On the Complete Elliptic Integrals and Babylonian Identity I: The $\frac{1}{\pi}$ Formulaes Involving Gamma Functions and Summations

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Abstract. I evaluate the constant $\frac{1}{\pi}$ using the Babylonian identity and complete elliptic integral of first kind. This resulted in two representations in terms of the Euler's gamma functions and summations.

1. Introduction

By means of the complete elliptic integral of the first kind and Babylonian identity, I demonstrated the identities following, among others:

$$\frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = 4\sqrt{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}.$$

2. Lemmas

Lemma 1. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = \frac{1}{2}\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2} \quad (1)$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}}. \quad (2)$$

Proof. I know the Babylonian identity [1, page 119]

$$ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]. \quad (3)$$

Make the following algebraic manipulation in (3)

$$ab = \left(\frac{a+b}{2}\right)^2 \left[1 - \left(\frac{a-b}{a+b}\right)^2\right],$$

hence,

$$a^{\frac{1}{2}}b^{\frac{1}{2}} = \frac{a+b}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2} \Leftrightarrow \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = \frac{1}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2},$$

and inverting both members, I have

$$a^{-\frac{1}{2}}b^{-\frac{1}{2}} = \frac{2}{a+b} \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}} \Leftrightarrow \frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}}. \square$$

Lemma 2. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 (2k-1)} \left(\frac{a-b}{a+b}\right)^{2k} \quad (4)$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} \left(\frac{a-b}{a+b}\right)^{2k}. \quad (5)$$

Proof. I calculate

$$\sqrt{1-z^2} = -\sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 (2k-1)} z^{2k} \quad (6)$$

and

$$\frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} z^{2k} \quad (7)$$

Take $z = \frac{a-b}{a+b}$ in (6) and (7), then replace in (1) and (2) respectively, completing the proof. $\square$

3. THEOREMS

Theorem 1. *I have*

$$K(k) = \frac{\sqrt{2}}{2} \int_0^{\pi} \frac{1}{\sqrt{2-k^2(1+\cos t)}} dt,$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. Putting $\frac{a-b}{a+b} = t$ in (5), I encounter

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} t^{2n}. \quad (8)$$

Multiplying (8) by $\frac{1}{\sqrt{1-k^2 t^2}}$ and integrating from 0 at 1 in t , I find

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2 t^2)}} dt &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} \int_0^1 \frac{t^{2n}}{\sqrt{1-k^2 t^2}} dt \Leftrightarrow \\ K(k) &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (2n+1) n!^2} {}_2F_1\left(\frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; k^2\right). \end{aligned} \quad (9)$$

On the one hand, in [2, page 21], I have

$${}_2F_1(a, b; c; z) = \frac{2^{1-c}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\pi \frac{(\sin t)^{2b-1}(1+\cos t)^{c-2b}}{\left(1-\frac{1}{2}z+\frac{1}{2}z\cos t\right)^a} dt, \quad (10)$$

for $\Re(c) > \Re(b) > 0$. Substituting (10) in (9), I encounter

$$\begin{aligned} K(k) &= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{3n}(2n+1)n!^2} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \frac{(\sin t)^{2n}(1+\cos t)^{-n+\frac{1}{2}}}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}} dt = \\ &= \frac{1}{\sqrt{2}} \int_0^\pi \sqrt{\frac{1+\cos t}{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}} \sum_{n=0}^{\infty} \frac{(2n)! \Gamma\left(n+\frac{3}{2}\right)}{2^{3n}(2n+1)n!^2 \Gamma\left(n+\frac{1}{2}\right)} (\sin t)^{2n}(1+\cos t)^{-n} dt = \\ &= \frac{1}{2} \int_0^\pi \frac{1}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}} dt \\ &= \frac{\sqrt{2}}{2} \int_0^\pi \frac{1}{\sqrt{2-k^2(1+\cos t)}} dt. \square \end{aligned}$$

Theorem 2. *I have*

$$k'K(k) = K\left(-i\frac{k}{k'}\right).$$

Proof. I leave to the reader. $\square$

Theorem 3. *For $0 < k < 1$, then*

$$\frac{K(k)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(\frac{2n+3}{2}\right) k^{2n}}{\left(\frac{3}{2}\right)_n n!^2},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I consider

$$\begin{aligned} K(k) &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; k^2\right) \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \left(n+\frac{1}{2}\right)_r}{\left(n+\frac{3}{2}\right)_r r!} k^{2r} \\ &= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)_r \left(\sum_{n=0}^{\infty} \frac{(2n)! \left(n+\frac{1}{2}\right)_r}{2^{2n}(2n+1)n!^2 \left(n+\frac{3}{2}\right)_r} \right) \frac{k^{2r}}{r!} \end{aligned}$$

$$= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right) k^{2r}}{\left(\frac{3}{2}\right)_r \Gamma(r+1) r!} = \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right) k^{2r}}{\left(\frac{3}{2}\right)_r r!^2}.$$

