

## About an even as the sum or the difference of two primes

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**Keywords:** Goldbach; De Polignac; Twin primes; Algebraic proof

**Abstract.** The present algebraic development begins by an exposition of the data of the problem. The definition of the primal radius  $r > 0$  is : For all positive integer  $x \geq 3$  exists a finite number of integers called the primal radius  $r > 0$ , for which  $x + r$  and  $x - r$  are prime numbers. The corollary is that  $2x = (x + r) + (x - r)$  is always the sum of a finite number of primes. Also, for all positive integer  $x \geq 0$ , exists an infinity of integers  $r > 0$ , for which  $+ r$  and  $r - x$  are prime numbers. The conclusion is that  $2x = (x - r) - (r - x)$  is always an infinity of differences of primes.

### Introduction

There is a similarity between the assertion : “an even number is always the sum of two primes” and the assertion : “an even number is always the difference of two primes”. The present article gives the proof that the two assertions are the consequences of the same concept by the introduction of the notion of the primal radius.

### The proof

Let us suppose that exists an integer  $x \geq 3$  for which  $2x$  is never the sum of two primes, then for all  $p_1$  and  $p_2$  primes,  $3 \leq p_2 < p_1$ ,  $2x \neq p_1 + p_2$ , OR  $2x = p_1 + p_2 + 2b_{p_1, p_2} = p_1 + p_2 + 2b$

$$x = \frac{p_1 + p_2}{2} + b.$$

then

But for all  $p_1, p_2$  exists  $y$ , for which  $y = \frac{p_1 - p_2}{2} + b$

Let

$$x_1 = p_1 + 2b, \quad x_2 = p_2 - 2b, \quad x_3 = p_2 + 2b, \quad x_4 = p_1 - 2b$$

We deduce that

$$x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + x_2}{2} + 2b = \frac{x_1 + p_2}{2} = \frac{x_1 + x_2}{2} + b$$

$$= \frac{p_1 + x_3}{2} = \frac{x_1 + x_3}{2} - b = \frac{x_4 + x_3}{2} + b = \frac{x_4 + x_2}{2} + 3b$$

$$y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - x_2}{2} = \frac{x_1 - p_2}{2} = \frac{x_1 - x_2}{2} - b$$

$$= \frac{p_1 - x_3}{2} + 2b = \frac{x_1 - x_3}{2} + b = \frac{x_4 - x_3}{2} + 3b = \frac{x_4 - x_2}{2} + b$$

$$x_1 + x_2 = p_1 + p_2$$

$$x_1 - x_3 = p_1 - p_2$$

### Lemma 1

The following formula

$$x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + x_2}{2} + 2b = \frac{x_1 + p_2}{2} = \frac{x_1 + x_2}{2} + b$$

$$= \frac{p_1 + x_3}{2} = \frac{x_1 + x_3}{2} - b = \frac{x_4 + x_3}{2} + b = \frac{x_4 + x_2}{2} + 3b$$

$$y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - x_2}{2} = \frac{x_1 - p_2}{2} = \frac{x_1 - x_2}{2} - b$$

$$= \frac{p_1 - x_3}{2} + 2b = \frac{x_1 - x_3}{2} + b = \frac{x_4 - x_3}{2} + 3b = \frac{x_4 - x_2}{2} + b$$

Imply that exist  $p_1$  and  $p_2$  prime numbers for which  $b = 0$

**Proof of lemma 1**

If  $x$  is prime  $2x = x + x$  is the sum of two primes, then  $p_1 - p_2 \neq 0$

We will suppose firstly that  $(x_1 - x_2)(x_1 + x_3) \neq 0$

Let

$$\frac{x_1 - x_2}{p_1 - p_2} = \frac{p_1 - p_2 + 4b}{p_1 - p_2} = 1 + \frac{4b}{p_1 - p_2}$$

$$\frac{p_1 - p_2}{x_1 - x_2} = \frac{x_1 - x_2 - 4b}{x_1 - x_2} = 1 - \frac{4b}{x_1 - x_2}$$

