

## A proof of Beal's conjecture

Jamel Ghanouchi

RIME Dpt Mathematics

### Abstract

More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all ? Mihailescu's solution utilizes computation on machines, we propose here not really a proof of Catalan theorem as it is extended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. We generalize our approach to Beal equation and discuss the solutions. (Keywords: Diophantine equations, Catalan, Fermat-Catalan, Conjectures, Proofs, Algebraic resolution).

### Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2002 it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1,  $X > 1$  and  $Y > 1$ , which with exponents strictly greater than 1,  $p > 1$  and  $q > 1$ ,  $Y^p = X^q + 1$  but for  $(X, Y, p, q) = (2, 3, 2, 3)$ . We can verify that  $3^2 = 2^3 + 1$ . Euler has proved that the equation  $Y^2 = X^3 + 1$  has this only solution. We propose in this study a general solution. The particular cases already solved concern  $p=2$ , solved by Ko Chao in 1965, and  $q=3$  which has been solved in 2002. The case  $q=2$  has been solved by Lebesgue in 1850. We solve here the equation and prove that Beal equation is related to this problem.

### The proof

Catalan equation is  $Y^p = X^q + 1$  with  $p > 1$  and  $q > 1$  and  $c = \frac{X^q - 1}{Y^p - 1}$ ,  $c' = \frac{7 - X^p}{Y^2}$ , thus  $Y^{\frac{p}{2}} = \frac{6}{c + c'}$  and  $X^p = cY^{\frac{p}{2}} + 1 = \frac{7c + c'}{c + c'}$ ,  $X^q = \frac{36 - (c + c')^2}{(c + c')^2}$  but  $X^p > 7 \Rightarrow c' < 0$  and  $c > c'$

and  $Y^{\frac{p}{2}} \geq 3 \Rightarrow 0 < c + c' \leq 2$ . We will prove now that  $q + 1 = 2p$  and will discuss two cases :  $c^2 > 1$  and  $c^2 < 1$ .

1)  $c^2 > 1$

We have  $X^q - X^{2p} = \frac{36 - (c + c')^2 - (7c + c')^2}{(c + c')^2} < 36 \frac{1 - c^2}{(c + c')^2} < 0$  we deduce  $q + 1 \leq 2p$

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} > \frac{64c^2 - (7c + c')^2}{(c + c')^2} > 0$$

$$c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} < \frac{36c^2 - c^2(c + c')^2 - 36c^2}{(c + c')^2} < 0$$

But  $2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$

**Lemma :**

If we have

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} \text{ and } c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'}, X^p = cY^{\frac{p}{2}} + 1_a = \frac{7_a c + 1_a c'}{c + c'}, X^q = Y^p - 1_a 1_a = 1_a 1_a \frac{36 - (c + c')^2}{(c + c')^2}$$

$$c^2 X > 1 \quad X^2 > u^2 c^2$$

Then and

**Proof of the lemma :**

We have if  $49_a 1_a c^{4p} > 1$

$$\begin{aligned} (c^{4p} X^{2p} - 1)(c + c')^2 &= (7_a c + 1_a c')^2 c^{4p} - (c + c')^2 \\ &= (49_a 1_a c^{4p} - 1)c^2 + (14_a 1_a c^{4p} - 2)cc' + (1_a 1_a c^{4p} - 1)c'^2 \\ &> (49_a 1_a c^{4p} - 1)c'^2 + (14_a 1_a c^{4p} - 2)cc' + (1_a 1_a c^{4p} - 1)c'^2 \\ &= c'((50_a 1_a c^{4p} - 2)c' + (14_a 1_a c^{4p} - 2)c) \end{aligned}$$

And

$$\begin{aligned} &((50_a 1_a c^{4p} - 2)c' + (14_a 1_a c^{4p} - 2)c)Y^{\frac{p}{2}} \\ &= (50_a 1_a c^{4p} - 2)(7_a - X^p) + (14_a 1_a c^{4p} - 2)(X^p - 1_a) \\ &= -36_a 1_a c^{4p} X^p + 336_a 1_a 1_a c^{4p} - 12_a < 0 \end{aligned}$$

With  $X^p > 10_a > \frac{336_a}{36}$  then  $c^2 X > 1$  and if  $49_a 1_a c^{4p} < 1$  then

$$\begin{aligned} (c^{4p} X^{2p} - 1)(c + c')^2 &= (7_a c + 1_a c')^2 c^{4p} - (c + c')^2 \\ &= (49_a 1_a c^{4p} - 1)c^2 + (14_a 1_a c^{4p} - 2)cc' + (1_a 1_a c^{4p} - 1)c'^2 > 0 \end{aligned}$$

$$\Delta' = ((7_a 1_a c^{4p} - 1)^2 - (14_a 1_a c^{4p} - 2)(1_a 1_a c^{4p} - 1))c'^2 = 36_a 1_a c^{4p}$$

$$cY^{\frac{p}{2}} = X^p - 1_a \frac{1 - 1_a c^{4p} \pm 6_a c^{2p}}{49_a 1_a c^{4p} - 1} c' Y^{\frac{p}{2}} = \frac{1 - 7_a 1_a c^{4p} \pm 6_a c^{2p}}{49_a 1_a c^{4p} - 1} (7_a - X^p)$$

