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A PROOF OF THE RIEMANN HYPOTHESIS

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Abstract
In this paper, we present the Riemann problem and define the real primes. It allows to generalize the Riemann hypothesis to the reals. A calculus of integral solves the problem. We generalize the proof to the integers.

The Riemann hypothesis

The Riemann conjecture is a conjecture which has been formulated in 1859 by Bernhard Riemann in the subjet of the Riemann funtion zeta or ζ . It is called the zeta Riemann function.

This function is defined as follows

$$\zeta(s)=\sum_{n=1}^{n=\infty}(\frac{1}{n^s})=1+\frac{1}{2^s}+\frac{1}{3^s}+...$$

The first result is the divergence of the harmonic serie

$$\zeta(1)=\sum_{n=1}^{n=\infty}(\frac{1}{n})=1+\frac{1}{2}+\frac{1}{3}+...$$

It has been proved in the middle age by Nicole Oresme.
In the XVIII century, Leonard Euler has discovered the main proprieties of the ζ function.

In the 1730's he conjectured after numerical calculus the following equality, which is often called the Basel problem.

$$\zeta(2)=\sum_{n=1}^{n=\infty}(\frac{1}{n^2})=1+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

Euler proved it in 1748 and introduced the ζ function. He calculated its value for the positive even numbers.

$$\zeta(2k)=\sum_{n=1}^{n=\infty}(\frac{1}{n^{2k}})=1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+...=\frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}$$

Where B_{2k} are the Bernoulli numbers.

Thereafter, he proved in 1744 the Euler idendity where prime numbers are related to the ζ function.

$$\zeta(s)=\sum_{n=1}^{n=\infty}(\frac{1}{n^s})=1+\frac{1}{2^s}+\frac{1}{3^s}+...=\prod_{primes}\frac{1}{1-p^{-s}}$$

Consequently he deduced the divergence of the serie of the inverse of primes.

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With Bernard Riemann, s can be complex number. Riemann proved the following formula

$$\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{\frac{-(1-s)}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Where

$$\Gamma(s)=\int_0^{\infty}t^{s-1}e^{-t}dt$$

This formula demonstrates that this equation does not change if we replace s by 1-s.

Thus it is symmetric $| \; s=\frac{1}{2}$

Riemann demonstrates that the only zeros in the $R(s)<0$ are the trivial zero[illegible] negativ[illegible]ven numbers and that there is no zero in the $R(s)>1$.

The other zeros are the non trivial zeros. They are in the critical zone $0\leq R($[illegible]$)\leq 1$.

Riemann conjectured theu are all in the critical line $R(s)=\frac{1}{2}$.
This conjecture is called the Riemann hypothesis.
They calculated numerically one billion zeros of the Rieman[illegible] t[illegible]y are [illegible] located in the critical line.

Definition

A real number is compound if it can be written as $\prod_j p_j$[illegible]ere p_j are primes and n_j are rationals.
This decomposition in prime factors is unique[illegible]ime real [illegible]mber can be written only as p=p.1 .

Thus we define other real prime numbers lik[illegible], e, [illegible](2) . Thus $\sqrt[q]{p}=p^{\frac{1}{q}}$ is compound. Also $\sqrt[q]{p}+1=p^{\frac{1}{q}}$ is prime when p is p[illegible]me an[illegible] we hav[illegible] $\sqrt[2^i]{p}-1=(p-1)(\sqrt[2^i]{p}+1)^{-1}(\sqrt[2^{i-1}]{p}+1)^{-1}...(\sqrt{p}+1)^{-1}$ compound for p prime, for exa[illegible]ple.

The approach of the Ri[illegible]ma[illegible] hypothes[illegible]s

The Rieman hyp[illegible]hesis states that [illegible]e non trivial zeros of the Riemann zeta function $\zeta(z)=\sum_{t=1}^{\infty}\frac{1}{t^z}$ lie on the critical line $\frac{1}{}$[illegible]$+iy$.

