left(\frac{3}{2}\right)_n n!^2}, \quad (1)$$

for $0 < k < 1$.

I know that

$$\left(\frac{1}{2}\right)_n^2 = \frac{[(2n)!]^2}{2^{4n} n!^2}, \quad (2)$$

$$\left(\frac{3}{2}\right)_n = \frac{(2(n+1))!}{2^{2n+1} (n+1)!} \quad (3)$$

And

$$\Gamma\left(n + \frac{3}{2}\right) = \sqrt{\pi} \frac{(2n+1)!}{2^{2n+1} n!}. \quad (4)$$

From (1), (2), (3) and (4), it follows that

$$\begin{aligned} \frac{K(k)}{\sqrt{\pi}} &= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{[(2n)!]^2 2^{2n+1} (n+1)! (2n+1)! k^{2n}}{2^{4n} (2(n+1))! n!^2 2^{2n+1} n! n!^2} \\ &= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{[(2n)!]^2 (n+1)! (2n+1)! k^{2n}}{n!^2 (2(n+1))! n! 2^{4n} n!^2} \\ &= \sqrt{\pi} \sum_{n=0}^{\infty} \binom{n+1}{n+2} \frac{\binom{2n}{n}^2 \binom{2n+1}{n}}{\binom{2n+2}{n}} \left(\frac{k}{4}\right)^{2n}, \end{aligned}$$

ergo,

$$\begin{aligned} K(k) &= \pi \sum_{n=0}^{\infty} \binom{n+1}{n+2} \frac{\binom{2n}{n}^2 \binom{2n+1}{n}}{\binom{2n+2}{n}} \left(\frac{k}{4}\right)^{2n} \\ &= \pi \sum_{n=0}^{\infty} \binom{n}{n+2} \frac{\binom{2n}{n}^2 \binom{2n+1}{n}}{\binom{2n+2}{n}} \left(\frac{k}{4}\right)^{2n} + \pi \sum_{n=0}^{\infty} \binom{1}{n+2} \frac{\binom{2n}{n}^2 \binom{2n+1}{n}}{\binom{2n+2}{n}} \left(\frac{k}{4}\right)^{2n} \\ &= \frac{\pi k^2}{16} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; k^2\right) + \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; k^2\right) \\ &= \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; k^2\right) + \frac{\pi k^2}{16} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; k^2\right). \square \end{aligned}$$

Note: For the reader's delight, I construct the Table 1.

Table 1

In this table, I have: first column: m ; second column: $k = 1/m$; third column: $\frac{\pi}{16m^2} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; \frac{1}{m^2}\right)$; fourth column: $\frac{1}{2}\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{1}{m^2}\right)$, fifth column: $\frac{\pi}{16m^2} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; \frac{1}{m^2}\right) + \frac{1}{2}\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{1}{m^2}\right)$; sixth column: $K(k)$.

2	$\frac{1}{2}$	0.06055480897	1.625195545	1.685750354	1.685750354
3	$\frac{1}{3}$	0.0238082898	1.593578445	1.617386735	1.617386735
4	$\frac{1}{4}$	0.01287669071	1.583365531	1.596242222	1.596242222
5	$\frac{1}{5}$	0.008097213128	1.578770634	1.586867847	1.586867847
6	$\frac{1}{6}$	0.005570305291	1.576308131	1.581878436	1.581878436
7	$\frac{1}{7}$	0.004069461643	1.574834457	1.578903918	1.578903918
8	$\frac{1}{8}$	0.003104358801	1.573882412	1.576986771	1.576986771
9	$\frac{1}{9}$	0.002446732173	1.57323169	1.575678422	1.575678422
10	$\frac{1}{10}$	0.00197833762	1.572767224	1.574745561	1.574745561
11	$\frac{1}{11}$	0.001632847064	1.572424101	1.574056948	1.574056948
12	$\frac{1}{12}$	0.00137067899	1.572163429	1.573534108	1.573534108
13	$\frac{1}{13}$	0.001167011611	1.571960744	1.573127756	1.573127756
14	$\frac{1}{14}$	0.00100563207	1.571800032	1.572805664	1.572805664
15	$\frac{1}{15}$	0.0008755836452	1.571670449	1.572546032	1.572546032
16	$\frac{1}{16}$	0.0007692443155	1.571564443	1.572333687	1.572333687
17	$\frac{1}{17}$	0.000681178131	1.57147662	1.572157798	1.572157798
18	$\frac{1}{18}$	0.0006074233099	1.571403046	1.572010469	1.572010469
19	$\frac{1}{19}$	0.0005450369919	1.571340797	1.571885834	1.571885834
20	$\frac{1}{20}$	0.0004917960419	1.571287661	1.571779457	1.571779457
21	$\frac{1}{21}$	0.0004459956038	1.571241943	1.571687938	1.571687938
22	$\frac{1}{22}$	0.0004063105236	1.571202322	1.571608632	1.571608632
23	$\frac{1}{23}$	0.0003716981682	1.571167761	1.571539459	1.571539459
24	$\frac{1}{24}$	0.0003413290826	1.571137433	1.571478762	1.571478762
25	$\frac{1}{25}$	0.0003145367283	1.571110674	1.571425211	1.571425211
26	$\frac{1}{26}$	0.0002907805271	1.571086946	1.571377726	1.571377726
27	$\frac{1}{27}$	0.0002696183309	1.571065806	1.571335424	1.571335424
28	$\frac{1}{28}$	0.0002506856666	1.571046892	1.571297578	1.571297578
29	$\frac{1}{29}$	0.0002336799135	1.571029902	1.571263582	1.571263582
30	$\frac{1}{30}$	0.0002183481196	1.571014584	1.571232932	1.571232932
31	$\frac{1}{31}$	0.0002044775275	1.571000724	1.571205202	1.571205202
32	$\frac{1}{32}$	0.0001918881458	1.570988144	1.571180032	1.571180032

REFERENCES

- [1] Guedes, Edigles, *On the complete elliptic integrals and babylonian Identity 1: the $\frac{1}{\pi}$ formulaes involving gamma functions and summations*, 2013.