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Are There Infinitely Many twin Primes?

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ABSTRACT. We prove that are there infinitely many twin primes.

1. INTRODUCTION

One of the most famous and beautiful open problem in number theory is the twin prime conjecture: are there infinitely many twin primes?

In this paper, we demonstrate that yes, using an analytic proof by contradiction, Rosser's theorem and Dusart's inequality.

2. PRELIMINARES

The Rosser's theorem [1] states that p_n is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + \log\log n - 1 < \frac{p_n}{n} < \log n + \log\log n, \tag{2.1}$$

for $n \geq 6$.

3. THEOREM

THEOREM 1. *Are there infinitely many twin primes.*

Proof. We will use the *reductio ad impossibilem*. Hence, we assume, by hypothesis, which are there finitely many twin prime numbers.

Step 1. We let, by Rosser's theorem,

$$\log(n+1) + \log\log(n+1) - 1 < \frac{p_{n+1}}{n+1} < \log(n+1) + \log\log(n+1) \tag{3.1}$$

and

$$\log n + \log\log n - 1 + \frac{2}{n} < \frac{p_n+2}{n} < \log n + \log\log n + \frac{2}{n} \tag{3.2}$$

for $n \geq 6$.

Dividing (3.1) by (3.2), member by member, we have

$$\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n-1+\frac{2}{n}} < \frac{p_{n+1}}{p_n+2} \cdot \frac{n}{n+1} < \frac{\log(n+1)+\log\log(n+1)}{\log n+\log\log n+\frac{2}{n}}, \tag{3.3}$$

Multiplying (3.3) by $\frac{n+1}{n}$, we encounter

$$\frac{n+1}{n} \cdot \frac{\log(n+1) + \log\log(n+1) - 1}{\log n + \log\log n - 1 + \frac{2}{n}} < \frac{p_{n+1}}{p_n + 2} < \frac{n+1}{n} \cdot \frac{\log(n+1) + \log\log(n+1)}{\log n + \log\log n + \frac{2}{n}},$$

$$(n+1)\cdot\frac{\log(n+1)+\log\log(n+1)-1}{n\log n+n\log\log n-n+2}<\frac{p_{n+1}}{p_n+2}<(n+1)\cdot\frac{\log(n+1)+\log\log(n+1)}{n\log n+n\log\log n+2}, \tag{3.4}$$

From (3.4), we have

$$n+1<\frac{p_{n+1}}{p_n+2}\cdot\frac{n\log n+n\log\log n-n+2}{\log(n+1)+\log\log(n+1)-1} \tag{3.5}$$

and

$$\frac{p_{n+1}}{p_n+2}\cdot\frac{n\log n+n\log\log n+2}{\log(n+1)+\log\log(n+1)}<n+1. \tag{3.6}$$

Inverting (3.5) and (3.6), we obtain

$$\frac{p_n+2}{p_{n+1}}\cdot\frac{\log(n+1)+\log\log(n+1)-1}{n\log n+n\log\log n-n+2}<\frac{1}{n+1} \tag{3.7}$$

and

$$\frac{1}{n+1}<\frac{p_n+2}{p_{n+1}}\cdot\frac{\log(n+1)+\log\log(n+1)}{n\log n+n\log\log n+2}. \tag{3.8}$$

We set $n+1\to n$ in (3.7) and (3.8)

$$\frac{p_{n-1}+2}{p_n}\cdot\frac{\log n+\log\log n-1}{(n-1)\log(n-1)+(n-1)\log\log(n-1)-n+3}<\frac{1}{n}<\frac{p_{n-1}+2}{p_n}\cdot\frac{\log n+\log\log n}{(n-1)\log(n-1)+(n-1)\log\log(n-1)+2} \tag{3.9}$$

The summation in n from 6 to infinity, give us

$$\sum_{n=6}^{\infty}\frac{p_{n-1}+2}{p_n}\cdot\frac{\log n+\log\log n-1}{(n-1)\log(n-1)+(n-1)\log\log(n-1)-n+3}<\sum_{n=6}^{\infty}\frac{1}{n}<\sum_{n=6}^{\infty}\frac{p_{n-1}+2}{p_n}\cdot\frac{\log n+\log\log n}{(n-1)\log(n-1)+(n-1)\log\log(n-1)+2}. \tag{3.10}$$