We pose  $k = \frac{2b}{p_1 - p_2}$ ,  $k' = -\frac{2b}{x_1 - x_2}$

if  $kk' = 0 \Rightarrow b = 0$ , we have supposed  $kk' \neq 0$

$$\forall (x, y); \exists \varphi | x = \varphi y$$

$$x + y = (\varphi + 1)y = x_1 \neq 0, \quad x - y = (\varphi - 1)y = p_2 \neq 0$$

$$\forall (k, k'); \exists \alpha | k = \alpha k'$$

$$\Rightarrow \frac{2b}{p_1 - p_2} = -\alpha \frac{2b}{x_1 - x_2}$$

$$\Rightarrow x_1 - x_2 = -\alpha(p_1 - p_2) \Rightarrow x_1 - x_2 - p_1 + p_2 = 4b = -(\alpha + 1)(p_1 - p_2)$$

$$\Rightarrow b = -\frac{\alpha + 1}{4}(p_1 - p_2)$$

$$\Rightarrow x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\alpha + 1}{4}(p_1 - p_2) = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = \frac{\varphi}{\varphi - 1} p_2$$

$$y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\alpha + 1}{4}(p_1 - p_2) = \frac{(1 - \alpha)(p_1 - p_2)}{4} = \frac{1}{\varphi - 1} p_2$$

Let

$$\frac{x_1 + x_3}{p_1 + p_2} = \frac{p_1 + p_2 + 4b}{p_1 + p_2} = 1 + \frac{4b}{p_1 + p_2}$$

$$\frac{p_1 + p_2}{x_1 + x_3} = \frac{x_1 + x_3 - 4b}{x_1 + x_3} = 1 - \frac{4b}{x_1 + x_3}$$

We pose  $m = \frac{2b}{p_1 + p_2}$ ,  $m' = -\frac{2b}{x_1 + x_3}$

$mm' = 0 \Rightarrow b = 0$ , we have supposed  $mm' \neq 0$

$$\forall (m, m'); \exists \beta | m = \beta m'$$

$$\Rightarrow \frac{2b}{p_1 + p_2} = -\beta \frac{2b}{x_1 + x_3}$$

$$\Rightarrow x_1 + x_3 = -\beta(p_1 + p_2) \Rightarrow x_1 + x_3 - p_1 - p_2 = 4b = -(\beta + 1)(p_1 + p_2)$$

$$\Rightarrow b = -\frac{\beta + 1}{4}(p_1 + p_2)$$

$$\Rightarrow x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\beta + 1}{4}(p_1 + p_2) = \frac{(1 - \beta)(p_1 + p_2)}{4} = \frac{\varphi}{\varphi - 1} p_2$$

$$y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\beta + 1}{4}(p_1 + p_2) = \frac{(1 - \beta)p_1 - (\beta + 3)p_2}{4} = \frac{1}{\varphi - 1} p_2$$

$$b = -\frac{\alpha + 1}{4}(p_1 - p_2) = -\frac{\beta + 1}{4}(p_1 + p_2) \Rightarrow (\beta - \alpha)p_1 = (-2 - \alpha - \beta)p_2$$

As  $p_1, p_2$  are primes then  $\beta = \alpha = -1 \Rightarrow b = 0$

But  $(2k + 1)(2k' + 1) = 1 \Rightarrow 2kk' + k + k' = 0 \Rightarrow (2k + 1)k' = -k$

$$2k+1 = \frac{-k}{k'} = \frac{k+1}{k'+1} = \frac{a(k+1)+a'(2k+1)}{a(k'+1)+a'}$$

$$2k'+1 = \frac{-k'}{k} = \frac{k'+1}{k+1} = \frac{c(k'+1)+c'(2k'+1)}{c(k+1)+c'}, \quad \forall (a, a', c, c')$$

$$\frac{2k+1}{2k'+1} = \frac{((a+2a')k+a+a')(ck+c+c')}{((c+2c')k'+c+c')(ak'+a+a')}$$

$$= \frac{(ac+2a'c)k^2+(2ac+ac'+3a'c+2a'c')k+ac+ac'+a'c+a'c'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'}$$

$$\Rightarrow \frac{2k+1}{2k'+1} - 1 = (k-k') \frac{2}{2k'+1}$$

$$= \frac{ac(k^2-k'^2)+2a'ck^2-2ac'k'^2+(2ac+ac'+a'c+2a'c')(k-k')+2a'ck-2ac'k'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'}$$

$\forall (a, a', c, c')$ , particularly  $(a, a', c, c') | ac' = \delta k^2, a'c = \delta k'^2$

$$\Rightarrow (k-k') \frac{2}{2k'+1}$$

$$= \frac{ac(k+k')(k-k')+(2ac+ac'+a'c+2a'c')(k-k')+2\delta k k'^2-2\delta k^2 k'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'}$$

$$= (k-k') \frac{-2ackk'+2ac+ac'+a'c+2a'c'-2\delta k k'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'}$$

If  $k = k' \Rightarrow \alpha = 1 \Rightarrow x = \frac{(1-\alpha)p_1 + (3+\alpha)p_2}{4} = p_2$ , trivial solution : it is impossible.

