A proof of Beal’s conjecture
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Abstract
More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all ? Mihailescu's solution utilizes computation on machines, we propose here not really a proof of Catalan theorem as it is intended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. We generalize our approach to Beal equation and discuss the solutions. (Keywords: Diophantine equations, Catalan, Fermat-Catalan, Conjectures, Proofs, Algebraic resolution).

Introduction
Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, X>1 and Y>1, for which with exponents strictly greater than 1, p>1 and q>1, \( Y^p = X^q + 1 \) but for (X,Y,p,q)=(2,3,2,3), we can verify that \( 3^2 = 2^3 + 1 \). Euler has proved that the equation \( Y^2 = X^3 + 1 \) has this only solution. We propose in this study a general solution. The particular cases already solved concern p=2, solved by Ko Chao in 1965, and q=3 which has been solved in 2002. The case q=2 has been solved by Lebesgue in 1850. We solve here the equation and prove that Beal equation is related to this problem.

The proof

Catalan equation is

\[ Y^p = X^q + 1 \]

\[ X^p = cY^q + 1 \]

\[ c = \frac{7 - X^p}{Y^2} \]

\[ c' = \frac{6}{c + c'} \]

\[ X^q = 36 - (c + c')^2 \]

\[ X^p > 7 \Rightarrow c' < 0 \text{ and } c > c' \]

\[ c^2 > 1 \text{ and } c' > 1 \]

We have

\[ q + 1 = 2p \]

and will discuss two cases :

1) \( c^2 X^q > X^{2p} \)

We deduce \( q + 1 \leq 2p \)

\[ 2c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c + e)^2 - (7c + e)^2}{(c + e)^2} < 0 \]

But \( 2c^2 X^{q+1} \geq 2c^2 X^q \Rightarrow 2c^2 > X \)
Lemma:
If we have 
\[ c = \frac{X^p - 1}{a} \quad \text{and} \quad c' = \frac{7 \cdot a - X^p}{Y^2} \]

\[ Y^2 = \frac{6}{c + c'} \quad X^p = cY^2 + 1 \quad X^q = \frac{7 \cdot a + 1 \cdot a}{c + c'} \]

Then 
\[ c^2 X > 1 \quad X^2 > u^2 c^2 \]

Proof of the lemma:
We have if \( 49 \cdot a \cdot c^{4p} > 1 \)

\[ (c^{4p} X^{2p} - 1)(c + c')^2 = (7 \cdot a + 1 \cdot a c^{2p} - c + c')^2 \]

\[ = (49 \cdot a \cdot c^{4p} - 1)c^2 + (14 \cdot a \cdot c^{4p} - 2)cc' + (2 \cdot a \cdot c^{4p} - 1)c^2 \]

\[ > (49 \cdot a \cdot c^{4p} - 1)c^2 + (14 \cdot a \cdot c^{4p} - 2)cc' + (2 \cdot a \cdot c^{4p} - 1)c^2 \]

\[ = c'((50 \cdot a \cdot c^{4p} - 2)c' + (14 \cdot a \cdot c^{4p} - 2)c) \]

And 
\[ (50 \cdot a \cdot c^{4p} - 2)c' + (14 \cdot a \cdot c^{4p} - 2)c)c_1 Y^2 \]

\[ = (50 \cdot a \cdot c^{4p} - 2)(7 \cdot a - X^p) + (14 \cdot a \cdot c^{4p} - 2)(X^p - a) \]

\[ = -36 \cdot a \cdot c^{4p} X^p + 336 \cdot a \cdot a \cdot c^{4p} - 12a < 0 \]

With 
\[ \frac{X^p}{36} > 10a > \frac{336a}{36} \quad \text{then} \quad c^2 X > 1 \quad \text{and if} \quad 49 \cdot a \cdot c^{4p} < 1 \]

then 
\[ (c^{4p} X^{2p} - 1)(c + c')^2 = (7 \cdot a + 1 \cdot a c^{2p} - c + c')^2 \]

\[ = (49 \cdot a \cdot c^{4p} - 1)c^2 + (14 \cdot a \cdot c^{4p} - 2)cc' + (2 \cdot a \cdot c^{4p} - 1)c^2 > 0 \]

\[ \Delta' = ((7 \cdot a \cdot c^{4p} - 1) + (49 \cdot a \cdot c^{4p} - 1)(1 \cdot a \cdot c^{4p} - 1))c^2 = 36 \cdot a \cdot c^{4p} \]

\[ c Y^2 = X^p - a \quad c' Y^2 = \frac{1 - 49 \cdot a \cdot c^{4p} \pm 6 \cdot a \cdot c^{2p}}{49 \cdot a \cdot c^{4p} - 1}(7 \cdot a - X^p) \]

