

On the Complete Elliptic Integrals and Babylonian Identity III: An Approximation for the Complete Elliptic Integral of the first kind

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Abstract. Using the Corollary 3 of previous paper, I evaluate the complete elliptic integral of the first kind in an approximately equal analytical closed form, by means of Bessel functions.

1. Introduction

By means of the Corollary 3 of previous paper, I proved the following approximately equal analytical closed form:

$$K(k) \cong \frac{\pi}{2} I_0(2k) + \frac{\pi k^2 (10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} + \frac{\pi (24e^{k^2} k^2 + 24e^{k^2} - 5k^8 - 16k^6 - 36k^4 - 48k^2 - 24)}{48},$$

for $0 < k \leq 1/2$.

2. THEOREM

Theorem 1. For $0 < k \leq 1/2$, then

$$K(k) \cong \frac{\pi}{2} I_0(2k) + \frac{\pi k^2 (10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} + \frac{\pi (24e^{k^2} k^2 + 24e^{k^2} - 5k^8 - 16k^6 - 36k^4 - 48k^2 - 24)}{48},$$

where $K(k)$ denotes the complete elliptic integral of first kind and $I_\nu(z)$ denotes the modified Bessel function of the first kind.

Proof. I will prove the equality above, developing the left hand side of the equation below

$$K(k) - \frac{\pi}{2} I_0(2k) = I_k. \tag{1}$$

First, in [1], Corollary 3, I proved the identity

$$K(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n}. \tag{2}$$

On the other hand, I calculated

$$\sqrt{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n} = \frac{\pi}{2} I_0(2k), \tag{3}$$

I subtract (3) of (2), and obtain

$$\begin{aligned}
K(k) - \frac{\pi}{2} I_0(2k) &= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n} - \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n} \\
&= \sqrt{\pi} \sum_{m=0}^{\infty} \left[\left(\frac{1}{2}\right)_n^2 - 1 \right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2m} \\
&= \sqrt{\pi} \sum_{m=0}^4 \left[\left(\frac{1}{2}\right)_n^2 - 1 \right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2m} + \sqrt{\pi} \sum_{m=5}^{\infty} \left[\left(\frac{1}{2}\right)_n^2 - 1 \right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2m} \\
&= \frac{\pi k^2 (10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} \\
&\quad + \sqrt{\pi} \sum_{m=5}^{\infty} \left[\left(\frac{1}{2}\right)_n^2 - 1 \right] \frac{\Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2m}. \tag{5}
\end{aligned}$$

I note that, if $m \geq 5$, then

$$\left(\frac{1}{2}\right)_n^2 - 1 > (n+1)!. \tag{6}$$

I take (6) into (5)

$$\begin{aligned}
&K(k) - \frac{\pi}{2} I_0(2k) \\
&> \frac{\pi k^2 (10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} + \sqrt{\pi} \sum_{m=5}^{\infty} \frac{(n+1)! \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2m} \\
&= \frac{\pi k^2 (10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} + \sqrt{\pi} \sum_{m=5}^{\infty} \frac{(n+1) \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n} k^{2m}. \tag{7}
\end{aligned}$$

I calculate

$$\sqrt{\pi} \sum_{m=5}^{\infty} \frac{(n+1) \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n} k^{2m} = \frac{\pi (24e^{k^2} k^2 + 24e^{k^2} - 5k^8 - 16k^6 - 36k^4 - 48k^2 - 24)}{48}. \tag{8}$$

From (7) and (8), I obtain

$$K(k) - \frac{\pi}{2}I_0(2k) > \frac{\pi k^2(10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} \\ + \frac{\pi(24e^{k^2}k^2 + 24e^{k^2} - 5k^8 - 16k^6 - 36k^4 - 48k^2 - 24)}{48},$$

hence,

$$K(k) > \frac{\pi}{2}I_0(2k) + \frac{\pi k^2(10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} \\ + \frac{\pi(24e^{k^2}k^2 + 24e^{k^2} - 5k^8 - 16k^6 - 36k^4 - 48k^2 - 24)}{48},$$

since $0 < k < 1/2$, I conclude easily that

$$K(k) \cong \frac{\pi}{2}I_0(2k) + \frac{\pi k^2(10769k^6 + 10304k^4 - 16128k^2 - 110592)}{294912} \\ + \frac{\pi(24e^{k^2}k^2 + 24e^{k^2} - 5k^8 - 16k^6 - 36k^4 - 48k^2 - 24)}{48}.$$

