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On an Infinite Product for the Ratio of Consecutive Prime Numbers

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ABSTRACT. The main objective of this paper is to develop an infinite product formula for the ratio of consecutive prime numbers, using Jacobi elliptic functions.

1. INTRODUCTION

The Rosser's theorem [2] states that p_n is larger than $n\log n$. This can be improved by the following pair of bounds:

$$\log n+\log\log n-1<\frac{p_n}{n}<\log n+\log\log n, \tag{1}$$

for $n>6$.

2. THEOREM

THEOREM 1. *For* $n>9$, *we have*

$$\begin{aligned}\frac{p_n}{p_{n+1}}+\theta_n &= \frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}}\times\frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}}\times\frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}}\times\frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}}\\
&\times\frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}}\times\frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}}\times\frac{9988}{9987}\times\frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}}\times\frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}}\\
&\times\frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}}\times\frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}}\times\frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}}\times\frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}}\times\frac{1}{1+e^{-539\pi/256}}\\
&\times e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4,\end{aligned}$$

where

$$\theta_n<\ 0.153$$

and p_n *is the n-th prime number.*

Proof. Firstly, we consider the sequence of prime numbers

$$2<3<5<7<11<13<17<19<\cdots p_{n-2}<p_{n-1}<p_n<p_{n+1}. \tag{2}$$

Second, we note that

$$0<\frac{2}{3}<1{,}0<\frac{3}{5}<1{,}0<\frac{5}{7}<1{,}0<\frac{7}{11}<1{,}0<\frac{11}{13}<1{,}0<\frac{13}{17}<1{,}0<\frac{17}{19}<1,\ldots,$$
$$0<\frac{p_{n-2}}{p_{n-1}}<1{,}0<\frac{p_{n-1}}{p_n}<1{,}0<\frac{p_n}{p_{n+1}}<1. \tag{3}$$

Then, we define that

$$k:=k_{n,n+1}=\frac{p_n}{p_{n+1}}, \tag{4}$$

where $k_{n,n+1}$ is the k modulus.

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \tag{5}$$

where τ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

On the other hand, in [2, p. 85], we have

$$\theta_2 = 2q^{1/4} G \prod_{l=1}^{\infty} (1+q^{2l})^2 \tag{6}$$

and

$$\theta_3 = G \prod_{l=1}^{\infty} (1+q^{2l-1})^2. \tag{7}$$

So, we obtain

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{2q^{1/4} G \prod_{l=1}^{\infty}(1+q^{2l})^2}{G \prod_{l=1}^{\infty}(1+q^{2l-1})^2} = 2q^{1/4} \prod_{l=1}^{\infty} \left(\frac{1+q^{2l}}{1+q^{2l-1}}\right)^2, \tag{8}$$

multiplying by $k^{1/2}$ in both members of (8), we have

$$k = 4q^{1/2} \prod_{l=1}^{\infty} \left(\frac{1+q^{2l}}{1+q^{2l-1}}\right)^4, \tag{9}$$

In [3], we encounter

$$q(k) = e^{\pi i \tau} = e^{-\pi K\prime(k)/K(k)} = e^{-\pi K\left(\sqrt{1-k^2}\right)/K(k)}. \tag{10}$$

We knew that [4] $K(k)$ can be expressed as a power series

$$K(k) = \frac{\pi}{2} \sum_{l=0}^{\infty} \left[\frac{(2l)!}{2^{2l}(l!)^2}\right]^2 k^{2l} = \frac{\pi}{2} \sum_{l=0}^{\infty} \left[\frac{(2l-1)!!}{(2l)!!}\right]^2 k^{2l}. \tag{11}$$

It can be expressed by asymptotic expansion

$$K(k) \approx \frac{\pi}{2} + \frac{\pi}{8}\frac{k^2}{1-k^2} - \frac{\pi}{16}\frac{k^4}{1-k^2}. \tag{12}$$

If we substitute (12) in (10), then we find

$$q(k) \approx e^{-\pi\left(\frac{\pi}{2}+\frac{\pi}{8}\frac{1-k^2}{k^2}-\frac{\pi}{16}\frac{\left(1-k^2\right)^2}{k^2}\right)\Big/\left(\frac{\pi}{2}+\frac{\pi}{8}\frac{k^2}{1-k^2}-\frac{\pi}{16}\frac{k^4}{1-k^2}\right)} = e^{\pi\frac{k^6-9k^4+7k^2+1}{k^2(k^4+6k^2-8)}}. \tag{13}$$

