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Investigations on the Theory of Riemann Zeta Function II: On the Riemann-Siegel Integral and Hardy's Z-Function

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ABSTRACT. We create new formulas for Riemann-Siegel Integral and Hardy's Z-function.

1. INTRODUCTION

We define Hardy's Z-function to be

$$Z(t) := e^{i\theta(t)}\,\zeta\left(\tfrac{1}{2}+it\right), \tag{1.1}$$

where $\theta(t)$ is given by

$$\theta(t) := \Im\left(\log\Gamma\left(\tfrac{1}{4}+\tfrac{it}{2}\right)\right) - \tfrac{t}{2}\log\pi, \tag{1.2}$$

see [1, p. 47].

In this paper, we proved a new formula for Hardy's Z-function:

$$Z(t) = \frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it}-1}\left[2^{\frac{1}{2}+it} + \zeta\left(\tfrac{1}{2}+it, \tfrac{3}{2}\right)\right], \tag{1.3}$$

and a new formula for Riemann-Siegel integral:

$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it}e^{-Nx}}{e^x-1}\,dx = \frac{e^{2i\theta(t)}(2\pi)^{\frac{1}{2}+it}e^{\frac{t\pi}{2}-\frac{i\pi}{4}}\left(1-ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it}-1\right)}\left[2^{\frac{1}{2}+it}+\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)\right]$$

$$-2(2\pi)^{\frac{1}{2}+it}e^{i\theta(t)}e^{\frac{t\pi}{2}-\frac{i\pi}{4}}\left(1-ie^{-\frac{i\pi}{4}}\right)\sum_{n=1}^{N}\left(\frac{\cos(\theta(t)-t\log n)}{n^{\frac{1}{2}}}\right). \tag{1.3}$$

2. PRELIMINARES

THEOREM 1. *Let* $\operatorname{Re}(s) > 0$ *and* $s \neq 1$, *then*

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{\zeta\left(s,\frac{3}{2}\right)}{2^s-1}, \tag{2.1}$$

where $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s,a)$*is the Hurwitz zeta function.*

Proof. See 2. □

COROLLARY 1. *For* $\operatorname{Re}(s) > 0$, *then*

$$(2^s-1)\Gamma\left(\tfrac{1-s}{2}\right)\pi^{s-1/2}\,\zeta(1-s) = \Gamma\left(\tfrac{s}{2}\right)\left[2^s+\zeta\left(s,\tfrac{3}{2}\right)\right], \tag{2.2}$$

and, for $0 < Re(s) < 1$, *then*

$$(2^{1-s}-1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\,\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s}+\zeta\left(1-s,\frac{3}{2}\right)\right], \tag{2.3}$$

where $\Gamma(s)$ *is the gamma function,* $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s,a)$ *is the Hurwitz zeta function.*

Proof. See [2]. □

THEOREM 2 (Approximate Functional Equation). *Let* $x,y\in\mathbb{R}^{+}$ *with* $2\pi xy=|t|$; *then for* $s=\sigma+it$, *with* $0\leq\sigma\leq1$, *we have*

$$\zeta(s)=\sum_{n\leq x}\frac{1}{n^{s}}+\chi(s)\sum_{n\leq y}\frac{1}{n^{1-s}}+\mathcal{O}(x^{-\sigma})+\mathcal{O}\left(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}\right), \tag{2.1}$$

where $\chi(s)$ *is given by*

$$\chi(s)=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{2.2}$$

Proof. See [1, p. 47]. □

For the Hardy's Z-function, we have the approximate

$$Z(t)=2\sum_{n=1}^{[x]}\frac{\cos(\theta(t)-t\log n)}{n^{\frac{1}{2}}}+\mathcal{O}\left(t^{-\frac{1}{4}}\right), \tag{2.3}$$

see [1, p. 48].

The Riemann-Siegel Formula. THEOREM 3. *For all* $t\in\mathbb{R}$,

$$Z(t)=2\sum_{n=1}^{N}\left(\frac{\cos(\theta(t)-t\log n)}{n^{\frac{1}{2}}}\right)+\frac{e^{-i\theta(t)}e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it}e^{-\frac{i\pi}{4}}\left(1-ie^{-\frac{i\pi}{4}}\right)}\int_{C_N}\frac{(-x)^{-\frac{1}{2}+it}e^{-Nx}}{e^{x}-1}dx, \tag{2.4}$$

where C_N *is positively oriented closed contour containing all of the points* $\pm2\pi iN,\pm2\pi i(N-1),\dots,\pm2\pi i,$ *and* 0. See [1, p. 48].

3. LEMMA AND THEOREM

Hardy's Z-Function. LEMMA 1. *For all* $t\in\mathbb{R}$, *then*

$$Z(t)=\frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it}-1}\left[2^{\frac{1}{2}+it}+\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)\right]. \tag{3.1}$$

Proof. Let $s=\frac{1}{2}+it$ in (2.1)

$$\zeta\left(\frac{1}{2}+it\right)=\frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1}+\frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1}. \tag{3.2}$$

Substituting (3.2) in (1.1), we find

$$Z(t)=\frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it}-1}\left[2^{\frac{1}{2}+it}+\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)\right].\ \square$$

The Riemann-Siegel Integral. THEOREM 4. *For all $t \in \mathbb{R}$, then*

$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2}-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right]$$

$$-2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2}-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right) \sum_{n=1}^{N} \left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right), \tag{3.3}$$

where C_N is positively oriented closed contour containing all of the points $\pm 2\pi iN, \pm 2\pi i(N-1), \ldots, \pm 2\pi i,$ and 0.

Proof. We set (3.3) in the left-hand side of (2.4)

$$\frac{e^{i\theta(t)}}{2^{\frac{1}{2}+it} - 1} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right] = 2\sum_{n=1}^{N} \left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right)$$

$$+ \frac{e^{-i\theta(t)} e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it} e^{-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)} \int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx, \tag{3.4}$$

thereby,

$$\int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 1} dx = \frac{e^{2i\theta(t)} (2\pi)^{\frac{1}{2}+it} e^{\frac{t\pi}{2}-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right)}{\left(2^{\frac{1}{2}+it} - 1\right)} \left[2^{\frac{1}{2}+it} + \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right)\right]$$

$$-2(2\pi)^{\frac{1}{2}+it} e^{i\theta(t)} e^{\frac{t\pi}{2}-\frac{i\pi}{4}} \left(1 - ie^{-\frac{i\pi}{4}}\right) \sum_{n=1}^{N} \left(\frac{\cos(\theta(t) - t\log n)}{n^{\frac{1}{2}}}\right). \square$$

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