Thus 
\[ (50 \cdot a \cdot c^{4p} - 2 c') X^p > -6a \pm 42 \cdot a \cdot c^{2p} \quad \text{and two cases} \quad c > 1 \quad \text{then} \quad c^2 X > 1 \quad \text{and} \quad c < 1 \]

then 
\[ (42 \cdot a \cdot c^{4p} - 2 c') X^p < 42 \cdot a \cdot c^{4p} + 6 \cdot c^{2p} < a + 42 \cdot a \cdot c^{2p} \quad \text{thus} \quad c^2 X > 1 \quad \text{in all cases}. \]

We have also if 
\[ 49 \cdot a \cdot c^{4p} - (u^2 c^{2})^{2p} < 0 \]

\[ (X^{2p} - (u^2 c^{2})^{2p})(c + c')^2 = (7 \cdot a + 1 \cdot a c^{2p} - (u^2 c^{2})^{2p})(c + c')^2 \]

\[ = (49 \cdot a \cdot c^{4p} - (u^2 c^{2})^{2p})c^2 + (14 \cdot a \cdot c^{4p} - 2(u^2 c^{2})^{2p})cc' + (2 \cdot a \cdot c^{4p} - 1(u^2 c^{2})^{2p})c^2 > 0 \]

\[ \Delta = 36 \cdot a \cdot (u^2 c^{2})^{2p} \]

\[ c Y^2 = X^p - a \quad c' Y^2 = \frac{1 - 49 \cdot a \cdot (u^2 c^{2})^{2p} \pm 6 \cdot a \cdot (u^2 c^{2})^{2p}}{49 \cdot a \cdot (u^2 c^{2})^{2p} - 1}(7 \cdot a - X^p) \]
If $c<1$ then $X^2 > u^2c^2$ else $c>1$ then two cases

$$(42a_1 + 6x(u^2c^2)^p) X^p < (42a_1 + 6x(u^2c^2)^p)(u^2c^2)^p < 6a_1(u^2c^2)^2p + 42a_1(u^2c^2)^p$$
or

$$(42a_1 - 6x(u^2c^2)^p) X^p < 0 < 6a_1(u^2c^2)^2p - 42a_1(u^2c^2)^p$$
Else $X^{2p} > 49a_1 > (u^2c^2)^2p$  \(\Rightarrow X > u^2c^2\) thus $X^2 > u^2c^2$ in all cases

$$\Rightarrow X^{q+1} > u^2c^2 X^q > X^{2p} \geq X^{q+1}$$

Here

$$\Rightarrow 2c^2 X^q > X^{2p} \geq X^{q+1} \Rightarrow q+1 = 2p$$

$$\Rightarrow 2c^2 Y^p = 2(X^p - 1)^2 = X^{q+1} + X$$

$$\Rightarrow \frac{2}{X} = X^q + 1 - 2X^{2p-1} + 4X^{q-1} \in N \Rightarrow X = 2$$

$$\Rightarrow 2^{p+1} - 2^{2p-1} = 0 \Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm3$$

Now let

$$2) \quad c^2 > 1$$

$$2X^q - X^{2p} = \frac{72 - 2(c+c')^2 - (7c + c')^2}{(c+c')^2} > \frac{64 - (7c + c')^2}{(c+c')^2} > 0$$

We have

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c+c')^2 - (7c + c')^2}{(c+c')^2} > \frac{64c^2 - (7c + c')^2}{(c+c')^2} < 0$$

