

A PROOF OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we present the Riemann problem and define the real primes. It allows to generalize the Riemann hypothesis to the reals. A calculus of integral solves the problem. We generalize the proof to the integers.

The Riemann hypothesis

The Riemann conjecture is a conjecture which has been formulated in 1859 by Bernhard Riemann in the subject of the Riemann function zeta or ζ . It is called the zeta Riemann function.

This function is defined as follows

$$\zeta(s) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^s}\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

The first result is the divergence of the harmonic serie

$$\zeta(1) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

It has been proved in the middle age by Nicole Oresme

In the XVIII century, Leonard Euler has discovered the main proprieties of the ζ function.

In the 1730's he conjectured after numerical calculus the following equality, which is often called the Basel problem.

$$\zeta(2) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^2}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Euler proved it in 1743 and introduced the ζ function. He calculated its value for the positive even numbers.

$$\zeta(2k) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^{2k}}\right) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \dots = \frac{|B_{2k}| (2\pi)^{2k}}{2(2k)!}$$

Where B_{2k} are the Bernoulli numbers.

Thereafter, he proved in 1744 the Euler identity where prime numbers are related to the ζ function.

$$\zeta(s) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^s}\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \prod_{primes} \frac{1}{1-p^{-s}}$$

Consequently he deduced the divergence of the serie of the inverse of primes.

With Bernard Riemann, s can be complex number. Riemann proved the following formula

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{-(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Where

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

This formula demonstrates that this equation does not change if we replace s by $1-s$.

Thus it is symmetric $|s = \frac{1}{2}$

Riemann demonstrates that the only zeros in the $R(s) < 0$ are the trivial zero, negative even numbers and that there is no zero in the $R(s) > 1$.

The other zeros are the non trivial zeros. They are in the critical zone $0 \leq R(s) \leq 1$.

Riemann conjectured they are all in the critical line $R(s) = \frac{1}{2}$.

This conjecture is called the Riemann hypothesis.

They calculated numerically one billion zeros of the Riemann zeta function and they are all located in the critical line.

Definition

A real number is compound if it can be written as $\prod p_j^{n_j}$ where p_j are primes and n_j are rationals. This decomposition in prime factors is unique. Every prime real number can be written only as $p = p \cdot 1$.

Thus we define other real prime numbers like π , e , $\ln(2)$. Thus $\sqrt[p]{p} = p^{\frac{1}{p}}$ is compound. Also $\sqrt[p]{p} + 1 = p^{\frac{1}{p} + \frac{1}{p}}$ is prime when p is prime and we have $\sqrt[p]{p} - 1 = (p-1)(\sqrt[p]{p} + 1)^{-1}(\sqrt[p]{p} + 1)^{-1} \dots (\sqrt[p]{p} + 1)^{-1}$ compound for p prime, for example.

The approach of the Riemann hypothesis

The Riemann hypothesis states that the non trivial zeros of the Riemann zeta function $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$ lie on the critical line $\frac{1}{2} + iy$.

For t real, Euler has proved that $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \prod_{primes} \frac{1}{1-p^{-z}} = \prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha z}} \right)$ it is the Euler

identity. For t real, it is still true and it becomes $\prod_{primes} \left(\sum_{\alpha \in \mathbb{Q}} \frac{1}{p^{\alpha z}} \right) = \int_1^\infty \frac{dt}{t^z} = \frac{1}{1-z} [t^{1-z}]_1^\infty$ But there are the

trivial zeros : we have $\zeta(-2k) = 0, \forall k \in \mathbb{N}$ and $[t^{1+2k}]_1^\infty = 0$ but if $[t^z]^\infty$ is the limit in the infinity,

$$\lim_{t \rightarrow \infty} (t^{1+2k}) = 1, \forall k \in \mathbb{N} \text{ and}$$

