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On the Complete Elliptic Integrals and Babylonian Identity II: An Approximation for the Complete Elliptic Integral of the first kind

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Abstract. Using the Theorem 4 of previous paper, I evaluate the complete elliptic integral of the first kind in approximate analytical closed form, by means of Bessel functions.

1. Introduction

By means of the theorem 4 of previous paper, I demonstrated the following approximations:

$$K(k) \approx \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{\sqrt{\pi}}{2} k I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) - \frac{\sqrt{\pi}(4+k^2)}{8}$$

and

$$K(k) \approx \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \pi\left[I_1\left(\frac{k}{2}\right)\right]^2 + \frac{7\sqrt{\pi}}{8} k I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) - \left(\frac{2\pi k^2 + 7\sqrt{\pi}k^2 + 16\sqrt{\pi}}{32}\right),$$

for $0 < k \leq 1/2$.

2. THEOREMS

Theorem 1. *For* $0 < k \leq 1/2$*, then*

$$K(k) \approx \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{\sqrt{\pi}}{2} k I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) - \frac{\sqrt{\pi}(4+k^2)}{8},$$

where $K(k)$ *denotes the complete elliptic integral of first kind and* $I_v(z)$ *denotes the modified Bessel function of the first kind.*

Proof. I will prove the equality above, developing the left hand side of the equation below

$$K(k) - \frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 = I_k. \tag{1}$$

First, I encounter

$$\frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\Gamma\left(m+\frac{1}{2}\right)\frac{k^{2m}}{2^{2m}(m!)^3} = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^3} = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{(2m)!k^{2m}}{2^{4m}(m!)^4} \tag{2}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{k^{2m}}{2^{4m}(m!)^2}.$$

On the other hand, in [1], I proved the identity

$$K(k) = \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\Gamma\left(m+\frac{3}{2}\right)}{(2m+1)}\frac{k^{2m}}{2^{2m}(m!)^3} = \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!(2m+2)!}{(2m+1)2^{2m+2}(m+1)!}\frac{k^{2m}}{2^{2m}(m!)^3} \tag{3}$$

$$= \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,(2m+2)(2m+1)(2m)!}{(2m+1)2^{2m+2}(m+1)m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^3}$$

$$= \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,2(m+1)(2m+1)(2m)!}{(2m+1)2^{2m+2}(m+1)m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^3}$$

$$= \sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,(2m)!}{2^{2m+1}m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^3} = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!}\cdot\frac{(2m)!\,k^{2m}}{2^{2m}(m!)^3}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{[(2m)!]^2k^{2m}}{2^{4m}(m!)^4} = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}^2\frac{k^{2m}}{2^{4m}}.$$

provided that $0 < k < 1$. I subtract (3) of (2), and obtain

$$K(k) - \frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}^2\frac{k^{2m}}{2^{4m}} - \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{k^{2m}}{2^{4m}(m!)^2} \quad (4)$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\left[\binom{2m}{m} - \frac{1}{(m!)^2}\right]\binom{2m}{m}\frac{k^{2m}}{2^{4m}}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\left[\frac{(2m)!}{(m!)^2} - \frac{1}{(m!)^2}\right]\frac{(2m)!}{(m!)^2}\frac{k^{2m}}{2^{4m}}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}[(2m)! - 1]\frac{(2m)!}{(m!)^4}\frac{k^{2m}}{2^{4m}}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=1}^{\infty}[(2m)! - 1]\frac{(2m)!}{(m!)^4}\frac{k^{2m}}{2^{4m}} = \frac{\sqrt{\pi}k^2}{2^4} + \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}[(2m)! - 1]\frac{(2m)!}{(m!)^4}\frac{k^{2m}}{2^{4m}}.$$

I note that, if $m \geq 2$, then

$$(2m)! - 1 \geq 5m + 1. \quad (5)$$

I take (5) into (4)

$$K(k) - \frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 > \frac{\sqrt{\pi}k^2}{2^4} + \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}(5m+1)\frac{(2m)!}{(m!)^4}\frac{k^{2m}}{2^{4m}}$$

$$= \frac{\sqrt{\pi}k^2}{2^4} + \frac{5\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\frac{(2m)!\,mk^{2m}}{(m!)^4 2^{4m}} + \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\frac{(2m)!\,k^{2m}}{(m!)^4 2^{4m}}$$

$$= \frac{\sqrt{\pi}k^2}{2^4} + \frac{5\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m-1)!m!2^{4m}} + \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m!)^2 2^{4m}}. \quad (6)$$