Summing $\sum_{n=1}^{5}\frac{1}{n}$ in each member of (3.10), we find

$$\sum_{n=1}^{5}\frac{1}{n}+\sum_{n=6}^{\infty}\frac{p_{n-1}+2}{p_n}\cdot\frac{\log n+\log\log n-1}{(n-1)\log(n-1)+(n-1)\log\log(n-1)-n+3}$$

$$<\sum_{n=1}^{\infty}\frac{1}{n}<\sum_{n=1}^{5}\frac{1}{n}+\sum_{n=6}^{\infty}\frac{p_{n-1}+2}{p_n}\cdot\frac{\log n+\log\log n}{(n-1)\log(n-1)+(n-1)\log\log(n-1)+2},$$

so

$$H_5+\sum_{n=6}^{\infty}\frac{p_{n-1}+2}{p_n(n-1)}\cdot\frac{\log n+\log\log n-1}{\log(n-1)+\log\log(n-1)-\left(\frac{n-3}{n-1}\right)} \tag{3.11}$$

$$<\sum_{n=1}^{\infty}\frac{1}{n}<H_5+\sum_{n=6}^{\infty}\frac{p_{n-1}+2}{p_n(n-1)}\cdot\frac{\log n+\log\log n}{\log(n-1)+\log\log(n-1)+\frac{2}{n-1}},$$

where H_n denote the n-th harmonic number.

The maximum of $\frac{\log n+\log\log n}{\log(n-1)+\log\log(n-1)+\frac{2}{n-1}}$, for $n\in\mathbb{N}_{\geq 6}$, is $\lim_{n\to\infty}\frac{\log n+\log\log n}{\log(n-1)+\log\log(n-1)+\frac{2}{n-1}}=1$. In other words,

$$\max_{n\in\mathbb{N}_{\geq 6}} \frac{\log n+\log\log n}{\log(n-1)+\log\log(n-1)+\frac{2}{n-1}} \tag{3.12}$$

$$= \lim_{n\to\infty} \frac{\log n + \log\log n}{\log(n-1) + \log\log(n-1) + \frac{2}{n-1}} = 1.$$

The minimum of $\frac{\log n+\log\log n-1}{\log(n-1)+\log\log(n-1)-\left(\frac{n-3}{n-1}\right)}$, for $n \in \mathbb{N}_{\geq 6}$, is $\frac{\log 6+\log\log 6-1}{\log 5+\log\log 5-\frac{3}{5}} = 0.925696050889805$ ... In other words,

$$\min_{n\in\mathbb{N}_{\geq 6}} \frac{\log n+\log\log n-1}{\log(n-1)+\log\log(n-1)-\left(\frac{n-3}{n-1}\right)} \tag{3.13}$$

$$= \frac{\log 6 + \log\log 6 - 1}{\log 5 + \log\log 5 - \frac{3}{5}} = 0.925696050889805 \ldots =: C_1$$

From (3.11), (3.12) and (3.13), we encounter

$$H_5 + C_1 \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n(n-1)} = H_5 + C_1 \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)}\right] < \sum_{n=1}^{\infty} \frac{1}{n} < H_5 + \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n(n-1)} = H_5 + \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)}\right]. \tag{3.14}$$

In [2], we have

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \frac{1}{1-p_1^{-1}} \cdot \frac{1}{1-p_2^{-1}} \cdot \frac{1}{1-p_3^{-1}} \cdot \frac{1}{1-p_4^{-1}} \cdot \frac{1}{1-p_5^{-1}} \cdot \ldots \tag{3.15}$$

We can factor (3.15) in prime numbers that are not twin primes and twin primes without repetition in each member, that is, not a pair of twin primes, just every twin prime number. Explicitly, $\widetilde{\mathbb{P}}_{\text{twin}} = \{2,23,31,37,47,53,67,79, \ldots\}$ and $\mathbb{P}_{\text{twin}} = \{3,5,7,11,13,17,19,29,31, \ldots\}$. After, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}} \frac{1}{1-p^{-1}} \cdot \prod_{p\in\mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}}. \tag{3.16}$$

Substituting (3.16) in (3.14), we obtain

$$H_5 + C_1 \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)}\right] < \sum_{n=1}^{\infty} \frac{1}{n} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}} \frac{1}{1-p^{-1}} \cdot \prod_{p\in\mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}} < H_5 + \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)}\right]. \tag{3.17}$$

Multiplying both members of (3.17) by $\prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}}(1-p^{-1})$, then

$$\prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}} (1-p^{-1}) \cdot \left\{H_5 + C_1 \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)}\right]\right\} < \prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}} (1-p^{-1}) \cdot \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= \prod_{p\in\mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}} < \prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}} (1-p^{-1}) \cdot \left\{H_5 + \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)}\right]\right\}. \tag{3.18}$$