This finished the proof. The reader may see the Table 1. \square

Table 1

In this table, I have: first column: m ; second column: $k = 1/m$; third column: G_1^* ; fourth column: $K\left(\frac{1}{m}\right)$; fifth column: $\frac{K\left(\frac{1}{m}\right)}{G_1^*}$, where $G_1^* = \frac{1}{2}\pi I_0\left(\frac{2}{m}\right) + \frac{\pi}{294912m^2}\left(\frac{10769}{m^6} + \frac{10304}{m^4} - \frac{16128}{m^2} - 110592\right) + \frac{\pi}{48}\left[-\frac{5}{m^8} - \frac{16}{m^6} - \frac{36}{m^4} + \frac{24em^{\frac{1}{2}}}{m^2} - \frac{48}{m^2} + 24e^{\frac{1}{m^2}} - 24\right]$.

2	$\frac{1}{2}$	1.6857132328367148405	1.6857503548125960428	1.0000220215248703562
3	$\frac{1}{3}$	1.6173863193872715507	1.6173867356247324265	1.0000002573519114675
4	$\frac{1}{4}$	1.5962422022969758972	1.5962422221317835101	1.0000000124259386102
5	$\frac{1}{5}$	1.586867845490274469	1.5868678474541662373	1.0000000012375899945
6	$\frac{1}{6}$	1.58187843601454314	1.5818784363176980222	1.00000000019164233815
7	$\frac{1}{7}$	1.5789039187966027092	1.5789039188597221945	1.00000000003997677416
8	$\frac{1}{8}$	1.576986771199506925	1.5769867712158131421	1.00000000001034011023
9	$\frac{1}{9}$	1.5756784226192184351	1.575678422624177276	1.00000000000314711483
10	$\frac{1}{10}$	1.5747455615156424328	1.5747455615173559526	1.00000000000108812486
11	$\frac{1}{11}$	1.574056948196294348	1.5740569481969505752	1.00000000000041690181
12	$\frac{1}{12}$	1.5735341080426167014	1.5735341080428901972	1.00000000000017380992
13	$\frac{1}{13}$	1.5731277560113064953	1.5731277560114288442	1.0000000000000777427
14	$\frac{1}{14}$	1.5728056640230651054	1.5728056640231232325	1.00000000000003695756
15	$\frac{1}{15}$	1.5725460328830295064	1.5725460328830585891	1.00000000000001849406
16	$\frac{1}{16}$	1.5723336873166913615	1.5723336873167065823	1.00000000000000968041
17	$\frac{1}{17}$	1.5721577982816248128	1.5721577982816330998	1.00000000000000527106
18	$\frac{1}{18}$	1.572010469744483189	1.5720104697444878613	1.00000000000000297215
19	$\frac{1}{19}$	1.5718858341463978701	1.5718858341464005877	1.00000000000000172885
20	$\frac{1}{20}$	1.5717794574832929031	1.5717794574832945285	1.00000000000000103411
21	$\frac{1}{21}$	1.5716879385035409094	1.5716879385035419064	1.00000000000000063431
22	$\frac{1}{22}$	1.5716086328456097972	1.5716086328456104228	1.00000000000000039806
23	$\frac{1}{23}$	1.5715394594947145977	1.5715394594947149985	1.00000000000000025505
24	$\frac{1}{24}$	1.5714787626279961544	1.5714787626279964161	1.00000000000000016655
25	$\frac{1}{25}$	1.5714252114413812699	1.5714252114413814438	1.00000000000000011067
26	$\frac{1}{26}$	1.5713777264737403402	1.5713777264737404576	1.00000000000000007473
27	$\frac{1}{27}$	1.5713354247087490081	1.5713354247087490886	1.00000000000000005122
28	$\frac{1}{28}$	1.571297578176380422	1.571297578176380478	1.00000000000000003559
29	$\frac{1}{29}$	1.5712635823885120066	1.571263582388512046	1.00000000000000002505
30	$\frac{1}{30}$	1.5712329320261925689	1.5712329320261925969	1.00000000000000001784
31	$\frac{1}{31}$	1.5712052020348712169	1.5712052020348712371	1.00000000000000001285
32	$\frac{1}{32}$	1.5711800327950414235	1.5711800327950414382	1.00000000000000000935

REFERENCES

- [1] Guedes, Edigles, *On the Complete Elliptic Integrals and Babylonian Identity I: The $1/\pi$ formulaes Involving Gamma functions and Summations*, 2013.