

On the Complete Elliptic Integrals and Babylonian Identity I: The $\frac{1}{\pi}$ Formulaes Involving Gamma Functions and Summations

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Abstract. I evaluate the constant $\frac{1}{\pi}$ using the Babylonian identity and complete elliptic integral of first kind. This resulted in two representations in terms of the Euler's gamma functions and summations.

1. Introduction

By means of the complete elliptic integral of the first kind and Babylonian identity, I demonstrated the identities following, among others:

$$\frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = 4\sqrt{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}.$$

2. Lemmas

Lemma 1. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = \frac{1}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2} \quad (1)$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}}. \quad (2)$$

Proof. I know the Babylonian identity [1, page 119]

$$ab = \frac{1}{4} [(a+b)^2 - (a-b)^2]. \quad (3)$$

Make the following algebraic manipulation in (3)

$$ab = \left(\frac{a+b}{2}\right)^2 \left[1 - \left(\frac{a-b}{a+b}\right)^2\right],$$

hence,

$$a^{\frac{1}{2}}b^{\frac{1}{2}} = \frac{a+b}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2} \Leftrightarrow \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = \frac{1}{2} \sqrt{1 - \left(\frac{a-b}{a+b}\right)^2},$$

and inverting both members, I have

$$a^{-\frac{1}{2}}b^{-\frac{1}{2}} = \frac{2}{a+b} \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}} \Leftrightarrow \frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \frac{1}{\sqrt{1 - \left(\frac{a-b}{a+b}\right)^2}}. \square$$

Lemma 2. For a and b any number, then

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 (2k-1)} \left(\frac{a-b}{a+b}\right)^{2k} \tag{4}$$

and

$$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}} = 2 \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} \left(\frac{a-b}{a+b}\right)^{2k}. \tag{5}$$

Proof. I calculate

$$\sqrt{1 - z^2} = - \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 (2k-1)} z^{2k} \tag{6}$$

and

$$\frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2} z^{2k} \tag{7}$$

Take $z = \frac{a-b}{a+b}$ in (6) and (7), then replace in (1) and (2) respectively, completing the proof. \square

3. THEOREMS

Theorem 1. *I have*

$$K(k) = \frac{\sqrt{2}}{2} \int_0^{\pi} \frac{1}{\sqrt{2 - k^2(1 + \cos t)}} dt,$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. Putting $\frac{a-b}{a+b} = t$ in (5), I encounter

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} t^{2n}. \tag{8}$$

Multiplying (8) by $\frac{1}{\sqrt{1-k^2t^2}}$ and integrating from 0 at 1 in t , I find

$$\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} \int_0^1 \frac{t^{2n}}{\sqrt{1-k^2t^2}} dt \Leftrightarrow$$

$$K(k) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (2n+1) n!^2} {}_2F_1 \left(\frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; k^2 \right). \tag{9}$$

On the one hand, in [2, page 21], I have

$${}_2F_1(a, b; c; z) = \frac{2^{1-c}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\pi \frac{(\sin t)^{2b-1}(1+\cos t)^{c-2b}}{\left(1-\frac{1}{2}z+\frac{1}{2}z \cos t\right)^a} dt, \quad (10)$$

for $\Re(c) > \Re(b) > 0$. Substituting (10) in (9), I encounter

$$\begin{aligned} K(k) &= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{3n}(2n+1)n!^2} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \frac{(\sin t)^{2n}(1+\cos t)^{-n+\frac{1}{2}}}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2 \cos t}} dt = \\ &= \frac{1}{\sqrt{2}} \int_0^\pi \sqrt{\frac{1+\cos t}{1-\frac{1}{2}k^2+\frac{1}{2}k^2 \cos t}} \sum_{n=0}^{\infty} \frac{(2n)! \Gamma\left(n+\frac{3}{2}\right)}{2^{3n}(2n+1)n!^2 \Gamma\left(n+\frac{1}{2}\right)} (\sin t)^{2n}(1+\cos t)^{-n} dt = \\ &= \frac{1}{2} \int_0^\pi \frac{1}{\sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2 \cos t}} dt \\ &= \frac{\sqrt{2}}{2} \int_0^\pi \frac{1}{\sqrt{2-k^2(1+\cos t)}} dt. \square \end{aligned}$$

Theorem 2. I have

$$k'K(k) = K\left(-i\frac{k}{k'}\right).$$

Proof. I leave to the reader. \square

Theorem 3. For $0 < k < 1$, then

$$\frac{K(k)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(\frac{2n+3}{2}\right) k^{2n}}{\left(\frac{3}{2}\right)_n n!^2},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I consider

$$\begin{aligned} K(k) &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; k^2\right) \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \left(n+\frac{1}{2}\right)_r}{\left(n+\frac{3}{2}\right)_r r!} k^{2r} \\ &= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)_r \left(\sum_{n=0}^{\infty} \frac{(2n)! \left(n+\frac{1}{2}\right)_r}{2^{2n}(2n+1)n!^2 \left(n+\frac{3}{2}\right)_r}\right) \frac{k^{2r}}{r!} \end{aligned}$$

$$= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right) k^{2r}}{\left(\frac{3}{2}\right)_r \Gamma(r+1) r!} = \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right) k^{2r}}{\left(\frac{3}{2}\right)_r r!^2}.$$