$$\lim_{i \rightarrow \infty} (t^{\frac{1}{2} + iy}) = \lim_{i \rightarrow \infty} (t^{\frac{1}{2} - iy}) = \lim_{i \rightarrow \infty} (t^{\frac{1}{2} + iy}) = \frac{1}{2} (\lim_{i \rightarrow \infty} (t^{\frac{1}{2} - iy}) + \lim_{i \rightarrow \infty} (t^{\frac{1}{2} + iy})) = a$$

$$\Rightarrow \lim_{i \rightarrow \infty} (t^{2(\frac{1}{2} + iy)} - 2at^{\frac{1}{2} + iy} + t) = \lim_{i \rightarrow \infty} (t^{2(\frac{1}{2} + iy)} - 2at^{\frac{1}{2} + iy} + 1) = 0 \Rightarrow \lim_{i \rightarrow \infty} (t^{\frac{1}{2} + iy}) = a = a + \sqrt{a^2 - 1} = 1$$

it means that $\zeta\left(\frac{1}{2} + iy\right) = \left[\frac{1}{1 - \frac{1}{2} - iy} t^{1 - \frac{1}{2} - iy} \right]_1^\infty = 0$

Let now $\zeta(x + iy) = 0 = \left[\frac{t^{1-x-iy}}{1-x-iy} \right]_1^\infty = \left[\frac{t^{\frac{1-iy}{2} - x}}{1-x-iy} \right]_1^\infty = \left[\frac{t^{\frac{1-x}{2}}}{1-x-iy} \right]_1^\infty = 0 \Rightarrow x = \frac{1}{2}$

We have proved that the non trivial zeros of the Riemann function for the reals lie in the critical line ! So the hypothesis is proved for the real numbers. The Riemann hypothesis is important because it gives information about the zeros of the Riemann function and the distribution of those zeros are related to real primes !

The generalization to the integers

We have

$$\int_1^\infty \frac{dt}{t^z} = \frac{1}{1-z} \left[t^{1-z} \right]_1^\infty = \sum_{t=1}^\infty \frac{1}{t^z} + \sum_{\alpha \in \mathbb{Q}^-} \frac{1}{t^{\alpha z}} + \sum_{\alpha \in \mathbb{Q}^+ \setminus \mathbb{N}} \frac{1}{t^{\alpha z}}$$

$$= \sum_{t=1}^\infty \frac{1}{t^z} \cdot B = \prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha z}} \right) \cdot B = \sum_{t=1}^\infty \frac{1}{t^z} + A$$

Let now

$$\prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha(x+iy)}} \right) = 0$$

We have : If B is finite, the zeta function for the reals will be equal to zero. Else Let $\zeta_1(z)$ the Riemann function for the reals and $\zeta(z)$ the Riemann function for the integers, if B = +∞ and if A is finite and

$$A > 1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^n \zeta(z) + A)^2 = A^2 = A^2 + (A^n \zeta(z))^2 + 2A^{n+1} \zeta(z) \Rightarrow A^{n+1} \zeta(z) = 0, \forall n$$

$$\exists \alpha | B \quad A^\alpha < A \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^n \zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow 1 \geq A$$

And if

$$1 \geq A > 0$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1} \zeta(z) = A^{-2} + 2A^{-1} \zeta(z) \Rightarrow A^{-1} \zeta(z) = 0$$

$$(A^{-n} \zeta(z) + A^{-1})^2 = A^{-2} = A^{-2} + (A^{-n} \zeta(z))^2 + 2A^{-n-1} \zeta(z) \Rightarrow A^{-n-1} \zeta(z) = 0, \forall n$$

$$\exists \alpha | B \quad A^{-\alpha} < A^{-1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^{-n} \zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

$$0 \geq A \geq -1$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1} \zeta(z) = A^{-2} + 2A^{-1} \zeta(z) \Rightarrow A^{-1} \zeta(z) = 0$$

$$(A^{-2n-1} \zeta(z) + A^{-1})^2 = A^{-2} = A^{-2} + (A^{-2n-1} \zeta(z))^2 + 2A^{-2n-3} \zeta(z) \Rightarrow A^{-2n-3} \zeta(z) = 0, \forall n$$

$$\exists \alpha | B \quad -A^{-\alpha} < -A^{-2n-1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < -A^{-2n-1} \zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$A < -1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^{2n+1}\zeta(z) + A^2)^2 = A^4 = A^4 + (A^{2n+1}\zeta(z))^2 + 2A^{2n+3}\zeta(z) \Rightarrow A^{2n+3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B^{-A^\alpha} < -A^{2n+1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^{2n+1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow A \geq -1$$