$$k - k' \neq 0 \Rightarrow \frac{2}{2k'+1} = \frac{-2ackk'+2ac+ac'+a'c+2a'c'-2\delta k k'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'}$$

Also  $(2m+1)(2m'+1) = 1 \Rightarrow 2mm'+m+m' = 0 \Rightarrow (2m+1)m' = -m$

$$2m+1 = \frac{-m}{m'} = \frac{m+1}{m'+1} = \frac{a(m+1)+a'(2m+1)}{a(m'+1)+a'}$$

$$2m'+1 = \frac{-m'}{m} = \frac{m'+1}{m+1} = \frac{c(m'+1)+c'(2m'+1)}{c(m+1)+c'}, \quad \forall (a, a', c, c')$$

$$\frac{2m+1}{2m'+1} = \frac{((a+2a')m+a+a')(cm+c+c')}{((c+2c')m'+c+c')(am'+a+a')}$$

$$= \frac{(ac+2a'c)m^2+(2ac+ac'+3a'c+2a'c')m+ac+ac'+a'c+a'c'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'}$$

$$\Rightarrow \frac{2m+1}{2m'+1} - 1 = (m-m') \frac{2}{2m'+1}$$

$$= \frac{ac(m^2-m'^2)+2a'cm^2-2ac'm'^2+(2ac+ac'+a'c+2a'c')(m-m')+2a'cm-2ac'm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'}$$

$\forall (a, a', c, c')$ , particularly  $(a, a', c, c') | ac' = \delta k^2 = \gamma m^2, a'c = \delta k'^2 = \gamma' m'^2$

$$\Rightarrow (m-m') \frac{2}{2m'+1}$$

$$= \frac{ac(m+m')(m-m')+2(\gamma'-\gamma)m^2m'^2+(2ac+ac'+a'c+2a'c')(m-m')+2\gamma'mm'^2-2\gamma m^2m'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'}$$

$$= (m-m') \frac{-2acmm'+2\left(\frac{\gamma'-\gamma}{m-m'}\right)m^2m'^2+2ac+ac'+a'c+2a'c'-2\left(\frac{\gamma m-\gamma' m'}{m-m'}\right)mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'}$$

$m = m' \Rightarrow \beta = 1 \Rightarrow x = \frac{(1-\beta)(p_1+p_2)}{4} = 0$ , it is impossible

$$\begin{aligned} \text{But } \frac{2}{2k'+1} &= \frac{-2ackk'+2ac+ac'+a'c+2a'c'-2\delta kk'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'} \\ \text{For } (a, a', c, c') \mid ac' &= \delta k^2 = \gamma m^2, \quad a'c = \delta k'^2 = \gamma' m'^2 \\ \Rightarrow \frac{2}{2m'+1} &= \frac{-2acmm'+2ac+ac'+a'c+2a'c'-2\gamma mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'} \\ &= \frac{-2acmm'+2ac+ac'+a'c+2a'c'-2\gamma' mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'} \\ \Rightarrow \frac{2}{2m'+1} &= \frac{-2acmm'+2(\frac{\gamma'-\gamma}{m-m'})m^2m'^2+2ac+ac'+a'c+2a'c'-2(\frac{\gamma m-\gamma' m'}{m-m'})mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'} \\ &= \frac{-2acmm'+2ac+ac'+a'c+2a'c'-2\gamma mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'} \\ &= \frac{-2acmm'+2(\frac{\gamma-\gamma'}{m-m'})m^2m'^2+2ac+ac'+a'c+2a'c'-2\gamma mm'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'} \\ \Rightarrow \gamma &= \gamma' \Rightarrow ac' = \delta k^2 = \gamma m^2, \quad a'c = \delta k'^2 = \gamma' m'^2 = \gamma m'^2 \\ \Rightarrow \frac{\delta}{\gamma} &= \frac{m^2}{k^2} = \frac{m'^2}{k'^2} \Rightarrow \frac{k^2}{k'^2} = (2k+1)^2 = \alpha^2 = \frac{m^2}{m'^2} = (2m+1)^2 = \beta^2 \end{aligned}$$

$$\text{If } \alpha = -\beta \Rightarrow (\beta - \alpha)p_1 = -2\alpha p_1 = 2\beta p_1 = (-2 - \alpha - \beta)p_2 = -2p_2 \Rightarrow \alpha = -\beta = \frac{p_2}{p_1}$$

$$b = -(\alpha + 1)\frac{p_1 - p_2}{4} = -(\beta + 1)\frac{p_1 + p_2}{4} = -\left(\frac{p_1 + p_2}{4p_1}\right)(p_1 - p_2) = \frac{p_2^2 - p_1^2}{4p_1}$$

$$\Rightarrow 4bp_1 = p_2^2 - p_1^2 \Rightarrow (4b + p_1)p_1 = p_2^2 \Rightarrow 4b + p_1 = \frac{p_2^2}{p_1} \quad \text{and it is impossible because } p_1 \text{ and } p_2 \text{ are primes}$$

$$4b + p_1 = \frac{p_2^2}{p_1}$$

and  $\frac{p_2^2}{p_1}$  can not be an integer.