In virtue of the lemma

$$X > 2c^2 \Rightarrow c^2X > 2 \Rightarrow X^{q+1} > c^2X^q > 2X^{q+1}$$

$$\Rightarrow \frac{X^{q+1}}{2} > X^q \geq X^{q+1}$$

For Catalan equation, $q$ and $2p$ do not have the same parity thus $q+1 = 2p$. But for Catalan

$$0 > (c^2 - 1)X^p = X^{2p} - 2X^p - X^q$$

$$= X^{2p} - 2X^{2p-1} + X^{2p-1} \geq X^{2p-1} - 2X^{p-1} \geq 0$$

$$\Rightarrow (c^2 - 1)X^p = X^{2p} - 2X^p - X^q = 0$$

$$= X^{2p} - 2X^{2p-1} + X^{2p-1} = 0 \Rightarrow \frac{2}{X} = X^{p-1} - X^{p-2} \in N$$

$$\Rightarrow (X, Y, p, q)(= (2, \pm3, 2, 3)$$

**Generalization to Beal or Fermat-Catalan equation**

We have Fermat-Catalan equation $Y^p = X^q + Z^c$ with $X^q > Z^c > \nu X^q$ we pose

$$c = \frac{X^p - 1}{Y^p}$$

$$Y^p = X^q + 1\cdot a_1$$ and
with almost the same equalities than for Catalan but discuss two cases

1) \( c^2 > 1 \)

\[ X^q - X^{2p} = \frac{36a_1a - 1_1a_1c^2 - (7a_1 + 1a_1c)^2}{(c + c')^2} < \frac{36a_1a - 1a_1c^2}{(c + c')^2} < 0 \]

\( \Rightarrow q + 1 \leq 2p \) and with \( u^2 = \frac{64}{18} \) we have

\[ u^2c^2X^q - X^{2p} = \frac{u^236a_1a^2 - u^21_1a_1c^2(e + c')^2 - (7a_1 + 1a_1c)^2}{(c + c')^2} > \frac{64a_1c^2 - (7a_1 + 1a_1c)^2}{(c + c')^2} > 0 \]

2) \( \frac{c^2}{X} > 1 \)

The proof is the same than for Catalan, we prove then that in virtue of the lemma, \( X = u^2c^2X^q > X^{2p} \Rightarrow u^2 > X \) and we prove then that in virtue of the lemma, \( X^q - X^{2p} = u^2(1 - a_1a_1X) < 0 \) \( \Rightarrow X^q \leq (X - u^2)^{1_1a_1} \)

And for Fermat-Catalan the solutions are \( q + 1 = 2p \) and \( q + 2 = 2p \)

By the same way \( u^2c^2X^{2p-1} \geq u^2c^2X^q > X^{2p} \Rightarrow u^2 > X \) and we prove then that in virtue of the lemma, \( X = u^2c^2X^q > X^{2p} \Rightarrow u^2 > X \) and we prove then that in virtue of the lemma, \( X^q - X^{2p} = u^2(1 - a_1a_1X) < 0 \) \( \Rightarrow X^q \leq (X - u^2)^{1_1a_1} \)

Thus \( q + 1 \geq 2p \) and

\[ u^2X^q - X^{2p} = \frac{u^236a_1a^2 - u^21_1a_1c^2(e + c')^2 - (7a_1 + 1a_1c)^2}{(c + c')^2} > \frac{64a_1c^2 - (7a_1 + 1a_1c)^2}{(c + c')^2} > 0 \]

We prove by the same calculus than higher in virtue of the lemma, then that \( c^2 < X \Rightarrow X^{q+1} < u^2c^2X^q < X^{2p} \leq X^{q+1} \Rightarrow X^{q+1} \) thus \( 2p = q + 2; q + 1 = 2p \)

\( \Rightarrow (c^2 - 1)Y^p = X^{2p} - 2a_1X^p - X^q = 0 \)

\( \Rightarrow \frac{2}{X} = X^{q+1} - X^{q+1} \in N \)
Thus Beal equation implies \( q+1=2p \) or \( q+2=2p \) or \( q=2p \)! With Fermat equation there are solutions for \( q=p=n=1 \) or \( q=p=n=2 \). But if we take \( X^r = Y^p - 1_a 1_a \Rightarrow p = q = 2p - 2 = 2q - 2 = 2; \) \( p = q = 2p - 1 = 2q - 1 = 1 \) then \( p \) and \( q \) cannot be greater than 2, it means that there are not solutions for \( p>2 \) or \( q>2 \).

**Conclusion**

Catalan equation implies two other equations, and an original solution of Catalan equation exists. By the same method we have proved that Beal equation does not have solutions for the exponents greater than 2.

**The bibliography**