I calculate

$$\frac{5\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m-1)!m!2^{4m}} = \frac{5\sqrt{\pi}k^2}{16}\,{}_1F_2\left(\frac{3}{2};2,2;\frac{k^2}{4}\right) - \frac{5\sqrt{\pi}k^2}{16} = \frac{5\sqrt{\pi}}{4}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \frac{5\sqrt{\pi}k^2}{16} \quad (7)$$

and

$$\frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m!)^2 2^{4m}} = \frac{\sqrt{\pi}}{2}\,{}_1F_2\left(\frac{1}{2};1,1;\frac{k^2}{4}\right) = -\frac{\sqrt{\pi}k^2}{16} + \frac{\sqrt{\pi}}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 - \frac{\sqrt{\pi}}{2}. \quad (8)$$

From (6), (7) and (8), I obtain

$$K(k) - \frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 > \frac{\sqrt{\pi}k^2}{16} + \frac{5\sqrt{\pi}}{4}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \frac{5\sqrt{\pi}k^2}{16} - \frac{\sqrt{\pi}k^2}{16} + \frac{\sqrt{\pi}}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 - \frac{\sqrt{\pi}}{2},$$

hence,

$$K(k) > \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{5\sqrt{\pi}}{4}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \sqrt{\pi}\left(\frac{5k^2+8}{16}\right),$$

since $0 < k < 1/2$, I conclude easily that

$$K(k) \approx \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{5\sqrt{\pi}}{4}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \sqrt{\pi}\left(\frac{5k^2+8}{16}\right).$$

This completed the proof. The reader can see the Table 1. □

Theorem 2. *For* $0 < k \leq 1/2$, *then*

$$K(k) \approx \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \pi\left[I_1\left(\frac{k}{2}\right)\right]^2 + \frac{7\sqrt{\pi}}{8}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \left(\frac{2\pi k^2 + 7\sqrt{\pi}k^2 + 16\sqrt{\pi}}{32}\right),$$

where $K(k)$ *denotes the complete elliptic integral of first kind and* $I_v(z)$ *denotes the modified Bessel function of the first kind.*

Proof. I will prove the equality above, developing the left hand side of the equation below

$$K(k) - \left\{\frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 + \pi\left[I_1\left(\frac{k}{2}\right)\right]^2\right\} = I_k. \quad (9)$$

First, I encounter

$$\frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\Gamma\left(m+\frac{1}{2}\right)\frac{k^{2m}}{2^{2m}(m!)^3} = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^3} = \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{(2m)!\,k^{2m}}{2^{4m}(m!)^4}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{k^{2m}}{2^{4m}(m!)^2} \quad (10)$$

And

$$\pi\left[I_1\left(\frac{k}{2}\right)\right]^2 = \sqrt{\pi}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{2m+1}{2}\right)}{\Gamma(m+1)\Gamma(m+2)\Gamma(m)}\frac{k^{2m}}{2^{2m}} = \sqrt{\pi}\sum_{m=0}^{\infty}\frac{\Gamma\left(m+\frac{1}{2}\right)}{(m-1)!\,m!\,(m+1)!}\frac{k^{2m}}{2^{2m}}$$

$$= \pi\sum_{m=0}^{\infty}\frac{(2m)!}{(m-1)!(m!)^2(m+1)!}\frac{k^{2m}}{2^{4m}} = \pi\sum_{m=0}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m-1)!(m+1)!2^{4m}}. \quad (11)$$

On the other hand, in [1], I proved the identity

$$K(k)=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,\Gamma\left(m+\frac{3}{2}\right)}{(2m+1)}\frac{k^{2m}}{2^{2m}(m!)^{3}}=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,(2m+2)!}{(2m+1)2^{2m+2}(m+1)!}\frac{k^{2m}}{2^{2m}(m!)^{3}}$$

$$=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,(2m+2)(2m+1)(2m)!}{(2m+1)2^{2m+2}(m+1)m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^{3}}$$

$$=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,2(m+1)(2m+1)(2m)!}{(2m+1)2^{2m+2}(m+1)m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^{3}}$$

