Are There Infinitely Many twin Primes?

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ABSTRACT. We prove that are there infinitely many twin primes.

1. INTRODUCTION

One of the most famous and beautiful open problem in number theory is the twin prime conjecture: are there infinitely many twin primes?

In this paper, we demonstrate that yes, using an analytic proof by contradiction, Rosser's theorem and Dusart's inequality.

2. PRELIMINARES

The Rosser's theorem [1] states that p_n is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + \log \log n - 1 < \frac{p_n}{n} < \log n + \log \log n, \quad (2.1)$$

for $n \geq 6$.

3. THEOREM

THEOREM 1. *Are there infinitely many twin primes.*

Proof. We will use the *reductio ad impossibile*. Hence, we assume, by hypothesis, which are there finitely many twin prime numbers.

Step 1. We let, by Rosser's theorem,

$$\log(n+1) + \log \log(n+1) - 1 < \frac{p_{n+1}}{n+1} < \log(n+1) + \log \log(n+1) \quad (3.1)$$

and

$$\log n + \log \log n - 1 + \frac{2}{n} < \frac{p_{n+2}}{n} < \log n + \log \log n + \frac{2}{n} \quad (3.2)$$

for $n \geq 6$.

Dividing (3.1) by (3.2), member by member, we have

$$\frac{\log(n+1) + \log \log(n+1) - 1}{\log n + \log \log n - 1 + \frac{2}{n}} < \frac{p_{n+1}}{p_{n+2}} \cdot \frac{n}{n+1} < \frac{\log(n+1) + \log \log(n+1)}{\log n + \log \log n + \frac{2}{n}}, \quad (3.3)$$

Multiplying (3.3) by $\frac{n+1}{n}$, we encounter

$$\frac{n+1}{n} \cdot \frac{\log(n+1) + \log \log(n+1) - 1}{\log n + \log \log n - 1 + \frac{2}{n}} < \frac{p_{n+1}}{p_{n+2}} < \frac{n+1}{n} \cdot \frac{\log(n+1) + \log \log(n+1)}{\log n + \log \log n + \frac{2}{n}},$$

$$(n+1) \cdot \frac{\log(n+1)+\log \log(n+1)-1}{n \log n+n \log \log n-n+2} < \frac{p_{n+1}}{p_{n+2}} < (n+1) \cdot \frac{\log(n+1)+\log \log(n+1)}{n \log n+n \log \log n+2}, \quad (3.4)$$

From (3.4), we have

$$n+1 < \frac{p_{n+1}}{p_{n+2}} \cdot \frac{n \log n+n \log \log n-n+2}{\log(n+1)+\log \log(n+1)-1} \quad (3.5)$$

and

$$\frac{p_{n+1}}{p_{n+2}} \cdot \frac{n \log n+n \log \log n+2}{\log(n+1)+\log \log(n+1)} < n+1. \quad (3.6)$$

Inverting (3.5) and (3.6), we obtain

$$\frac{p_{n+2}}{p_{n+1}} \cdot \frac{\log(n+1)+\log \log(n+1)-1}{n \log n+n \log \log n-n+2} < \frac{1}{n+1} \quad (3.7)$$

and

$$\frac{1}{n+1} < \frac{p_{n+2}}{p_{n+1}} \cdot \frac{\log(n+1)+\log \log(n+1)}{n \log n+n \log \log n+2}. \quad (3.8)$$

We set $n+1 \rightarrow n$ in (3.7) and (3.8)

$$\frac{p_{n-1}+2}{p_n} \cdot \frac{\log n+\log \log n-1}{(n-1) \log(n-1)+(n-1) \log \log(n-1)-n+3} < \frac{1}{n} < \frac{p_{n-1}+2}{p_n} \cdot \frac{\log n+\log \log n}{(n-1) \log(n-1)+(n-1) \log \log(n-1)+2} \quad (3.9)$$

The summation in n from 6 to infinity, give us

$$\sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n} \cdot \frac{\log n+\log \log n-1}{(n-1) \log(n-1)+(n-1) \log \log(n-1)-n+3} < \sum_{n=6}^{\infty} \frac{1}{n} < \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n} \cdot \frac{\log n+\log \log n}{(n-1) \log(n-1)+(n-1) \log \log(n-1)+2}. \quad (3.10)$$