Thus  $(7_a c^{4p} - 42_a c^{2p})X^p > -6_a \mp 42_a 1_a c^{2p}$  and two cases  $c > 1$  then  $c^2 X > 1$  and  $c < 1$

then  $((7_a c^{4p} - 42_a 1_a c^{4p})X^p < 0 < 6_a - 42_a 1_a c^{2p}$  or

$$(42_a 1_a c^{4p} - 6_a c^{2p})X^p < 42_a 1_a c^{3p} + 6_a c^p < 6_a + 42_a 1_a c^{2p} \text{ thus } c^2 X > 1 \text{ in all cases.}$$

We have also if

$$49_a 1_a - (u^2 c^2)^{2p} < 0$$

$$\begin{aligned} (X^{2p} - (u^2 c^2)^{2p})(c + c')^2 &= (7_a c + 1_a c')^2 - (u^2 c^2)^{2p} (c + c')^2 \\ &= (49_a 1_a - (u^2 c^2)^{2p})c^2 + (14_a 1_a - 2(u^2 c^2)^{2p})cc' + (1_a 1_a - (u^2 c^2)^{2p})c'^2 > 0 \end{aligned}$$

$$\Delta' = 36_a 1_a (u^2 c^2)^p$$

$$cY^{\frac{p}{2}} = X^p - 1_a > \frac{(u^2 c^2)^{2p} - 7_a 1_a \pm 6_a (u^2 c^2)^p}{49_a 1_a - (u^2 c^2)^{2p}} c' Y^{\frac{p}{2}} = \frac{(u^2 c^2)^{2p} - 7_a 1_a \pm 6_a (u^2 c^2)^p}{49_a 1_a - (u^2 c^2)^{2p}} (7_a - X^p)$$

If  $c < 1$  thus  $X^2 > u^2 c^2$  else  $c > 1$  then two cases

$$(42_a 1_a + 6_a (u^2 c^2)^p) X^p < (42_a 1_a + 6_a (u^2 c^2)^p) (u^2 c^2)^p < 6_a (u^2 c^2)^{2p} + 42_a 1_a (u^2 c^2)^p \text{ or}$$

$$(42_a 1_a - 6_a (u^2 c^2)^p) X^p < 0 < 6_a (u^2 c^2)^{2p} - 42_a 1_a (u^2 c^2)^p \text{ Else } X^{2p} > 49_a 1_a > (u^2 c^2)^{2p} \Rightarrow X > u^2 c^2 \text{ thus}$$

$$X^2 > u^2 c^2 \text{ in all cases}$$

$$\Rightarrow X^{q+3} > u^4 X^{q+1} > u^2 c^2 X^q > X^{2p} \geq X^{q+1}$$

Here

$$\Rightarrow X^{q+1} > 2c^2 X^q > X^{2p} \geq X^{q+1} \Rightarrow q+1 = 2p$$

$$\Rightarrow 2c^2 Y^p = 2(X^p - 1)^2 = X^{q+1} + X$$

$$\Rightarrow \frac{2}{X} = X^q + 1 - 2X^{2p-1} + 4X^{p-1} \in N \Rightarrow X = 2$$

$$\Rightarrow 2^{p+1} - 2^{2p-1} = 0 \Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3$$

Now let

$$2) c^{-2} > 1$$

$$2X^q - X^{2p} = \frac{72 - 2(c+c')^2 - (7c+c')^2}{(c+c')^2} > \frac{64 - (7c+c')^2}{(c+c')^2} >$$

$$\Rightarrow X^{2p-q-1} < \frac{2}{X} < 1 \Rightarrow 2p \leq q+1$$

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} > \frac{64c^2 - (7c+c')^2}{(c+c')^2} > 0$$

We have

$$c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} < \frac{36c^2 - c^2(c+c')^2 - 36c^2}{(c+c')^2} < 0$$

In virtue of the lemma

$$X > 2c^{-2} \Rightarrow c^2 X > 2 \Rightarrow X^{2p} > c^{2p} > 2^{2p}$$

$$\Rightarrow \frac{X^{2p+1}}{2} > X^q \geq X^{2p-1}$$

For Catalan equation,  $q$  and  $2p$  do not have the same parity thus  $q+1=2p$ . But for Catalan

$$0 > (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q$$

$$= X^{2p} - 2X^p - X^{q-1} \geq Y^{p-1} - 2X^p \geq 0$$

$$\Rightarrow (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = 0$$

$$= X^{2p} - 2X^p - X^{2p-1} = 0 \Rightarrow \frac{2}{X} = X^{p-1} - X^{p-2} \in N$$

$$\Rightarrow (X, Y, p, q) = (2, \pm 3, 2, 3)$$

### Generalization to Beal or Fermat-Catlan equation

We have Fermat-Catalan equation  $Y^p = X^q + Z^c$  with  $X^q > Z^c > vX^q$  we pose