We calculate the asymptotic expansion of $\frac{k^6-9k^4+7k^2+1}{k^2(k^4+6k^2-8)}$, that is,

$$\frac{k^6-9k^4+7k^2+1}{k^2(k^4+6k^2-8)} = -\frac{1}{8k^2} - \frac{31}{32} + \frac{49k^2}{128} + \frac{21k^4}{512} + O(k^5) = -\left[\frac{1}{8k^2} + \frac{31}{32} - \frac{49k^2}{128} - \frac{21k^4}{512} - O(k^5)\right] \tag{14}$$

We take (14) in (13)

$$q(k) \approx e^{-\pi\left(\frac{1}{8k^2}+\frac{31}{32}-\frac{49k^2}{128}-\frac{21k^4}{512}\right)}. \tag{15}$$

Substituting (15) in (9), it follows that

$$k \approx 4e^{-\frac{\pi}{2}\left(\frac{1}{8k^2}+\frac{31}{32}-\frac{49k^2}{128}-\frac{21k^4}{512}\right)} \prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{1}{8k^2}+\frac{31}{32}-\frac{49k^2}{128}-\frac{21k^4}{512}\right)}}{1+e^{-(2l-1)\pi\left(\frac{1}{8k^2}+\frac{31}{32}-\frac{49k^2}{128}-\frac{21k^4}{512}\right)}}\right]^4. \quad (16)$$

We put (4) in (16)

$$\frac{p_n}{p_{n+1}} \approx 4e^{-\frac{\pi}{2}\left(\frac{p_{n+1}^2}{8p_n^2}+\frac{31}{32}-\frac{49p_n^2}{128p_{n+1}^2}-\frac{21p_n^4}{512p_{n+1}^4}\right)} \prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{p_{n+1}^2}{8p_n^2}+\frac{31}{32}-\frac{49p_n^2}{128p_{n+1}^2}-\frac{21p_n^4}{512p_{n+1}^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{p_{n+1}^2}{8p_n^2}+\frac{31}{32}-\frac{49p_n^2}{128p_{n+1}^2}-\frac{21p_n^4}{512p_{n+1}^4}\right)}}\right]^4. \quad (17)$$

Using (1), we discover that

$$\frac{(n+1)^2[\log(n+1)+\log\log(n+1)]^2}{8n^2(\log n+\log\log n-1)^2}+\frac{31}{32}-\frac{49n^2(\log n+\log\log n)^2}{128(n+1)^2[\log(n+1)+\log\log(n+1)-1]^4}$$
$$-\frac{21n^4(\log n+\log\log n)^4}{512(n+1)^4[\log(n+1)+\log\log(n+1)-1]^4}$$
$$<\frac{p_{n+1}^2}{8p_n^2}+\frac{31}{32}-\frac{49p_n^2}{128p_{n+1}^2}-\frac{21p_n^4}{512p_{n+1}^4}$$
$$<\frac{(n+1)^2[\log(n+1)+\log\log(n+1)]^2}{8n^2(\log n+\log\log n-1)^2}+\frac{31}{32}-\frac{49n^2(\log n+\log\log n-1)^2}{128(n+1)^2[\log(n+1)+\log\log(n+1)]^2}$$
$$-\frac{21n^4(\log n+\log\log n-1)^4}{512(n+1)^4[\log(n+1)+\log\log(n+1)]^4},$$

whereas factors involving logarithms tend to 1, we have the approximation

$$\frac{p_{n+1}^2}{8p_n^2}+\frac{31}{32}-\frac{49p_n^2}{128p_{n+1}^2}-\frac{21p_n^4}{512p_{n+1}^4}\sim\frac{(n+1)^2}{8n^2}+\frac{31}{32}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}$$
$$=\frac{n^2+2n+1}{8n^2}+\frac{31}{32}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}$$
$$=\frac{1}{8}+\frac{31}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}$$
$$=\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}. \quad (18)$$

We substitute (18) in (17)

$$\frac{p_n}{p_{n+1}} \sim 4e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)} \prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4.$$

Ergo, we can consider that

$$\frac{p_n}{p_{n+1}} = 4G_n e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)} \prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4. \quad (19)$$

