Bulletin of Mathematical Sciences and Applications
ISSN: 2278-9634, Vol. 6, pp 40-46
doi:10.18052/www.scipress.com/BMSA.6.40

Online: 2013-11-04

On the Complete Elliptic Integrals and Babylonian Identity I: The $\frac{1}{\pi}$ Formulaes Involving Gamma Functions and Summations

Edigles Guedes[1] and Prof. Dr. K. Raja Rama Gandhi[2]

Number Theorist, Brazil[1]
and
Resource person in Math for Oxford University Press and Professor in Math at BITS-Vizag[2]
edigles.guedes@gmail.com

Abstract. I evaluate the constant $\frac{1}{\pi}$ using the Babylonian identity and complete elliptic integral of first kind. This resulted in two representations in terms of the Euler's gamma functions and summations.

1. Introduction

By means of the complete elliptic integral of the first kind and Babylonian identity, I demonstrated the identities following, among others:

$$\frac{1}{\pi}=\frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi}=4\sqrt{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^{2}\Gamma\left(\frac{3}{4}\right)}.$$

2. Lemmas

Lemma 1. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^{2}} \tag{1}$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^{2}}}. \tag{2}$$

Proof. I know the Babylonian identity [1, page 119]

$$ab=\frac{1}{4}[(a+b)^{2}-(a-b)^{2}]. \tag{3}$$

Make the following algebraic manipulation in (3)

$$ab=\left(\frac{a+b}{2}\right)^{2}\left[1-\left(\frac{a-b}{a+b}\right)^{2}\right],$$

hence,

$$a^{\frac{1}{2}}b^{\frac{1}{2}}=\frac{a+b}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^{2}}\Leftrightarrow\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^{2}},$$

and inverting both members, I have

$$a^{-\frac{1}{2}}b^{-\frac{1}{2}} = \frac{2}{a+b}\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}} \Leftrightarrow \frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}.\square$$

Lemma 2. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2(2k-1)}\left(\frac{a-b}{a+b}\right)^{2k} \tag{4}$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}\left(\frac{a-b}{a+b}\right)^{2k}. \tag{5}$$

Proof. I calculate

$$\sqrt{1-z^2} = -\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2(2k-1)}z^{2k} \tag{6}$$

and

$$\frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}z^{2k} \tag{7}$$

Take $z = \frac{a-b}{a+b}$ in (6) and (7), then replace in (1) and (2) respectively, completing the proof.□

3. THEOREMS

Theorem 1. *I have*

$$K(k) = \frac{\sqrt{2}}{2}\int_0^{\pi}\frac{1}{\sqrt{2-k^2(1+\cos t)}}dt,$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. Putting $\frac{a-b}{a+b} = t$ in (5), I encounter

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}t^{2n}. \tag{8}$$

Multiplying (8) by $\frac{1}{\sqrt{1-k^2t^2}}$ and integrating from 0 at 1 in t, I find

$$\int_0^1\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_0^1\frac{t^{2n}}{\sqrt{1-k^2t^2}}dt \Leftrightarrow$$

$$K(k) = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(2n+1)n!^2}\,{}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; k^2\right). \tag{9}$$

On the one hand, in [2, page 21], I have

$$ {}_2F_1(a,b;c;z)=\frac{2^{1-c}\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^{\pi}\frac{(\sin t)^{2b-1}(1+\cos t)^{c-2b}}{\left(1-\frac{1}{2}z+\frac{1}{2}z\cos t\right)^{a}}dt, \tag{10} $$

for $\Re(c)>\Re(b)>0$. Substituting (10) in (9), I encounter

$$ K(k)=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{3n}(2n+1)n!^2}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^{\pi}\frac{(\sin t)^{2n}(1+\cos t)^{-n+\frac{1}{2}}}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}}dt= $$

$$ =\frac{1}{\sqrt{2}}\int_0^{\pi}\sqrt{\frac{1+\cos t}{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}}\sum_{n=0}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{3n}(2n+1)n!^2\,\Gamma\left(n+\frac{1}{2}\right)}(\sin t)^{2n}(1+\cos t)^{-n}dt= $$

$$ =\frac{1}{2}\int_0^{\pi}\frac{1}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}}dt $$

$$ =\frac{\sqrt{2}}{2}\int_0^{\pi}\frac{1}{\sqrt{2-k^2(1+\cos t)}}dt. \square $$

Theorem 2. *I have*

$$ k'K(k)=K\left(-i\frac{k}{k'}\right). $$

Proof. I leave to the reader. □

Theorem 3. *For* $0<k<1$, *then*

$$ \frac{K(k)}{\sqrt{\pi}}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2\Gamma\left(\frac{2n+3}{2}\right)}{\left(\frac{3}{2}\right)_n}\frac{k^{2n}}{n!^2}, $$

where $K(k)$ *is the complete elliptic integral of first kind.*

Proof. I consider

$$ K(k)=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(2n+1)n!^2}\,{}_2F_1\left(\frac{1}{2},n+\frac{1}{2};n+\frac{3}{2};k^2\right) $$

$$ =\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(2n+1)n!^2}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\left(n+\frac{1}{2}\right)_r}{\left(n+\frac{3}{2}\right)_r r!}k^{2r} $$

$$ =\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)_r\left(\sum_{n=0}^{\infty}\frac{(2n)!\left(n+\frac{1}{2}\right)_r}{2^{2n}(2n+1)n!^2\left(n+\frac{3}{2}\right)_r}\right)\frac{k^{2r}}{r!} $$

$$= \sqrt{\pi}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r \Gamma(r+1)}\frac{k^{2r}}{r!} = \sqrt{\pi}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r}\frac{k^{2r}}{r!^2}.$$

