

Investigations on the Theory of Riemann Zeta Function II: On the Riemann-Siegel Integral and Hardy's Z-Function

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ABSTRACT. We create new formulas for Riemann-Siegel Integral and Hardy's Z-function.

1. INTRODUCTION

We define Hardy's Z-function to be

$$Z(t) := e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right), \quad (1.1)$$

where $\theta(t)$ is given by

$$\theta(t) := \Im\left(\log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{t}{2} \log \pi, \quad (1.2)$$

see [1, p. 47].

In this paper, we proved a new formula for Hardy's Z-function:

$$Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it}-1} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) \right], \quad (1.3)$$

and a new formula for Riemann-Siegel integral:

$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2} - \frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) \right] - 2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2} - \frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right) \sum_{n=1}^N \left(\frac{\cos(\theta(t)-t \log n)}{n^{\frac{1}{2}}}\right). \quad (1.3)$$

2. PRELIMINARES

THEOREM 1. Let $\text{Re}(s) > 0$ and $s \neq 1$, then

$$\zeta(s) = \frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1}, \quad (2.1)$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta(s, a)$ is the Hurwitz zeta function.

Proof. See 2. \square

COROLLARY 1. For $\text{Re}(s) > 0$, then

$$(2^s - 1) \Gamma\left(\frac{1-s}{2}\right) \pi^{s-1/2} \zeta(1-s) = \Gamma\left(\frac{s}{2}\right) \left[2^s + \zeta\left(s, \frac{3}{2}\right) \right], \quad (2.2)$$

and, for $0 < \text{Re}(s) < 1$, then

$$(2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1-s, \frac{3}{2}\right)\right], \quad (2.3)$$

where $\Gamma(s)$ is the gamma function, $\zeta(s)$ is the Riemann zeta function and $\zeta(s, a)$ is the Hurwitz zeta function.

Proof. See [2]. \square

THEOREM 2 (Approximate Functional Equation). *Let $x, y \in \mathbb{R}^+$ with $2\pi xy = |t|$; then for $s = \sigma + it$, with $0 \leq \sigma \leq 1$, we have*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + \mathcal{O}(x^{-\sigma}) + \mathcal{O}\left(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1}\right), \quad (2.1)$$

where $\chi(s)$ is given by

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s). \quad (2.2)$$

Proof. See [1, p. 47]. \square

For the Hardy's Z-function, we have the approximate

$$Z(t) = 2 \sum_{n=1}^{[x]} \frac{\cos(\theta(t) - t \log n)}{n^{\frac{1}{2}}} + \mathcal{O}\left(t^{-\frac{1}{4}}\right), \quad (2.3)$$

see [1, p. 48].

The Riemann-Siegel Formula. **THEOREM 3.** *For all $t \in \mathbb{R}$,*

$$Z(t) = 2 \sum_{n=1}^N \left(\frac{\cos(\theta(t) - t \log n)}{n^{\frac{1}{2}}} \right) + \frac{e^{-i\theta(t)} e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it} e^{-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)} \int_{C_N} \frac{(-x)^{\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx, \quad (2.4)$$

where C_N is positively oriented closed contour containing all of the points $\pm 2\pi iN, \pm 2\pi i(N-1), \dots, \pm 2\pi i$, and 0. See [1, p. 48].

3. LEMMA AND THEOREM

Hardy's Z-Function. **LEMMA 1.** *For all $t \in \mathbb{R}$, then*

$$Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) \right]. \quad (3.1)$$

Proof. Let $s = \frac{1}{2} + it$ in (2.1)

$$\zeta\left(\frac{1}{2} + it\right) = \frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it} - 1} + \frac{\zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)}{2^{\frac{1}{2}+it} - 1}. \quad (3.2)$$

Substituting (3.2) in (1.1), we find

$$Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) \right]. \square$$

The Riemann-Siegel Integral. THEOREM 4. For all $t \in \mathbb{R}$, then

$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2} - \frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right] - 2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2} - \frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right) \sum_{n=1}^N \left(\frac{\cos(\theta(t) - t \log n)}{n^{\frac{1}{2}}}\right), \quad (3.3)$$

where C_N is positively oriented closed contour containing all of the points $\pm 2\pi iN, \pm 2\pi i(N - 1), \dots, \pm 2\pi i$, and 0.

Proof. We set (3.3) in the left-hand side of (2.4)

$$\frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right] = 2 \sum_{n=1}^N \left(\frac{\cos(\theta(t) - t \log n)}{n^{\frac{1}{2}}}\right) + \frac{e^{-i\theta(t)} e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it} e^{-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)} \int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx, \quad (3.4)$$

thereby,

$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2} - \frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right] - 2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2} - \frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right) \sum_{n=1}^N \left(\frac{\cos(\theta(t) - t \log n)}{n^{\frac{1}{2}}}\right). \square$$

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