For t in[illegible]ger, [illegible]as proved that $\zeta(z)=\sum_{t=1}^{\infty}\frac{1}{t^z}=\prod_{primes}\frac{1}{1-p^{-z}}=\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha z}})$ it is the Euler identity. For t real, it is still true and it becomes $\prod_{primes}(\sum_{\alpha\in Q}\frac{1}{p^{\alpha z}})=\int_1^{\infty}\frac{dt}{t^z}=\frac{1}{1-z}[t^{1-z}]_1^{\infty}$ But there are the tivial zeros : we have $\zeta(-2k)=0,\forall k\in N$ and $[t^{1+2k}]_1^{\infty}=0$ but if $[t^z]^{\infty}$ s the limit in the infinity,

$\lim_{t\to\infty}(t^{1+2k})=1,\forall k\in N$ and

$$\lim_{i\to\infty}(t^{1-\frac{1}{2}-iy})=\lim_{i\to\infty}(t^{\frac{1}{2}-iy})=\lim_{i\to\infty}(t^{\frac{1}{2}+iy})=\frac{1}{2}(\lim_{i\to\infty}(t^{\frac{1}{2}-iy})+\lim_{i\to\infty}(t^{\frac{1}{2}+iy}))=a$$

$$\Rightarrow\lim_{i\to\infty}(t^{2(\frac{1}{2}+iy)}-2at^{\frac{1}{2}+iy}+t)=\lim_{i\to\infty}(t^{2(\frac{1}{2}+iy)}-2at^{\frac{1}{2}+iy}+1)=0\Rightarrow\lim_{i\to\infty}(t^{\frac{1}{2}+iy})=a=a+\sqrt{a^2-1}=1$$

it means that $\zeta(\frac{1}{2}+iy)=\left[\frac{1}{1-\frac{1}{2}-iy}t^{1-\frac{1}{2}-iy}\right]_1^{\infty}=0$

Let now $\zeta(x+iy)=0=\left[\frac{t^{1-x-iy}}{1-x-iy}\right]_1^{\infty}=\left[\frac{t^{\frac{1}{2}-iy}t^{\frac{1}{2}-x}}{1-x-iy}\right]_1^{\infty}=\left[\frac{t^{\frac{1}{2}-x}}{1-x-iy}\right]_1^{\infty}=0\Rightarrow x=\frac{1}{2}$

We have proved that the non trivial zeros of the Riemann function for the reals lie in the critical line ! So the hypothesis is proved for the real numbers. The Riemann hypothesis is important because it gives information about the zeros of the Riemann function and the distribution of those zeros are related to real primes !

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The generalization to the integers

We have

$$\int_1^{\infty}\frac{dt}{t^z}=\frac{1}{1-z}\left[t^{1-z}\right]_1^{\infty}=\sum_{t=1}^{\infty}\frac{1}{t^z}+\sum_{\alpha\in Q^-}^{\infty}\frac{1}{t^{\alpha z}}+\sum_{\alpha\in Q^+\setminus N}^{\infty}\frac{1}{t^{\alpha z}}$$

$$=\sum_{t=1}^{\infty}\frac{1}{t^z}.B=\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha z}}).B=\sum_{t=1}^{\infty}\frac{1}{t^z}+A$$

Let now

$$\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha(x+iy)}})=0$$

We have : If B is finite, the zeta function for the reals will be equal to zero. Else Let ζ_1 (z) the Riemann function for the reals and ζ (z) the Riemann function fot the integers, if B = +∞ and if A is finite and

$A>1$

$(A+\zeta(z))^2=A^2=A^2+\zeta(z)^2+2A\zeta(z)=A^2+2A\zeta(z)\Rightarrow A\zeta(z)=0$

$(A^n\zeta(z)+A)^2=A^2=A^2+(A^n\zeta(z))^2+2A^{n+1}\zeta(z)\Rightarrow A^{n+1}\zeta(z)=0,\forall n$