Summing $\sum_{n=1}^5 \frac{1}{n}$ in each member of (3.10), we find

$$\begin{aligned} & \sum_{n=1}^5 \frac{1}{n} + \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n} \cdot \frac{\log n+\log \log n-1}{(n-1) \log(n-1)+(n-1) \log \log(n-1)-n+3} \\ & < \sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^5 \frac{1}{n} + \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n} \cdot \frac{\log n+\log \log n}{(n-1) \log(n-1)+(n-1) \log \log(n-1)+2}, \end{aligned}$$

so

$$\begin{aligned} & H_5 + \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n(n-1)} \cdot \frac{\log n+\log \log n-1}{\log(n-1)+\log \log(n-1)-\left(\frac{n-3}{n-1}\right)} \\ & < \sum_{n=1}^{\infty} \frac{1}{n} < H_5 + \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n(n-1)} \cdot \frac{\log n+\log \log n}{\log(n-1)+\log \log(n-1)+\frac{2}{n-1}}, \end{aligned} \quad (3.11)$$

where H_n denote the n -th harmonic number.

The maximum of $\frac{\log n+\log \log n}{\log(n-1)+\log \log(n-1)+\frac{2}{n-1}}$, for $n \in \mathbb{N}_{\geq 6}$, is $\lim_{n \rightarrow \infty} \frac{\log n+\log \log n}{\log(n-1)+\log \log(n-1)+\frac{2}{n-1}} =$

1. In other words,

$$\max_{n \in \mathbb{N}_{\geq 6}} \frac{\log n + \log \log n}{\log(n-1) + \log \log(n-1) + \frac{2}{n-1}} \quad (3.12)$$

$$= \lim_{n \rightarrow \infty} \frac{\log n + \log \log n}{\log(n-1) + \log \log(n-1) + \frac{2}{n-1}} = 1.$$

The minimum of $\frac{\log n + \log \log n - 1}{\log(n-1) + \log \log(n-1) - \frac{(n-3)}{(n-1)}}$, for $n \in \mathbb{N}_{\geq 6}$, is $\frac{\log 6 + \log \log 6 - 1}{\log 5 + \log \log 5 - \frac{3}{5}} = 0.925696050889805 \dots$ In other words,

$$\min_{n \in \mathbb{N}_{\geq 6}} \frac{\log n + \log \log n - 1}{\log(n-1) + \log \log(n-1) - \frac{(n-3)}{(n-1)}} \quad (3.13)$$

$$= \frac{\log 6 + \log \log 6 - 1}{\log 5 + \log \log 5 - \frac{3}{5}} = 0.925696050889805 \dots =: C_1$$

From (3.11), (3.12) and (3.13), we encounter

$$H_5 + C_1 \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n(n-1)} = H_5 + C_1 \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \right] < \sum_{n=1}^{\infty} \frac{1}{n} < H_5 + \sum_{n=6}^{\infty} \frac{p_{n-1}+2}{p_n(n-1)} = H_5 + \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \right]. \quad (3.14)$$

In [2], we have

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \frac{1}{1-p_1^{-1}} \cdot \frac{1}{1-p_2^{-1}} \cdot \frac{1}{1-p_3^{-1}} \cdot \frac{1}{1-p_4^{-1}} \cdot \frac{1}{1-p_5^{-1}} \cdot \dots \quad (3.15)$$

We can factor (3.15) in prime numbers that are not twin primes and twin primes without repetition in each member, that is, not a pair of twin primes, just every twin prime number. Explicitly, $\tilde{\mathbb{P}}_{\text{twin}} = \{2, 23, 31, 37, 47, 53, 67, 79, \dots\}$ and $\mathbb{P}_{\text{twin}} = \{3, 5, 7, 11, 13, 17, 19, 29, 31, \dots\}$. After, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} \frac{1}{1-p^{-1}} \cdot \prod_{p \in \mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}}. \quad (3.16)$$

Substituting (3.16) in (3.14), we obtain

$$H_5 + C_1 \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \right] < \sum_{n=1}^{\infty} \frac{1}{n} = \prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} \frac{1}{1-p^{-1}} \cdot \prod_{p \in \mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}} < H_5 + \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \right]. \quad (3.17)$$

Multiplying both members of (3.17) by $\prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1})$, then

$$\prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1}) \cdot \left\{ H_5 + C_1 \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \right] \right\} < \prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1}) \cdot \sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p \in \mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}} < \prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1}) \cdot \left\{ H_5 + \left[\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} + \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \right] \right\}. \quad (3.18)$$