Multiply both sides by $\frac{1}{\sqrt{\pi}}$ and let $r \to n$, so the result follows. □

Corollary 1. *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}.$$

Proof. Let $k = \frac{\sqrt{2}}{2}$ in Theorem 3

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\pi}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2}\left(\frac{1}{2}\right)^r$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)}. \tag{11}$$

On the other hand, in [3], I find

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}. \tag{12}$$

I substitute (12) into (11) and obtain

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}. \square$$

Corollary 2. *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = 4\sqrt{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2\Gamma\left(\frac{3}{4}\right)}.$$

Proof. From Theorem 2 and $k=\frac{\sqrt{2}}{2}$, I find

$$\frac{\sqrt{2}}{2}K\left(\frac{\sqrt{2}}{2}\right)=K\left(-i\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)\Rightarrow K\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}\,K(-i) \tag{13}$$

Using the Theorem (3), I discover that

$$K(-i)=\sqrt{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{1}{2}\right)_r^2\Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r(r!)^2}$$

$$=\sqrt{\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \tag{14}$$

I set (14) in (13)

$$K\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \tag{15}$$

I put (12) into (15), and have

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}=\sqrt{2\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\Rightarrow\frac{1}{\pi}=4\sqrt{2}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2\Gamma\left(\frac{3}{4}\right)}.\square$$

Theorem 4. *For* $0<k<1$, *then*

$$K(k)=\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}(2n+1)n!^3}k^{2n},$$

where $K(k)$ *is the complete elliptic integral of first kind.*

Proof. I put $\frac{a-b}{a+b}=t$ in (5) and encounter

$$\frac{1}{\sqrt{1-t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}t^{2n}. \tag{16}$$

Multiplying (16) by $\frac{1}{\sqrt{1-k^2t^2}}$ and integrating from 0 at 1 in t, I find

$$\int_0^1\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_0^1\frac{t^{2n}}{\sqrt{1-k^2t^2}}dt. \tag{17}$$

Let $t\to kt$ in (16)

$$\frac{1}{\sqrt{1-k^2t^2}}=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}k^{2m}t^{2m}. \tag{18}$$

I put (18) in (17)

$$\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \int_0^1 t^{2n} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} k^{2m} t^{2m}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\int_0^1 t^{2(m+n)}\, dt\right] k^{2m}$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \frac{k^{2m}}{2m+2n+1}$$

$$= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2m+2n+1)n!^2}\right] k^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\frac{\sqrt{\pi}\Gamma\left(\frac{2m+3}{2}\right)}{(2m+1)\Gamma(m+1)}\right] k^{2m}$$

$$= \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(2m)!\Gamma\left(\frac{2m+3}{2}\right)}{2^{2m}(2m+1)m!^3} k^{2m}. \tag{19}$$

Let $m \to n$, this concludes the proof. □

Corollary 3. *I have*

$$K(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n+\frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I know [5, page 884] that

$$\Gamma(z) \overset{\text{def}}{=} \int_0^{\infty} e^{-t} t^{z-1}\, dt, \tag{20}$$

for $\Re(z) > 0$. I substitute (20) in Theorem 4

$$K(k) = \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^3} \int_0^{\infty} e^{-t} t^{n+\frac{3}{2}}\, dt k^{2m}$$

$$= \sqrt{\pi} \int_0^{\infty} e^{-t} t^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^3} (tk^2)^n\, dt$$

$$= \sqrt{\pi} \int_0^{\infty} e^{-t} t^{\frac{3}{2}}\, {}_2F_2\left(\frac{1}{2},\frac{1}{2};1,\frac{3}{2};k^2 t\right) dt$$

$$= \sqrt{\pi} \int_0^{\infty} e^{-t} \sqrt{t} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} (tk^2)^n \, dt$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} \left(\int_0^{\infty} e^{-t} \, t^{n+\frac{3}{2}} dt \right) k^{2n}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n}. \square$$

REFERENCES

[1] Havil, Julian, *Gamma: Exploring the Euler's Constant*, Princeton University Press, 2003.

[2] Slater, Lucy Joan, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.

[3] http://en.wikipedia.org/wiki/Elliptic_integral, available in July 02, 2012.

[4] Armitage, J. V. and Eberlein, W. F., *Elliptic Functions*, Cambridge University Press, 2006.

[5] Gradshteyn, I. S. and Ryzhik, I. M., Table of Integrals, Series and Products, Academic Press, 2000.