$$=\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,(2m)!}{2^{2m+1}m!}\cdot\frac{k^{2m}}{2^{2m}(m!)^{3}}=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!}\cdot\frac{(2m)!\,k^{2m}}{2^{2m}(m!)^{3}}$$

$$=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\frac{[(2m)!]^{2}k^{2m}}{2^{4m}(m!)^{4}}=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}^{2}\frac{k^{2m}}{2^{4m}}. \tag{12}$$

provided that $0<k<1$. I subtract (12) of (11) and (10), to obtain

$$K(k)-\left\{\frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2+\pi\left[I_1\left(\frac{k}{2}\right)\right]^2\right\}$$

$$=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}^{2}\frac{k^{2m}}{2^{4m}}-\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\binom{2m}{m}\frac{k^{2m}}{2^{4m}(m!)^{2}}-\pi\sum_{m=0}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m-1)!\,(m+1)!\,2^{4m}}$$

$$=\sqrt{\pi}\sum_{m=0}^{\infty}\left[\frac{1}{2}\binom{2m}{m}-\frac{1}{2(m!)^{2}}-\frac{\sqrt{\pi}}{(m-1)!\,(m+1)!}\right]\binom{2m}{m}\frac{k^{2m}}{2^{4m}}$$

$$=\sqrt{\pi}\sum_{m=0}^{\infty}\left[\frac{1}{2}\frac{(2m)!}{(m!)^{2}}-\frac{1}{2(m!)^{2}}-\frac{\sqrt{\pi}}{(m-1)!\,(m+1)!}\right]\frac{(2m)!}{(m!)^{2}}\frac{k^{2m}}{2^{4m}}$$

$$=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\left[\frac{(2m)!}{(m!)^{2}}-\frac{1}{(m!)^{2}}-\frac{2m\sqrt{\pi}}{m(m-1)!\,(m+1)m!}\right]\frac{(2m)!}{(m!)^{2}}\frac{k^{2m}}{2^{4m}}$$

$$=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\left[\frac{(2m)!}{(m!)^{2}}-\frac{1}{(m!)^{2}}-\frac{2m\sqrt{\pi}}{(m!)^{2}(m+1)}\right]\frac{(2m)!}{(m!)^{2}}\frac{k^{2m}}{2^{4m}}$$

$$=\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty}\left[(2m)!-1-\frac{2m\sqrt{\pi}}{m+1}\right]\frac{(2m)!}{(m!)^{4}}\frac{k^{2m}}{2^{4m}}$$

$$=\frac{\sqrt{\pi}}{2}\sum_{m=1}^{\infty}\left[(2m)!-1-\frac{2m\sqrt{\pi}}{m+1}\right]\frac{(2m)!}{(m!)^{4}}\frac{k^{2m}}{2^{4m}}$$

$$=-\left(\frac{\pi-\sqrt{\pi}}{16}\right)k^{2}+\frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\left[(2m)!-1-\frac{2m\sqrt{\pi}}{m+1}\right]\frac{(2m)!}{(m!)^{4}}\frac{k^{2m}}{2^{4m}}. \tag{13}$$

I note that, if $m \geq 2$, then

$$(2m)! - 1 - \frac{2m\sqrt{\pi}}{m+1} > \frac{7m}{2} + 1. \quad (14)$$

I take (14) into (13)

$$K(k) - \left\{\frac{\pi}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 + \pi\left[I_1\left(\frac{k}{2}\right)\right]^2\right\} > \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\left(\frac{7m}{2}+1\right)\frac{(2m)!}{(m!)^4}\frac{k^{2m}}{2^{4m}} - \left(\frac{\pi-\sqrt{\pi}}{16}\right)k^2$$

$$= \frac{7\sqrt{\pi}}{4}\sum_{m=2}^{\infty}\frac{(2m)!\,mk^{2m}}{(m!)^4 2^{4m}} + \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\frac{(2m)!\,k^{2m}}{(m!)^4 2^{4m}} - \left(\frac{\pi-\sqrt{\pi}}{16}\right)k^2$$

$$= \frac{7\sqrt{\pi}}{4}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m-1)!m!2^{4m}} + \frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m!)^2 2^{4m}} - \left(\frac{\pi-\sqrt{\pi}}{16}\right)k^2. \quad (15)$$