Step 2. We prove that the series

$$\sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} \quad (3.19)$$

is divergent. By Dusart's inequality [2], we set

$$p_n < n \log n + n \log \log n, \quad (3.20)$$

for $n \geq 6$. Thus,

$$\begin{aligned} \sum_{n=6}^{\infty} \frac{2}{p_n(n-1)} &= 2 \sum_{n=6}^{\infty} \frac{1}{p_n(n-1)} \geq 2 \sum_{n=6}^{\infty} \frac{1}{(n \log n + n \log \log n)(n-1)} \\ &\geq 2 \sum_{n=6}^{\infty} \frac{1}{2n \log n (n-1)} = \sum_{n=6}^{\infty} \frac{1}{n \log n (n-1)} \\ &= \sum_{n=6}^{\infty} \left\{ \frac{1}{n \log n} - \left[\frac{n}{n(n-1) \log n} - \frac{2}{n(n-1) \log n} \right] \right\} \\ &= \sum_{n=6}^{\infty} \left[\frac{1}{n \log n} - \frac{1}{(n-1) \log n} + \frac{2}{n(n-1) \log n} \right] \\ &= \sum_{n=6}^{\infty} \frac{1}{n \log n} - \sum_{n=6}^{\infty} \frac{1}{(n-1) \log n} + \sum_{n=6}^{\infty} \frac{2}{n(n-1) \log n}. \end{aligned} \quad (3.21)$$

Note 1: is easy to check that $\sum_{n=6}^{\infty} \frac{1}{n \log n}$ is divergent by the integral test of convergence, $\sum_{n=6}^{\infty} \frac{1}{(n-1) \log n}$ is divergent by the integral test of convergence, and $\sum_{n=6}^{\infty} \frac{2}{n(n-1) \log n}$ is divergent by the root test of convergence. This demonstrate that the series on the left diverges.

Step 3. We prove that the series

$$\sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} \quad (3.22)$$

is divergent. By Rosser's inequality [1], we get

$$n \log n + n \log \log n - n < p_n < n \log n + n \log \log n, \quad (3.23)$$

for $n \geq 6$. Thus,

$$\begin{aligned} \sum_{n=6}^{\infty} \frac{p_{n-1}}{p_n(n-1)} &\geq \sum_{n=6}^{\infty} \frac{(n-1) \log(n-1) + (n-1) \log \log(n-1) - n + 1}{(n \log n + n \log \log n)(n-1)} \\ &= \sum_{n=6}^{\infty} \frac{\log(n-1) + \log \log(n-1) - 1}{n \log n + n \log \log n} \geq \sum_{n=6}^{\infty} \frac{\log(n-1) + \log \log(n-1) - 1}{2n \log n} \\ &= \sum_{n=6}^{\infty} \left[\frac{\log(n-1)}{2n \log n} + \frac{\log \log(n-1)}{2n \log n} - \frac{1}{2n \log n} \right] \\ &= \frac{1}{2} \left[\sum_{n=6}^{\infty} \frac{\log(n-1)}{n \log n} + \sum_{n=6}^{\infty} \frac{\log \log(n-1)}{n \log n} - \sum_{n=6}^{\infty} \frac{1}{n \log n} \right]. \end{aligned} \quad (3.24)$$

Note 2: is easy to verify that $\sum_{n=6}^{\infty} \frac{\log(n-1)}{n \log n}$ is divergent by the root test of convergence, $\sum_{n=6}^{\infty} \frac{\log \log(n-1)}{n \log n}$ is divergent by the root test of convergence, and that $\sum_{n=6}^{\infty} \frac{1}{n \log n}$ is divergent by the integral test of convergence. This demonstrate that the series on the left diverges.

Step 4. Note 3: is easy to verify that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the integral test of convergence.

Step 5. By hypothesis, we note that $\prod_{p \in \mathbb{P}_{\text{twin}}} \frac{1}{1-p^{-1}}$ is convergent, since $\mathbb{P}_{\text{twin}}$ is finite. But, from (3.18), Conclusion 1, Conclusion 2 and Conclusion 3, we note that the left hand side and the right hand side of inequality (3.18) is divergent, independently if $\prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1})$ is convergent or divergent; and the left hand side of equality also is divergent, independently if $\prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1})$ is convergent or divergent. This is an absurd! The contradiction that we wanted is guaranteed. Thus, are there infinitely many twin prime numbers. $\square$

4. OPEN QUESTION

The first author, Edigles Guedes, leave to Prof. Dr. Raja Rama Gandhi the following problem:

$$\prod_{p \in \tilde{\mathbb{P}}_{\text{twin}}} (1 - p^{-1})$$

is convergent or divergent? what its value asymptotic?

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