$\exists\alpha\,|\,B\quad A^{\alpha}<A^n\Rightarrow 0<\zeta_1(z)=B\zeta(z)<A^n\zeta(z)=0\Rightarrow\zeta_1(z)=0\Rightarrow 1\geq A$

And if

$1\geq A>0$

$(A^{-1}+\zeta(z))^2=A^{-2}=A^{-2}+\zeta(z)^2+2A^{-1}\zeta(z)=A^{-2}+2A^{-1}\zeta(z)\Rightarrow A^{-1}\zeta(z)=0$

$(A^{-n}\zeta(z)+A^{-1})^2=A^{-2}=A^{-2}+(A^{-n}\zeta(z))^2+2A^{-n-1}\zeta(z)\Rightarrow A^{-n-1}\zeta(z)=0,\forall n$

$\exists\alpha\,|\,B\quad A^{-\alpha}<A^{-n}\Rightarrow 0<\zeta_1(z)=B\zeta(z)<A^{-n}\zeta(z)=0\Rightarrow\zeta_1(z)=0$

$0\geq A\geq -1$

$(A^{-1}+\zeta(z))^2=A^{-2}=A^{-2}+\zeta(z)^2+2A^{-1}\zeta(z)=A^{-2}+2A^{-1}\zeta(z)\Rightarrow A^{-1}\zeta(z)=0$

$(A^{-2n-1}\zeta(z)+A^{-2})^2=A^{-4}=A^{-4}+(A^{-2n-1}\zeta(z))^2+2A^{-2n-3}\zeta(z)\Rightarrow A^{-2n-3}\zeta(z)=0,\forall n$

$\exists\alpha\,|\,B\quad -A^{-\alpha}<-A^{-2n-1}\Rightarrow 0<\zeta_1(z)=B\zeta(z)<-A^{-2n-1}\zeta(z)=0\Rightarrow\zeta_1(z)=0$

And if

$$A<-1$$

$$(A+\zeta(z))^2=A^2=A^2+\zeta(z)^2+2A\zeta(z)=A^2+2A\zeta(z)\Rightarrow A\zeta(z)=0$$

$$(A^{2n+1}\zeta(z)+A^2)^2=A^4=A^4+(A^{2n+1}\zeta(z))^2+2A^{2n+3}\zeta(z)\Rightarrow A^{2n+3}\zeta(z)=0,\forall n$$

$$\exists\alpha\,|\,B\quad -A^{\alpha}<-A^{2n+1}\Rightarrow 0<\zeta_1(z)=B\zeta(z)<A^{2n+1}\zeta(z)=0\Rightarrow\zeta_1(z)=0\Rightarrow A\geq-1$$

If now B = −∞ and if A is finite and

$$A>1$$

$$(A+\zeta(z))^2=A^2=A^2+\zeta(z)^2+2A\zeta(z)=A^2+2A\zeta(z)\Rightarrow A\zeta(z)=0$$

$$(A^n\zeta(z)+A)^2=A^2=A^2+(A^n\zeta(z))^2+2A^{n+1}\zeta(z)\Rightarrow A^{n+1}\zeta(z)=0,\forall n$$

$$\exists\alpha\,|\,B\quad -A^{\alpha}>-A^{n}\Rightarrow 0>\zeta_1(z)=B\zeta(z)>-A^{n}\zeta(z)=0\Rightarrow\zeta_1(z)=0\Rightarrow 1\geq A$$

And if

$$1\geq A\geq 0$$

$$(A^{-1}+\zeta(z))^2=A^{-2}=A^{-2}+\zeta(z)^2+2A^{-1}\zeta(z)=A^{-2}+2A^{-1}\zeta(z)\Rightarrow A^{-1}\zeta(z)=0$$

$$(A^{-n}\zeta(z)+A^{-1})^2=A^{-2}=A^{-2}+(A^{-n}\zeta(z))^2+2A^{-n-1}\zeta(z)\Rightarrow A^{-n-1}\zeta(z)=0,\forall n$$

$$\exists\alpha\,|\,B\quad -A^{-\alpha}>-A^{-n}\Rightarrow 0>\zeta_1(z)=B\zeta(z)>-A^{-n}\zeta(z)=0\Rightarrow\zeta_1(z)=0$$