Multiply both sides by $\frac{1}{\sqrt{\pi}}$ and let $r \rightarrow n$, so the result follows. \square

Corollary 1. *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}.$$

Proof. Let $k = \frac{\sqrt{2}}{2}$ in Theorem 3

$$\begin{aligned} K\left(\frac{\sqrt{2}}{2}\right) &= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2} \left(\frac{1}{2}\right)^r \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)}. \end{aligned} \tag{11}$$

On the other hand, in [3], I find

$$K\left(\frac{\sqrt{2}}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}. \tag{12}$$

I substitute (12) into (11) and obtain

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}. \square$$

Corollary 2. *I have*

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

and

$$\frac{1}{\pi} = 4\sqrt{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}.$$

Proof. From Theorem 2 and $k = \frac{\sqrt{2}}{2}$, I find

$$\frac{\sqrt{2}}{2} K\left(\frac{\sqrt{2}}{2}\right) = K\left(-i \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) \Rightarrow K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} K(-i) \quad (13)$$

Using the Theorem (3), I discover that

$$\begin{aligned} K(-i) &= \sqrt{\pi} \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{1}{2}\right)_r^2 \Gamma\left(\frac{2r+3}{2}\right)}{\left(\frac{3}{2}\right)_r (r!)^2} \\ &= \sqrt{\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \end{aligned} \quad (14)$$

I set (14) in (13)

$$K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \quad (15)$$

I put (12) into (15), and have

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{\pi}} = \sqrt{2\pi} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \Rightarrow \frac{1}{\pi} = 4\sqrt{2} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{3}{4}\right)}. \quad \square$$

Theorem 4. For $0 < k < 1$, then

$$K(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2n)! \Gamma\left(n + \frac{3}{2}\right)}{2^{2n} (2n+1) n!^3} k^{2n},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I put $\frac{a-b}{a+b} = t$ in (5) and encounter

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} t^{2n}. \quad (16)$$

Multiplying (16) by $\frac{1}{\sqrt{1-k^2 t^2}}$ and integrating from 0 at 1 in t , I find

$$\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2 t^2)}} dt = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^2} \int_0^1 \frac{t^{2n}}{\sqrt{1-k^2 t^2}} dt. \quad (17)$$

Let $t \rightarrow kt$ in (16)

$$\frac{1}{\sqrt{1-k^2 t^2}} = \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m} m!^2} k^{2m} t^{2m}. \quad (18)$$

I put (18) in (17)

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \int_0^1 t^{2n} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} k^{2m} t^{2m} dt \\
 &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\int_0^1 t^{2(m+n)} dt \right] k^{2m} \\
 &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \frac{k^{2m}}{2m+2n+1} \\
 &= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2m+2n+1)n!^2} \right] k^{2m} \\
 &= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}m!^2} \left[\frac{\sqrt{\pi} \Gamma\left(\frac{2m+3}{2}\right)}{(2m+1)\Gamma(m+1)} \right] k^{2m} \\
 &= \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(2m)! \Gamma\left(\frac{2m+3}{2}\right)}{2^{2m}(2m+1)m!^3} k^{2m}. \tag{19}
 \end{aligned}$$

Let $m \rightarrow n$, this concludes the proof. \square

Corollary 3. *I have*

$$K(k) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n},$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. I know [5, page 884] that

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-t} t^{z-1} dt, \tag{20}$$

for $\Re(z) > 0$. I substitute (20) in Theorem 4

$$\begin{aligned}
 K(k) &= \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^3} \int_0^{\infty} e^{-t} t^{n+\frac{3}{2}} dt k^{2m} \\
 &= \sqrt{\pi} \int_0^{\infty} e^{-t} t^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^3} (tk^2)^n dt \\
 &= \sqrt{\pi} \int_0^{\infty} e^{-t} t^{\frac{3}{2}} {}_2F_2\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; k^2t\right) dt
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\pi} \int_0^\infty e^{-t} \sqrt{t} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} (tk^2)^n dt \\
&= \sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n \left(\frac{3}{2}\right)_n n!} \left(\int_0^\infty e^{-t} t^{n+\frac{3}{2}} dt \right) k^{2n} \\
&= \sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n^2 \Gamma\left(n + \frac{3}{2}\right)}{(1)_n \left(\frac{3}{2}\right)_n n!} k^{2n}. \square
\end{aligned}$$

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