I calculate

$$\frac{7\sqrt{\pi}}{4}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m-1)!m!2^{4m}} = \frac{7\sqrt{\pi}k^2}{32}\,{}_1F_2\left(\frac{3}{2};2,2;\frac{k^2}{4}\right) - \frac{7\sqrt{\pi}k^2}{32} = \frac{7\sqrt{\pi}}{8}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \frac{7\sqrt{\pi}k^2}{32} \quad (16)$$

and

$$\frac{\sqrt{\pi}}{2}\sum_{m=2}^{\infty}\binom{2m}{m}\frac{k^{2m}}{(m!)^2 2^{4m}} = \frac{\sqrt{\pi}}{2}\,{}_1F_2\left(\frac{1}{2};1,1;\frac{k^2}{4}\right) = -\frac{\sqrt{\pi}k^2}{16} + \frac{\sqrt{\pi}}{2}\left[I_0\left(\frac{k}{2}\right)\right]^2 - \frac{\sqrt{\pi}}{2}. \quad (17)$$

From (15), (16) and (17), I obtain

$$K(k) > \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \pi\left[I_1\left(\frac{k}{2}\right)\right]^2 + \frac{7\sqrt{\pi}}{8}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \left(\frac{2\pi k^2 + 7\sqrt{\pi}k^2 + 16\sqrt{\pi}}{32}\right),$$

since $0 < k < 1/2$, I conclude easily that

$$K(k) \approx \left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \pi\left[I_1\left(\frac{k}{2}\right)\right]^2 + \frac{7\sqrt{\pi}}{8}kI_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right) - \left(\frac{2\pi k^2 + 7\sqrt{\pi}k^2 + 16\sqrt{\pi}}{32}\right).$$

This completed the proof. The reader can see the Table 2. □

Table 1

In this table, I have: first column: m; second column: $k = 1/m$; third column: $\left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{1}{2m}\right)\right]^2 + \frac{5\sqrt{\pi}}{4m} I_0\left(\frac{1}{2m}\right) I_1\left(\frac{1}{2m}\right) - \frac{\sqrt{\pi}}{16}\left[5\left(\frac{1}{m}\right)^2 + 8\right]$; fourth column: $K\left(\frac{1}{m}\right)$; fifth column: $\frac{K\left(\frac{1}{m}\right)}{\left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{1}{2m}\right)\right]^2 + \frac{5\sqrt{\pi}}{4m} I_0\left(\frac{1}{2m}\right) I_1\left(\frac{1}{2m}\right) - \frac{\sqrt{\pi}}{16}\left[5\left(\frac{1}{m}\right)^2 + 8\right]}$.

2	$\frac{1}{2}$	1.651757078	1.685750354	1.02058007
3	$\frac{1}{3}$	1.6057434	1.617386735	1.007251055
4	$\frac{1}{4}$	1.590251421	1.596242222	1.003767203
5	$\frac{1}{5}$	1.583187698	1.586867847	1.002324518
6	$\frac{1}{6}$	1.579378879	1.581878436	1.00158262
7	$\frac{1}{7}$	1.577091885	1.578903918	1.001148971
8	$\frac{1}{8}$	1.5756114	1.576986771	1.000872912
9	$\frac{1}{9}$	1.574598142	1.575678422	1.000686067
10	$\frac{1}{10}$	1.57387424	1.574745561	1.000553615
11	$\frac{1}{11}$	1.573339105	1.574056948	1.000456254
12	$\frac{1}{12}$	1.572932358	1.573534108	1.000382565
13	$\frac{1}{13}$	1.572615974	1.573127756	1.000325433
14	$\frac{1}{14}$	1.572365032	1.572805664	1.000280234
15	$\frac{1}{15}$	1.57216265	1.572546032	1.000243857
16	$\frac{1}{16}$	1.571997057	1.572333687	1.000214141
17	$\frac{1}{17}$	1.571859847	1.572157798	1.000189553
18	$\frac{1}{18}$	1.571744884	1.572010469	1.000168974
19	$\frac{1}{19}$	1.571647605	1.571885834	1.000151578
20	$\frac{1}{20}$	1.571564561	1.571779457	1.00013674
21	$\frac{1}{21}$	1.571493103	1.571687938	1.000123981
22	$\frac{1}{22}$	1.571431171	1.571608632	1.000112929
23	$\frac{1}{23}$	1.571377145	1.571539459	1.000103293
24	$\frac{1}{24}$	1.571329734	1.571478762	1.000094841
25	$\frac{1}{25}$	1.571287901	1.571425211	1.000087387
26	$\frac{1}{26}$	1.571250803	1.571377726	1.000080778
27	$\frac{1}{27}$	1.571217751	1.571335424	1.000074892
28	$\frac{1}{28}$	1.571188179	1.571297578	1.000069628
29	$\frac{1}{29}$	1.571161614	1.571263582	1.000064899
30	$\frac{1}{30}$	1.571137662	1.571232932	1.000060637
31	$\frac{1}{31}$	1.57111599	1.571205202	1.000056782
32	$\frac{1}{32}$	1.571096319	1.571180032	1.000053283