And if

$$0\geq A\geq -1$$

$$(A^{-1}+\zeta(z))^2=A^{-2}=A^{-2}+\zeta(z)^2+2A^{-1}\zeta(z)=A^{-2}+2A^{-1}\zeta(z)\Rightarrow A^{-1}\zeta(z)=0$$

$$(A^{-2n-1}\zeta(z)+A^{-2})^2=A^{-4}=A^{-4}+(A^{-2n-1}\zeta(z))^2+2A^{-2n-3}\zeta(z)\Rightarrow A^{-2n-3}\zeta(z)=0,\forall n$$

$$\exists\alpha\,|\,B\quad A^{-\alpha}>A^{-2n-1}\Rightarrow 0>\zeta_1(z)=B\zeta(z)>A^{-2n-1}\zeta(z)=0\Rightarrow\zeta_1(z)=0$$

And if

$$A<-1$$

$$(A+\zeta(z))^2=A^2=A^2+\zeta(z)^2+2A\zeta(z)=A^2+2A\zeta(z)\Rightarrow A\zeta(z)=0$$

$$(A^{2n+1}\zeta(z)+A^2)^2=A^4=A^4+(A^{2n+1}\zeta(z))^2+2A^{2n+3}\zeta(z)\Rightarrow A^{2n+3}\zeta(z)=0,\forall n$$

$$\exists\alpha\,|\,B\quad A^{\alpha}>A^{2n+1}\Rightarrow 0>\zeta_1(z)=B\zeta(z)>A^{2n+1}\zeta(z)=0\Rightarrow\zeta_1(z)=0\Rightarrow A\geq-1$$

And if $A=\pm\infty$ and we have B $A^{\alpha+1}$ hence

$$A^{\alpha}\zeta(z)^2=\frac{A^{\alpha+2}}{(A^{\alpha+1}-1)^2}=0$$

$$A^{\alpha}\zeta(z)=1$$

And

$$(A^{\alpha}\zeta(z)+A^{\alpha}\zeta(z)^{2})^{2}=A^{2\alpha}\zeta(z)^{2}=A^{2\alpha}\zeta(z)^{2}+2A^{2\alpha}\zeta(z)^{3}\Rightarrow A^{2\alpha}\zeta(z)^{3}=0$$

If $A^{n\alpha}\zeta(z)^{n+1}=0$ then

$$(A^{n\alpha}\zeta(z)^{n+1}+A^{\alpha}\zeta(z))=A^{2\alpha}\zeta(z)^{2}=A^{2\alpha}\zeta(z)^{2}+2A^{(n+1)\alpha}\zeta(z)^{n+2}\Rightarrow A^{(n+1)\alpha}\zeta(z)^{n+2}=0,\forall n$$

$$\Rightarrow A^{\alpha}\zeta(z)^{\frac{n+1}{n}}=0,\forall n\Rightarrow A^{\alpha}\zeta(z)=0=\frac{A^{\alpha+1}}{A^{\alpha+1}-1}=1$$

It is impossible !

$$\Rightarrow A\neq\infty, A\neq-\infty$$

Thus A=0 And

$$\int_{1}^{\infty}\frac{dt}{t^{x+iy}}=\frac{1}{1-x-iy}\left[t^{1-x-iy}\right]_{1}^{\infty}=\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha(x+iy)}}).B=\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha(x+iy)}})+A=0\Rightarrow x=\frac{1}{2}$$

Thus the non trivial zeros of the Riemann funtion zeta lie in the critical line like for the reals ! It is the proof of the Riemann hypothesis !

Conclusion

We have generalized the concept of prime to the reals. It allowed to prove the conjecture to the reals. Then, we have proved the Riemann hypothesis.

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