Step 2. We prove that the series

$$\sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \tag{3.19}$$

is divergent. By Dusart's inequality [2], we set

$$p_n < n\log n + n\log\log n\,, \tag{3.20}$$

for $n \geq 6$. Thus,

$$\sum_{n=6}^{\infty}\frac{2}{p_n(n-1)} = 2\sum_{n=6}^{\infty}\frac{1}{p_n(n-1)} \geq 2\sum_{n=6}^{\infty}\frac{1}{(n\log n + n\log\log n)(n-1)}$$

$$\geq 2\sum_{n=6}^{\infty}\frac{1}{2n\log n\,(n-1)} = \sum_{n=6}^{\infty}\frac{1}{n\log n\,(n-1)}$$

$$= \sum_{n=6}^{\infty}\left\{\frac{1}{n\log n} - \left[\frac{n}{n(n-1)\log n} - \frac{2}{n(n-1)\log n}\right]\right\}$$

$$= \sum_{n=6}^{\infty}\left[\frac{1}{n\log n} - \frac{1}{(n-1)\log n} + \frac{2}{n(n-1)\log n}\right]$$

$$= \sum_{n=6}^{\infty}\frac{1}{n\log n} - \sum_{n=6}^{\infty}\frac{1}{(n-1)\log n} + \sum_{n=6}^{\infty}\frac{2}{n(n-1)\log n}. \tag{3.21}$$

Note 1: is easy to check that $\sum_{n=6}^{\infty}\frac{1}{n\log n}$ is divergent by the integral test of convergence, $\sum_{n=6}^{\infty}\frac{1}{(n-1)\log n}$ is divergent by the integral test of convergence, and $\sum_{n=6}^{\infty}\frac{2}{n(n-1)\log n}$ is divergent by the root test of convergence. This demonstrate that the series on the left diverges.

Step 3. We prove that the series

$$\sum_{n=6}^{\infty}\frac{p_{n-1}}{p_n(n-1)} \tag{3.22}$$

is divergent. By Rosser's inequality [1], we get

$$n\log n + n\log\log n - n < \; p_n < n\log n + n\log\log n\,, \tag{3.23}$$

for $n \geq 6$. Thus,

$$\sum_{n=6}^{\infty}\frac{p_{n-1}}{p_n(n-1)} \geq \sum_{n=6}^{\infty}\frac{(n-1)\log(n-1) + (n-1)\log\log(n-1) - n + 1}{(n\log n + n\log\log n)(n-1)}$$

$$= \sum_{n=6}^{\infty}\frac{\log(n-1) + \log\log(n-1) - 1}{n\log n + n\log\log n} \geq \sum_{n=6}^{\infty}\frac{\log(n-1) + \log\log(n-1) - 1}{2n\log n}$$

$$= \sum_{n=6}^{\infty}\left[\frac{\log(n-1)}{2n\log n} + \frac{\log\log(n-1)}{2n\log n} - \frac{1}{2n\log n}\right]$$

$$= \frac{1}{2}\left[\sum_{n=6}^{\infty}\frac{\log(n-1)}{n\log n} + \sum_{n=6}^{\infty}\frac{\log\log(n-1)}{n\log n} - \sum_{n=6}^{\infty}\frac{1}{n\log n}\right]. \tag{3.24}$$

Note 2: is easy to verify that $\sum_{n=6}^{\infty}\frac{\log(n-1)}{n\log n}$ is divergent by the root test of convergence, $\sum_{n=6}^{\infty}\frac{\log\log(n-1)}{n\log n}$ is divergent by the root test of convergence, and that $\sum_{n=6}^{\infty}\frac{1}{n\log n}$ is divergent by the integral test of convergence. This demonstrate that the series on the left diverges.

Step 4. Note 3: is easy to verify that $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent by the integral test of convergence.

Step 5. By hypothesis, we note that $\prod_{p\in\mathbb{P}_{\text{twin}}}\frac{1}{1-p^{-1}}$ is convergent, since $\mathbb{P}_{\text{twin}}$ is finite. But, from (3.18), Conclusion 1, Conclusion 2 and Conclusion 3, we note that the left hand side and the right hand side of inequality (3.18) is divergent, independently if $\prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}}(1-p^{-1})$ is convergent or divergent; and the left hand side of equality also is divergent, independently if $\prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}}(1-p^{-1})$ is convergent or divergent. This is an absurd! The contradiction that we wanted is guaranteed. Thus, are there infinitely many twin prime numbers. □

4. OPEN QUESTION

The first author, Edigles Guedes, leave to Prof. Dr. Raja Rama Gandhi the following problem:

$$\prod_{p\in\widetilde{\mathbb{P}}_{\text{twin}}}(1-p^{-1})$$

is convergent or divergent? what its value asymptotic?

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