$$\Rightarrow \alpha - \beta = 0 \Rightarrow (\beta - \alpha)p_1 = (-2 - \alpha - \beta)p_2 = -2(1 + \alpha)p_2 = -2(1 + \beta)p_2 = 0 \Rightarrow \alpha = \beta = -1$$

$$\Rightarrow b = -\frac{(\alpha + 1)}{4}(p_1 - p_2) = -\frac{(\beta + 1)}{4}(p_1 + p_2) = 0$$

$$\Rightarrow x = \frac{p_1 + p_2}{2}, \quad y = \frac{p_1 - p_2}{2}$$

$x + y = p_1$ ,  $x - y = p_2$  are primes  $y = r$  is the primal radius. As there is the condition  $p_2 < x < p_1$ , there is not an infinity of  $p_1, p_2$

$$\text{If } (x_1 - x_2)(x_1 + x_2) = 0 \Rightarrow (x_4 + x_2)(x_4 - x_3) \neq 0$$

$$\text{Let } \frac{x_4 + x_2}{p_1 + p_2} = \frac{p_1 + p_2 - 4b}{p_1 + p_2} = 1 - \frac{4b}{p_1 + p_2}$$

$$\frac{p_1 + p_2}{x_4 + x_2} = \frac{x_4 + x_2 + 4b}{x_4 + x_2} = 1 + \frac{4b}{x_4 + x_2}$$

$$k = -\frac{2b}{p_1 + p_2}, \quad k' = \frac{2b}{x_4 + x_2}$$

$$\frac{x_4 - x_3}{p_1 - p_2} = \frac{p_1 - p_2 - 4b}{p_1 - p_2} = 1 - \frac{4b}{p_1 - p_2}$$

$$\frac{p_1 - p_2}{x_4 - x_3} = \frac{x_4 - x_3 + 4b}{x_4 - x_3} = 1 + \frac{4b}{x_4 - x_3}$$

$$m = -\frac{2b}{p_1 - p_2}, \quad m' = \frac{2b}{x_4 - x_3}$$

With the same reasoning and calculus  $\Rightarrow b = 0$ . But  $b$  can not be equal to zero for all  $x, p_1, p_2$  then the initial assertion led to an impossibility, the conjecture is undecidable ! If it is undecidable and false it will exist  $x$  for which  $2x$  is never the sum of two primes, we would calculate in a computer  $x$  for which  $2x$  is never the sum of two primes so it contradicts the fact that the assertion is undecidable, so the conjecture is undecidable and true !

Now, if we suppose that for all  $p_1, p_2$  primes, exists  $x \mid 2x \neq p_1 - p_2$

$x = \frac{p_1 - p_2}{2} + b, y = \frac{p_1 + p_2}{2} + b$ , with the same reasoning, the same calculus but replacing  $x$  by  $y$  and  $y$  by  $x$ , we prove that  $b = 0$ , which means that for all positive integer  $x$ , exists  $p_1, p_2$  for which

$x = \frac{p_1 - p_2}{2}$  if we pose  $y = \frac{p_1 + p_2}{2}, x + y = p_1, y - x = p_2$  primes,  $y$  is the primal radius. As there is no

condition on  $x, y, p_1, p_2$ , there is an infinity of couples of primes  $(p_1, p_2)$ . For  $x = 1, p_1$  and  $p_2$  are twin primes. Let us prove it. Let us suppose that exists an integer  $x \geq 0$  for which  $2x$  is never the difference of two primes, then for all  $p_1$  and  $p_2$  primes,  $p_2 < p_1, 2x \neq p_1 - p_2,$

or  $2x = p_1 - p_2 + 2b_{p_1, p_2} = p_1 - p_2 + 2b$ , then  $x = \frac{p_1 - p_2}{2} + b$ .