Table 2

In this table, I have: first column: m; second column: $k = 1/m$; third column: $\left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{1}{2m}\right)\right]^2 + \pi\left[I_1\left(\frac{1}{2m}\right)\right]^2 + \frac{7\sqrt{\pi}}{8m} I_0\left(\frac{1}{2m}\right) I_1\left(\frac{1}{2m}\right) - \left(\frac{2\pi(1/m)^2+7\sqrt{\pi}(1/m)^2+16\sqrt{\pi}}{32}\right)$; fourth column: $K\left(\frac{1}{m}\right)$; fifth column: $\frac{K\left(\frac{1}{m}\right)}{\left(\frac{\pi+\sqrt{\pi}}{2}\right)\left[I_0\left(\frac{1}{2m}\right)\right]^2+\pi\left[I_1\left(\frac{1}{2m}\right)\right]^2+\frac{7\sqrt{\pi}}{8m}I_0\left(\frac{1}{2m}\right)I_1\left(\frac{1}{2m}\right)-\left(\frac{2\pi(1/m)^2+7\sqrt{\pi}(1/m)^2+16\sqrt{\pi}}{32}\right)}$.

2	$\frac{1}{2}$	1.651546952	1.685750354	1.020709918
3	$\frac{1}{3}$	1.605702276	1.617386735	1.007276852
4	$\frac{1}{4}$	1.590238451	1.596242222	1.00377539
5	$\frac{1}{5}$	1.583182393	1.586867847	1.002327876
6	$\frac{1}{6}$	1.579376323	1.581878436	1.001584241
7	$\frac{1}{7}$	1.577090506	1.578903918	1.001149847
8	$\frac{1}{8}$	1.575610592	1.576986771	1.000873425
9	$\frac{1}{9}$	1.574597637	1.575678422	1.000686387
10	$\frac{1}{10}$	1.573873909	1.574745561	1.000553825
11	$\frac{1}{11}$	1.573338879	1.574056948	1.000456398
12	$\frac{1}{12}$	1.572932199	1.573534108	1.000382666
13	$\frac{1}{13}$	1.572615858	1.573127756	1.000325507
14	$\frac{1}{14}$	1.572364946	1.572805664	1.000280289
15	$\frac{1}{15}$	1.572162584	1.572546032	1.000243898
16	$\frac{1}{16}$	1.571997006	1.572333687	1.000214173
17	$\frac{1}{17}$	1.571859807	1.572157798	1.000189578
18	$\frac{1}{18}$	1.571744852	1.572010469	1.000168995
19	$\frac{1}{19}$	1.57164758	1.571885834	1.000151595
20	$\frac{1}{20}$	1.57156454	1.571779457	1.000136753
21	$\frac{1}{21}$	1.571493086	1.571687938	1.000123992
22	$\frac{1}{22}$	1.571431157	1.571608632	1.000112938
23	$\frac{1}{23}$	1.571377134	1.571539459	1.000103301
24	$\frac{1}{24}$	1.571329725	1.571478762	1.000094848
25	$\frac{1}{25}$	1.571287892	1.571425211	1.000087392
26	$\frac{1}{26}$	1.571250796	1.571377726	1.000080783
27	$\frac{1}{27}$	1.571217745	1.571335424	1.000074896
28	$\frac{1}{28}$	1.571188174	1.571297578	1.000069631
29	$\frac{1}{29}$	1.571161609	1.571263582	1.000064902
30	$\frac{1}{30}$	1.571137657	1.571232932	1.00006064
31	$\frac{1}{31}$	1.571115987	1.571205202	1.000056784
32	$\frac{1}{32}$	1.571096316	1.571180032	1.000053285

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