Multiply both sides by $\frac{1}{\sqrt{\pi}}$ and let $r \rightarrow n$, so the result follows. $\square$

Corollary 1. *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{2\sqrt{2} \Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}.$$

Proof. Let $k = \frac{\sqrt{2}}{2}$ in Theorem 3

$$\begin{aligned} K\left(\frac{\sqrt{2}}{2}\right) &= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2} \left(\frac{1}{2}\right)^r \\ &= \frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{2\sqrt{2} \Gamma\left(\frac{3}{4}\right)}. \end{aligned} \quad (11)$$

On the other hand, in [3], I find

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}. \quad (12)$$

I substitute (12) into (11) and obtain

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{2\sqrt{2} \Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}. \square$$

Corollary 2. *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = 4\sqrt{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}.$$

Proof. From Theorem 2 and $k = \frac{\sqrt{2}}{2}$, I find

$$\frac{\sqrt{2}}{2} K\left(\frac{\sqrt{2}}{2}\right) = K\left(-i \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) \Rightarrow K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} K(-i) \quad (13)$$

Using the Theorem (3), I discover that

$$\begin{aligned} K(-i) &= \sqrt{\pi} \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2} \\ &= \sqrt{\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \end{aligned} \quad (14)$$

I set (14) in (13)

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \quad (15)$$

I put (12) into (15), and have

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = 4\sqrt{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}. \square$$

Theorem 4. For $0 < k < 1$, then

$$K(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2n)! \Gamma\left(n + \frac{3}{2}\right)}{2^{2n} (2n+1) n!^3} k^{2n},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I put $\frac{a-b}{a+b} = t$ in (5) and encounter

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} t^{2n}. \quad (16)$$

Multiplying (16) by $\frac{1}{\sqrt{1-k^2 t^2}}$ and integrating from 0 at 1 in t , I find

$$\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2 t^2)}} dt = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} \int_0^1 \frac{t^{2n}}{\sqrt{1-k^2 t^2}} dt. \quad (17)$$

Let $t \rightarrow kt$ in (16)

$$\frac{1}{\sqrt{1-k^2 t^2}} = \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m} m!^2} k^{2m} t^{2m}. \quad (18)$$

I put (18) in (17)

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \int_0^1 t^{2n} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} k^{2m} t^{2m} dt \\
 &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\int_0^1 t^{2(m+n)} dt \right] k^{2m} \\
 &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \frac{k^{2m}}{2m+2n+1} \\
 &= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2m+2n+1)n!^2} \right] k^{2m} \\
 &= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\frac{\sqrt{\pi} \Gamma\left(\frac{2m+3}{2}\right)}{(2m+1)\Gamma(m+1)} \right] k^{2m} \\
 &= \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(2m)! \Gamma\left(\frac{2m+3}{2}\right)}{2^{2m}(2m+1)m!^3} k^{2m}.
 \end{aligned} \tag{19}$$

Let $m \rightarrow n$, this concludes the proof. $\square$

Corollary 3. *I have*

$$K(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I know [5, page 884] that

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-t} t^{z-1} dt, \tag{20}$$

for $\Re(z) > 0$. I substitute (20) in Theorem 4

$$\begin{aligned}
 K(k) &= \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^3} \int_0^{\infty} e^{-t} t^{n+\frac{3}{2}} dt k^{2m} \\
 &= \sqrt{\pi} \int_0^{\infty} e^{-t} t^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^3} (tk^2)^n dt \\
 &= \sqrt{\pi} \int_0^{\infty} e^{-t} t^{\frac{3}{2}} {}_2F_2\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; k^2 t\right) dt
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\pi} \int_0^\infty e^{-t} \sqrt{t} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} (tk^2)^n dt \\
&= \sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} \left(\int_0^\infty e^{-t} t^{n+\frac{3}{2}} dt \right) k^{2n} \\
&= \sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n+\frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n}. \square
\end{aligned}$$

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