In other words,

$$G_n = \frac{p_n}{4p_{n+1}e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4}.$$

On the other hand, we suppose that

$$G_n = \alpha_n G, \tag{20}$$

where

$$G = \lim_{n\to\infty} \frac{p_n}{4p_{n+1}e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4}. \tag{21}$$

By Rosser's theorem (1), we put

$$\lim_{n\to\infty}\left\{\frac{n(\log n+\log\log n-1)}{4(n+1)[\log(n+1)+\log\log(n+1)]}\times\frac{1}{e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4}\right\} \le G \le$$

$$\lim_{n\to\infty}\left\{\frac{n(\log n+\log\log n)}{4(n+1)[\log(n+1)+\log\log(n+1)-1]}\times\frac{1}{e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4}\right\}.$$

We calculate the limit in both members for obtain of

$$\lim_{n\to\infty}\left\{\frac{n(\log n+\log\log n-1)}{4(n+1)[\log(n+1)+\log\log(n+1)]}\times\frac{1}{e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4}\right\}$$

$$\approx \frac{1}{4}\times\frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}}\times\frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}}\times\frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}}\times\frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}}$$
$$\times\frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}}\times\frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}}\times\frac{9989}{9987}\times\frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}}\times\frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}}$$
$$\times\frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}}\times\frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}}\times\frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}}\times\frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}}\times\frac{1}{1+e^{-539\pi/256}}$$

and

$$\lim_{n\to\infty}\left\{\frac{n(\log n+\log\log n)}{4(n+1)[\log(n+1)+\log\log(n+1)-1]}\times\frac{1}{e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}\prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4}\right\}$$

$$\approx \frac{1}{4}\times\frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}}\times\frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}}\times\frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}}\times\frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}}$$
$$\times\frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}}\times\frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}}\times\frac{9988}{9987}\times\frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}}\times\frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}}$$
$$\times\frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}}\times\frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}}\times\frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}}\times\frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}}\times\frac{1}{1+e^{-539\pi/256}},$$

hence, we can assume that

$$G \approx \frac{1}{4} \times \frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}} \times \frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}} \times \frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}}$$
$$\times \frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}} \times \frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}} \times \frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}} \times \frac{9988}{9987}$$
$$\times \frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}} \times \frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}} \times \frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}} \times \frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}}$$
$$\times \frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}} \times \frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}} \times \frac{1}{1+e^{-539\pi/256}}. \tag{22}$$

From (22), (21), (20) and (19), it follows that

$$\frac{p_n}{p_{n+1}} =$$
$$\alpha_n \times \frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}} \times \frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}} \times \frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}} \times \frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}}$$
$$\times \frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}} \times \frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}} \times \frac{9988}{9987} \times \frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}} \times \frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}}$$
$$\times \frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}} \times \frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}} \times \frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}} \times \frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}} \times \frac{1}{1+e^{-539\pi/256}}$$
$$\times e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)} \prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4, \tag{23}$$

in other words, we conclude that

$$\frac{p_n}{p_{n+1}} + \theta_n = \frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}} \times \frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}} \times \frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}} \times \frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}}$$
$$\times \frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}} \times \frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}} \times \frac{9988}{9987} \times \frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}} \times \frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}}$$
$$\times \frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}} \times \frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}} \times \frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}} \times \frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}} \times \frac{1}{1+e^{-539\pi/256}}$$
$$\times e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)} \prod_{l=1}^{\infty}\left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4,$$

where

$$\theta_n < 0.153$$

for $n > 9$, see table 1. □

Table 1. In this table, we have: first column: n; second column: p_n; third column: p_{n+1}; fourth column: $\frac{p_n}{p_{n+1}}$; fifth column: the fractional part of $\frac{p_n}{p_{n+1}}$; sixth column: ψ_n^*; seventh column: $\psi_n^* - \frac{p_n}{p_{n+1}}$.