If now $B = -\infty$ and if A is finite and

$$A > 1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^n\zeta(z) + A)^2 = A^2 = A^2 + (A^n\zeta(z))^2 + 2A^{n+1}\zeta(z) \Rightarrow A^{n+1}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B^{-A^\alpha} > -A^n \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > -A^n\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow 1 > A$$

And if

$$1 \geq A \geq 0$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-n}\zeta(z) + A^{-1})^2 = A^{-2} = A^{-2} + (A^{-n}\zeta(z))^2 + 2A^{-n-1}\zeta(z) \Rightarrow A^{-n-1}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B^{-A^\alpha} > -A^{-n} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > -A^{-n}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$0 \geq A \geq -1$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-2n-1}\zeta(z) + A^{-2})^2 = A^{-4} = A^{-4} + (A^{-2n-1}\zeta(z))^2 + 2A^{-2n-3}\zeta(z) \Rightarrow A^{-2n-3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B^{-A^\alpha} > A^{-2n-1} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > A^{-2n-1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$A < -1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^{2n+1}\zeta(z) + A^2)^2 = A^4 = A^4 + (A^{2n+1}\zeta(z))^2 + 2A^{2n+3}\zeta(z) \Rightarrow A^{2n+3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B^{-A^\alpha} > A^{2n+1} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > A^{2n+1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow A \geq -1$$

And if $A = \pm\infty$ and we have $B | A^{\alpha+1}$ hence

$$A^\alpha \zeta(z)^2 = \frac{A^{\alpha+2}}{(A^{\alpha+1} - 1)^2} = 0$$

$$A^\alpha \zeta(z) = 1$$

And

$$(A^\alpha \zeta(z) + A^\alpha \zeta(z)^2)^2 = A^{2\alpha} \zeta(z)^2 = A^{2\alpha} \zeta(z)^2 + 2A^{2\alpha} \zeta(z)^3 \Rightarrow A^{2\alpha} \zeta(z)^3 = 0$$

If $A^{n\alpha} \zeta(z)^{n+1} = 0$ then

$$(A^{n\alpha} \zeta(z)^{n+1} + A^\alpha \zeta(z)) = A^{2\alpha} \zeta(z)^2 = A^{2\alpha} \zeta(z)^2 + 2A^{(n+1)\alpha} \zeta(z)^{n+2} \Rightarrow A^{(n+1)\alpha} \zeta(z)^{n+2} = 0, \forall n$$

$$\Rightarrow A^\alpha \zeta(z)^{\frac{n+1}{n}} = 0, \forall n \Rightarrow A^\alpha \zeta(z) = 0 = \frac{A^{\alpha+1}}{A^{\alpha+1} - 1} = 1$$

It is impossible !

$$\Rightarrow A \neq \infty, A \neq -\infty$$

Thus A=0 And

$$\int_1^\infty \frac{dt}{t^{x+iy}} = \frac{1}{1-x-iy} [t^{1-x-iy}]_1^\infty = \prod_{primes} (\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha(x+iy)}}) \cdot B = \prod_{primes} (\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha(x+iy)}}) + A = \dots \quad x = \frac{1}{2}$$

Thus the non trivial zeros of the Riemann function zeta lie in the critical line $\text{Re}(s) = \frac{1}{2}$ for the reals ! It is the proof of the Riemann hypothesis !

Conclusion

We have generalized the concept of prime to the reals. It allowed to prove the conjecture to the reals. Then, we have proved the Riemann hypothesis.

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