But for all  $p_1, p_2$  exists  $y$ , for which  $y = \frac{p_1 + p_2}{2} + b$

Let

$$x_1 = p_1 + 2b, x_2 = p_2 - 2b, x_3 = p_2 + 2b, x_4 = p_1 - 2b$$

We deduce that

$$y = \frac{p_1 + p_2}{2} + b = \frac{p_1 + x_2}{2} + 2b = \frac{x_1 + p_2}{2} = \frac{x_1 + x_2}{2} + b$$

$$= \frac{p_1 + x_3}{2} = \frac{x_1 + x_3}{2} - b = \frac{x_4 + x_3}{2} + b = \frac{x_4 + x_2}{2} + 3b$$

$$x = \frac{p_1 - p_2}{2} + b = \frac{p_1 - x_2}{2} = \frac{x_1 - p_2}{2} = \frac{x_1 - x_2}{2} - b$$

$$= \frac{p_1 - x_3}{2} + 2b = \frac{x_1 - x_3}{2} + b = \frac{x_4 - x_3}{2} + 3b = \frac{x_4 - x_2}{2} + b$$

$$x_1 + x_2 = p_1 + p_2$$

$$x_1 - x_3 = p_1 - p_2$$

**Lemma 2**

The following formula

$$y = \frac{p_1 + p_2}{2} + b = \frac{p_1 + x_2}{2} + 2b = \frac{x_1 + p_2}{2} = \frac{x_1 + x_2}{2} + b$$

$$= \frac{p_1 + x_3}{2} = \frac{x_1 + x_3}{2} - b = \frac{x_4 + x_3}{2} + b = \frac{x_4 + x_2}{2} + 3b$$

$$x = \frac{p_1 - p_2}{2} + b = \frac{p_1 - x_2}{2} = \frac{x_1 - p_2}{2} = \frac{x_1 - x_2}{2} - b$$

$$= \frac{p_1 - x_3}{2} + 2b = \frac{x_1 - x_3}{2} + b = \frac{x_4 - x_3}{2} + 3b = \frac{x_4 - x_2}{2} + b$$

Imply that exist  $p_1$  and  $p_2$  prime numbers for which  $b = 0$

**Proof of lemma 2**

If  $x$  is prime  $0 = x - x$  is the sum of two primes, then  $p_1 - p_2 \neq 0$

We will suppose firstly that  $(x_1 - x_2)(x_1 + x_3) \neq 0$

Let

$$\frac{x_1 - x_2}{p_1 - p_2} = \frac{p_1 - p_2 + 4b}{p_1 - p_2} = 1 + \frac{4b}{p_1 - p_2}$$

$$\frac{p_1 - p_2}{x_1 - x_2} = \frac{x_1 - x_2 - 4b}{x_1 - x_2} = 1 - \frac{4b}{x_1 - x_2}$$

We pose  $k = \frac{2b}{p_1 - p_2}$ ,  $k' = -\frac{2b}{x_1 - x_2}$

$$kk' = 0 \Rightarrow b = 0, \text{ we have supposed } kk' \neq 0$$

$$\forall (x, y); \exists \varphi | y = \varphi x$$

$$x + y = (\varphi + 1)x = x_1 \neq 0, \quad y - x = (\varphi - 1)x = p_2 \neq 0$$

$$\forall (k, k'); \exists \alpha | k = \alpha k'$$

$$\Rightarrow \frac{2b}{p_1 - p_2} = -\alpha \frac{2b}{x_1 - x_2}$$

$$\Rightarrow x_1 - x_2 = -\alpha(p_1 - p_2) \Rightarrow x_1 - x_2 - p_1 + p_2 = 4b = -(\alpha + 1)(p_1 - p_2)$$

$$\Rightarrow b = -\frac{\alpha + 1}{4}(p_1 - p_2)$$

$$\Rightarrow y = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\alpha + 1}{4}(p_1 - p_2) = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = \frac{\varphi}{\varphi - 1} p_2$$

$$x = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\alpha + 1}{4}(p_1 - p_2) = \frac{(1 - \alpha)(p_1 - p_2)}{4} = \frac{1}{\varphi - 1} p_2$$

Let

$$\frac{x_1 + x_3}{p_1 + p_2} = \frac{p_1 + p_2 + 4b}{p_1 + p_2} = 1 + \frac{4b}{p_1 + p_2}$$

$$\frac{p_1 + p_2}{x_1 + x_3} = \frac{x_1 + x_3 - 4b}{x_1 + x_3} = 1 - \frac{4b}{x_1 + x_3}$$