10	29	31	$\frac{29}{31}$	0.935483870968	0.98897721869	0.0534933477223
11	31	37	$\frac{31}{37}$	0.837837837838	0.990490778545	0.152652940706
12	37	41	$\frac{37}{41}$	0.90243902439	0.991680519818	0.089241495428
13	41	43	$\frac{41}{43}$	0.953488372093	0.992635412886	0.0391470407925
14	43	47	$\frac{43}{47}$	0.914893617021	0.993415470002	0.0785218529805
15	47	53	$\frac{47}{53}$	0.88679245283	0.994062430525	0.107269977694
16	53	59	$\frac{53}{59}$	0.898305084746	0.994606098095	0.0963010133487
17	59	61	$\frac{59}{61}$	0.967213114754	0.995068244543	0.027855129789
18	61	67	$\frac{61}{67}$	0.910447761194	0.995465095077	0.0850173338824
19	67	71	$\frac{67}{71}$	0.943661971831	0.995808956326	0.0521469844949
20	71	73	$\frac{71}{73}$	0.972602739726	0.996109310172	0.0235065704456
21	73	79	$\frac{73}{79}$	0.924050632912	0.996373565309	0.0723229323973
22	79	83	$\frac{79}{83}$	0.951807228916	0.996607584107	0.0448003551913
23	83	89	$\frac{83}{89}$	0.932584269663	0.996816058669	0.0642317890063
24	89	97	$\frac{89}{97}$	0.917525773196	0.997002783666	0.0794770104697
25	97	101	$\frac{97}{101}$	0.960396039604	0.997170857231	0.0367748176272
26	101	103	$\frac{101}{103}$	0.980582524272	0.997322830911	0.0167403066389
27	103	107	$\frac{103}{107}$	0.96261682243	0.997460822969	0.0348440005384
28	107	109	$\frac{107}{109}$	0.981651376147	0.997586604994	0.0159352288466
29	109	113	$\frac{109}{113}$	0.964601769912	0.9977016688	0.0330998988881
30	113	127	$\frac{113}{127}$	0.889763779528	0.997807278603	0.108043499075

Table 2. In this table, we have: first column: n; second column: p_n; third column: p_{n+1}; fourth column: $\frac{p_n}{p_{n+1}}$; fifth column: the fractional part of $\frac{p_n}{p_{n+1}}$; sixth column: ψ_n^*; seventh column: $\psi_n^* - \frac{p_n}{p_{n+1}}$.

10	29	31	$\frac{29}{31}$	0.93548387	0.98897711	0.05349324
100	541	547	$\frac{541}{547}$	0.98903107	0.99957136	0.010540289
1000	7919	7927	$\frac{7919}{7927}$	0.99899079	1.0000534	0.0010626618
10000	104729	104743	$\frac{104729}{104743}$	0.99986633	1.0000954	0.00022909612
100000	1299709	1299721	$\frac{1299709}{1299721}$	0.99999076	1.000099	0.00010880395

Remark:

$$\psi_n^* = \frac{e^{539\pi/1024}}{1+e^{-336875\pi/16}} \times \frac{1+e^{-10779461\pi/512}}{1+e^{-5389461\pi/256}} \times \frac{1+e^{-10778383\pi/512}}{1+e^{-2694461\pi/128}} \times \frac{1+e^{-10777305\pi/512}}{1+e^{-5388383\pi/256}}$$

$$\times \frac{1+e^{-10776227\pi/512}}{1+e^{-1346961\pi/64}} \times \frac{1+e^{-10775149\pi/512}}{1+e^{-5387305\pi/256}} \times \frac{9988}{9987} \times \frac{1+e^{-5929\pi/512}}{1+e^{-3773\pi/256}} \times \frac{1+e^{-4851\pi/512}}{1+e^{-1617\pi/128}}$$

$$\times \frac{1+e^{-3773\pi/512}}{1+e^{-2695\pi/256}} \times \frac{1+e^{-2695\pi/512}}{1+e^{-539\pi/64}} \times \frac{1+e^{-1617\pi/512}}{1+e^{-1617\pi/256}} \times \frac{1+e^{-539\pi/512}}{1+e^{-539\pi/128}} \times \frac{1}{1+e^{-539\pi/256}}$$

$$\times e^{-\frac{\pi}{2}\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)} \prod_{l=1}^{\infty} \left[\frac{1+e^{-2l\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}{1+e^{-(2l-1)\pi\left(\frac{35}{32}+\frac{1}{4n}+\frac{1}{8n^2}-\frac{49n^2}{128(n+1)^2}-\frac{21n^4}{512(n+1)^4}\right)}}\right]^4.$$

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