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}$$

$$Y^p = X^q + 1_a 1_a \text{ and}$$

$$c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}, Y^{\frac{p}{2}} = \frac{6_a}{c+c'}, X^p = cY^{\frac{p}{2}} + 1_a = \frac{7_a c + 1_a c'}{c+c'}, X^q = Y^p - 1_a 1_a = 1_a 1_a \frac{36 - (c+c')^2}{(c+c')^2}$$

$$Y^{\frac{p}{2}} = \frac{6_a}{c+c'} > \sqrt{2_a} \Rightarrow \frac{6}{\sqrt{2}} > c+c', \text{ we}$$

with almost the same equalities than for Catalan but discuss two cases

1)  $c^2 > 1$

$$X^q - X^{2p} = \frac{36_a 1_a - 1_a 1_a (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < 36_a 1_a \frac{1-c^2}{(c+c')^2} < 0$$

$\Rightarrow q+1 \leq 2p$  and with  $u^2 = \frac{64}{18}$  we have

$$u^2 c^2 X^q - X^{2p} = \frac{u^2 36_a 1_a c^2 - u^2 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} > \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c+c')^2} > 0$$

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < \frac{36_a 1_a c^2 - (7_a c + 1_a c')^2}{(c+c')^2} < 0$$

$$\frac{1}{4} u^2 c^2 X^q - X^{2p} < \frac{u^2 9_a 1_a c^2 - \frac{1}{2} u^2 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < \frac{9_a 1_a c^2 - 36_a 1_a c^2}{(c+c')^2} < 0$$

By the same way  $u^2 c^2 X^{2p-1} \geq u^2 c^2 X^q > X^{2p} \Rightarrow u^2 c^2 > X$  and we prove then that in virtue of the lemma,  $X = u^2 c^2 \Rightarrow q+1 = 2p$

$$\Rightarrow u^2 c^2 Y^p = u^2 (X^p - 1_a)^2 = X^{q+1} + 1_a 1_a X$$

$$\Rightarrow \frac{u^2 1_a 1_a - 1_a 1_a X}{X^p} = X^{q-p+1} - u^2 Y^p \cdot 2_a \in N \Rightarrow X^p \leq (X - u^2) 1_a 1_a$$

And for Fermat-Catalan the solution is  $q+1=2p$  and  $q+2=2p$

2)  $c^{-2} > 1$

The proof is the same than for Catalan, we prove that

$$u^2 X^q - X^{2p} = \frac{u^2 36_a 1_a - u^2 1_a 1_a (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} > \frac{64_a 1_a - (7_a c + 1_a c')^2}{(c+c')^2} > 0$$

Thus  $q+1 > 2p$  and

$$u^2 c^2 X^q - X^{2p} = \frac{u^2 36_a 1_a c^2 - u^2 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} > \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c+c')^2} > 0$$

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < \frac{36_a 1_a c^2 - (7_a c + 1_a c')^2}{(c+c')^2} < 0$$

$$\frac{1}{4} u^2 c^2 X^q - X^{2p} < \frac{u^2 9_a 1_a c^2 - \frac{1}{2} u^2 1_a 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2} < \frac{9_a 1_a u^2 c^2 - 36_a 1_a c^2}{(c+c')^2} < 0$$

We prove by the same calculus than higher in virtue of the lemma, then that

$$c^{-2} < X \Rightarrow u^2 X^{q-1} < u^2 c^2 X^q < 4X^{2p} \leq 4X^{q+1} < X^{q+2} \text{ thus } 2p = q+2; q+1 = 2p$$

$$\Rightarrow (c^2 - 1)Y^p = X^{2p} - 2_a X^p - X^q = 0$$

$$= X^{2p} - 2_a X^p - X^{2p-1} = 0 \Rightarrow \frac{2_a}{X} = X^{p-1} - X^{p-2} \in N$$

Thus Beal equation implies  $q+1=2p$  or  $q+2=2p$  or  $q=2p$  ! With Fermat equation there are solutions for  $q=p=n=1$  or  $q=p=n=2$ . But if we take

$X^q = Y^p - 1_a 1_a \Rightarrow p = q = 2p - 2 = 2q - 2 = 2; p = q = 2p - 1 = 2q - 1 = 1$  then  $p$  and  $q$  can not be greater than 2, it means that there are not solutions for  $p > 2$  or  $q > 2$ .

### Conclusion

Catalan equation implies two other equations, and an original solution of Catalan equation exists. By the same method we have proved that Beal equation does not have solutions for the exponents greater than 2.

### The bibliography

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