We pose  $m = \frac{2b}{p_1 + p_2}$ ,  $m' = -\frac{2b}{x_1 + x_3}$

$$mm' = 0 \Rightarrow b = 0, \text{ we have supposed } mm' \neq 0$$

$$\forall (m, m'); \exists \beta | m = \beta m'$$

$$\Rightarrow \frac{2b}{p_1 + p_2} = -\beta \frac{2b}{x_1 + x_3}$$

$$\Rightarrow x_1 + x_3 = -\beta(p_1 + p_2) \Rightarrow x_1 + x_3 - p_1 - p_2 = 4b = -(\beta + 1)(p_1 + p_2)$$

$$\Rightarrow b = -\frac{\beta + 1}{4}(p_1 + p_2)$$

$$\Rightarrow y = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\beta + 1}{4}(p_1 + p_2) = \frac{(1 - \beta)(p_1 + p_2)}{4} = \frac{\varphi}{\varphi - 1} p_2$$

$$x = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\beta + 1}{4}(p_1 + p_2) = \frac{(1 - \beta)p_1 - (\beta + 3)p_2}{4} = \frac{1}{\varphi - 1} p_2$$

$$b = -\frac{\alpha + 1}{4}(p_1 - p_2) = -\frac{\beta + 1}{4}(p_1 + p_2) \Rightarrow (\beta - \alpha)p_1 = (-2 - \alpha - \beta)p_2$$

But  $(2k + 1)(2k' + 1) = 1 \Rightarrow 2kk' + k + k' = 0 \Rightarrow (2k + 1)k' = -k$

$$2k + 1 = \frac{-k}{k'} = \frac{k + 1}{k' + 1} = \frac{a(k + 1) + a'(2k + 1)}{a(k' + 1) + a'}$$

$$2k' + 1 = \frac{-k'}{k} = \frac{k' + 1}{k + 1} = \frac{c(k' + 1) + c'(2k' + 1)}{c(k + 1) + c'}, \quad \forall (a, a', c, c')$$

$$\frac{2k + 1}{2k' + 1} = \frac{((a + 2a')k + a + a')(ck + c + c')}{((c + 2c')k' + c + c')(ak' + a + a')}$$

$$\begin{aligned}
 &= \frac{(ac + 2a'c)k^2 + (2ac + ac' + 3a'c + 2a'c')k + ac + ac' + a'c + a'c'}{(ac + 2ac')k^2 + (2ac + 3ac' + a'c + 2a'c')k + ac + ac' + a'c + a'c'} \\
 &\Rightarrow \frac{2k+1}{2k'+1} - 1 = (k - k') \frac{2}{2k'+1} \\
 &= \frac{ac(k^2 - k'^2) + 2a'ck^2 - 2ac'k'^2 + (2ac + ac' + a'c + 2a'c')(k - k') + 2a'ck - 2ac'k'}{(ac + 2ac')k^2 + (2ac + 3ac' + a'c + 2a'c')k + ac + ac' + a'c + a'c'}
 \end{aligned}$$

$\forall (a, a', c, c')$ , particularly  $(a, a', c, c') | ac' = \delta k^2, a'c = \delta k'^2$

$$\begin{aligned}
 &\Rightarrow (k - k') \frac{2}{2k'+1} \\
 &= \frac{ac(k + k')(k - k') + (2ac + ac' + a'c + 2a'c')(k - k') + 2\delta k k'^2 - 2\delta k'^2 k'}{(ac + 2ac')k^2 + (2ac + 3ac' + a'c + 2a'c')k + ac + ac' + a'c + a'c'} \\
 &= (k - k') \frac{-2ackk' + 2ac + ac' + a'c + 2a'c' - 2\delta k k'}{(ac + 2ac')k^2 + (2ac + 3ac' + a'c + 2a'c')k + ac + ac' + a'c + a'c'}
 \end{aligned}$$

$k = k' \Rightarrow \alpha = 1 \Rightarrow y = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = p_2$  it is impossible.

$k - k' \neq 0 \Rightarrow \frac{2}{2k'+1} = \frac{-2ackk' + 2ac + ac' + a'c + 2a'c' - 2\delta k k'}{(ac + 2ac')k^2 + (2ac + 3ac' + a'c + 2a'c')k + ac + ac' + a'c + a'c'}$

Also  $(2m + 1)(2m' + 1) = 1 \Rightarrow 2mm' + m + m' = 0 \Rightarrow (2m + 1)m' = -m$

$2m + 1 = \frac{-m}{m'} = \frac{m + 1}{m' + 1} = \frac{a(m + 1) + a'(2m + 1)}{a(m' + 1) + a'}$

$2m' + 1 = \frac{-m'}{m} = \frac{m' + 1}{m + 1} = \frac{c(m' + 1) + c'(2m' + 1)}{c(m + 1) + c'}, \forall (a, a', c, c')$

$\frac{2m + 1}{2m' + 1} = \frac{((a + 2a')m + a + a')(cm + c + c')}{((c + 2c')m' + c + c')(am' + a + a')}$

$= \frac{(ac + 2a'c)m^2 + (2ac + ac' + 3a'c + 2a'c')m + ac + ac' + a'c + a'c'}{(ac + 2ac')m^2 + (2ac + 3ac' + a'c + 2a'c')m + ac + ac' + a'c + a'c'}$

$\Rightarrow \frac{2m + 1}{2m' + 1} - 1 = (m - m') \frac{2}{2m' + 1}$

$= \frac{ac(m^2 - m'^2) + 2a'cm^2 - 2ac'm'^2 + (2ac + ac' + a'c + 2a'c')(m - m') + 2a'cm - 2ac'm'}{(ac + 2ac')m^2 + (2ac + 3ac' + a'c + 2a'c')m + ac + ac' + a'c + a'c'}$

$\forall (a, a', c, c')$ , particularly  $(a, a', c, c') | ac' = \delta k^2 = \gamma m^2, a'c = \delta k'^2 = \gamma' m'^2$

$\Rightarrow (m - m') \frac{2}{2m' + 1}$

$= \frac{ac(m + m')(m - m') + 2(\gamma' - \gamma)m^2m'^2 + (2ac + ac' + a'c + 2a'c')(m - m') + 2\gamma' mm'^2 - 2\gamma m^2 m'}{(ac + 2ac')m^2 + (2ac + 3ac' + a'c + 2a'c')m + ac + ac' + a'c + a'c'}$

$= (m - m') \frac{-2acmm' + 2(\frac{\gamma' - \gamma}{m - m'})m^2m'^2 + 2ac + ac' + a'c + 2a'c' - 2(\frac{\gamma m - \gamma' m'}{m - m'})mm'}{(ac + 2ac')m^2 + (2ac + 3ac' + a'c + 2a'c')m + ac + ac' + a'c + a'c'}$

$m = m' \Rightarrow \beta = 1 \Rightarrow y = \frac{(1 - \beta)(p_1 + p_2)}{4} = 0$ , it is impossible

But  $\frac{2}{2k'+1} = \frac{-2ackk' + 2ac + ac' + a'c + 2a'c' - 2\delta k k'}{(ac + 2ac')k^2 + (2ac + 3ac' + a'c + 2a'c')k + ac + ac' + a'c + a'c'}$

For  $(a, a', c, c') | ac' = \delta k^2 = \gamma m^2, a'c = \delta k'^2 = \gamma' m'^2$

$\Rightarrow \frac{2}{2m' + 1} = \frac{-2acmm' + 2ac + ac' + a'c + 2a'c' - 2\gamma mm'}{(ac + 2ac')m^2 + (2ac + 3ac' + a'c + 2a'c')m + ac + ac' + a'c + a'c'}$

$= \frac{-2acmm' + 2ac + ac' + a'c + 2a'c' - 2\gamma mm'}{(ac + 2ac')m^2 + (2ac + 3ac' + a'c + 2a'c')m + ac + ac' + a'c + a'c'}$

$$\begin{aligned}
\Rightarrow \frac{2}{2m'+1} &= \frac{-2acmm'+2\left(\frac{\gamma'-\gamma}{m-m'}\right)m^2m'^2+2ac+ac'+a'c+2a'c'-2\left(\frac{\gamma m-\gamma'm'}{m-m'}\right)mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'} \\
&= \frac{-2acmm'+2ac+ac'+a'c+2a'c'-2\gamma mm'}{(ac+2ac')m'^2+(2ac+3ac'+a'c+2a'c')m'+ac+ac'+a'c+a'c'} \\
&= \frac{-2acmm'+2\left(\frac{\gamma-\gamma'}{m-m'}\right)m^2m'^2+2ac+ac'+a'c+2a'c'-2\gamma mm'}{(ac+2ac')k'^2+(2ac+3ac'+a'c+2a'c')k'+ac+ac'+a'c+a'c'} \\
\Rightarrow \gamma &= \gamma' \Rightarrow ac' = \delta k^2 = \gamma m^2, \quad a'c = \delta k'^2 = \gamma' m'^2 = \gamma m'^2 \\
\Rightarrow \frac{\delta}{\gamma} &= \frac{m^2}{k^2} = \frac{m'^2}{k'^2} \Rightarrow \frac{k^2}{k'^2} = (2k+1)^2 = \alpha^2 = \frac{m^2}{m'^2} = (2m+1)^2 = \beta^2
\end{aligned}$$

$$\text{If } \alpha = -\beta \Rightarrow (\beta - \alpha)p_1 = -2\alpha p_1 = 2\beta p_1 = (-2 - \alpha - \beta)p_2 = -2p_2 \Rightarrow \alpha = -\beta = \frac{p_2}{p_1}$$

$$b = -(\alpha + 1)\frac{p_1 - p_2}{4} = -(\beta + 1)\frac{p_1 + p_2}{4} = -\left(\frac{p_1 + p_2}{4p_1}\right)(p_1 - p_2) = \frac{p_2^2 - p_1^2}{4p_1}$$

$$\Rightarrow 4bp_1 = p_2^2 - p_1^2 \Rightarrow (4b + p_1)p_1 = p_2^2 \Rightarrow 4b + p_1 = \frac{p_2^2}{p_1} \quad \text{and it is impossible because } p_1 \text{ and } p_2 \text{ are primes}$$

$$\text{and } 4b + p_1 = \frac{p_2^2}{p_1} \text{ can not be an integer.}$$

$$\Rightarrow \alpha - \beta = 0 \Rightarrow (\beta - \alpha)p_1 = (-2 - \alpha - \beta)p_2 = -2(1 + \alpha)p_2 = -2(1 + \beta)p_2 = 0 \Rightarrow \alpha = \beta = -1$$

$$\Rightarrow b = -\frac{(\alpha + 1)}{4}(p_1 - p_2) = -\frac{(\beta + 1)}{4}(p_1 + p_2) = 0$$

$$\Rightarrow y = \frac{p_1 + p_2}{2}, \quad x = \frac{p_1 - p_2}{2}$$

$x + y = p_1, \quad y - x = p_2$  are primes  $y = r$  is the primal radius. As there is no condition, there is an infinity of  $p_1, p_2$

$$\text{If } (x_1 - x_2)(x_1 + x_3) = 0 \Rightarrow (x_4 + x_2)(x_4 - x_3) \neq 0$$

$$\text{Let } \frac{x_4 + x_2}{p_1 + p_2} = \frac{p_1 + p_2 - 4b}{p_1 + p_2} = 1 - \frac{4b}{p_1 + p_2}$$

$$\frac{p_1 + p_2}{x_4 + x_2} = \frac{x_4 + x_2 + 4b}{x_4 + x_2} = 1 + \frac{4b}{x_4 + x_2}$$

$$k = -\frac{2b}{p_1 + p_2}, \quad k' = \frac{2b}{x_4 + x_2}$$

$$\frac{x_4 - x_3}{p_1 - p_2} = \frac{p_1 - p_2 - 4b}{p_1 - p_2} = 1 - \frac{4b}{p_1 - p_2}$$

$$\frac{p_1 - p_2}{x_4 - x_3} = \frac{x_4 - x_3 + 4b}{x_4 - x_3} = 1 + \frac{4b}{x_4 - x_3}$$

$$m = -\frac{2b}{p_1 - p_2}, \quad m' = \frac{2b}{x_4 - x_3}$$

With the same calculus and reasoning, it implies that  $b = 0$ .

For  $2 = p_1 - p_2$  is a difference of an infinity of couples of primes. There is an infinity of consecutive primes. And for all  $x = 2104$ , exists  $p_1 = p_2 + 2$ ,  $p_2$  primes for which  $2x = p_1 + p_2$



## Conclusion

The notion of the primal radius as defined in this study allows to confirm that for all integer  $x \geq 3$  exists a number  $r > 0$  for which  $x + r$  and  $x - r$  are primes and that for all integer  $x \geq 0$  exists a number  $r > 0$  for which  $x + r$  and  $r - x$  are primes and that exists an infinity of such primes.  $r$  is called the primal radius The corollary is the proof of the Goldbach conjecture and de Polignac conjecture which stipulate, the first that an even number is always the sum of two prime numbers, the second that an even number is always the difference between two primes and that there is an infinity of such couples of primes. Another corollary is the proof of the twin primes conjecture which stipulates that there is an infinity of consecutive primes.

## References

- [1] J. R. Chen, 2002, On the representation of a larger even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica* 16, 157-176.
- [2] D. R. Heath-Brown, J. C. Puchta, 2002, Integers represented as a sum of primes and powers of two. *The Asian Journal of Mathematics*, 6, no. 3, pages 535-565.
- [3] H.L. Montgomery, Vaughan, R. C., 1975, The exceptional set in Goldbach's problem. *Collection of articles in memory of Jurii Vladimirovich Linnik. Acta Arith.* 27, 353–370.
- [4] J. Richstein, 2001, Verifying the goldbach conjecture up to  $4 \cdot 10^{14}$ , *Math. Comp.*, 70:236, 1745-1749.
- [5] L. E. Dickson, 2005, *History of The Theory of Numbers*, Vol1, New York Dover.