

Investigations on the Theory of Riemann Zeta Function I: New Functional Equation, Integral Representation and Laurent Expansion for Riemann's Zeta Function

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ABSTRACT. We developed a new functional equation and a new integral representation for the Riemann zeta function.

1. INTRODUCTION

Our main goal is the development of these formulas:

$$(2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1-s, \frac{3}{2}\right)\right], \quad (1.1)$$

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{3^{1-s}}{(2-2^{1-s})(s-1)} + \frac{2^{s-1}}{3^s(2^s-1)} + \frac{2}{2^s-1} \int_0^\infty \frac{\sin\left[s \tan^{-1}\left(\frac{2t}{3}\right)\right]}{\left[\left(\frac{3}{2}\right)^2 + t^2\right]^{s/2}} e^{2\pi t} dt, \quad (1.2)$$

and

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{1}{(s-1)(2^s-1)} + \frac{1}{2^s-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \gamma_n\left(\frac{3}{2}\right) (s-1)^n. \quad (1.3)$$

which converges more rapidly than the known Laurent expansion [see 5]:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^\infty \frac{(-1)^n}{n!} \gamma_n(1) (s-1)^n. \quad (1.3)$$

2. LEMMAS AND THEOREMS

LEMMA 1. For $n \in \mathbb{Z}_{\geq 1}$ and $\alpha \in \mathbb{R} - \{0,1\}$, then

$$\frac{1}{n} = \frac{\alpha}{(\alpha-1)n} - \frac{\alpha}{\alpha(\alpha-1)n} \quad (2.1)$$

Proof. We expand the right-hand side of (2.1)

$$\begin{aligned} \frac{\alpha}{(\alpha-1)n} - \frac{\alpha}{\alpha(\alpha-1)n} &= \frac{\alpha}{\alpha-1} \left(\frac{1}{n} - \frac{1}{\alpha n}\right) \\ &= \frac{\alpha}{(\alpha-1)} \left(\frac{\alpha-1}{\alpha n}\right) = \frac{1}{n}. \quad \square \end{aligned}$$

LEMMA 2. For $n \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{>1}$, then

$$\frac{1}{n} = \frac{1}{n-1} - \frac{1}{(n-1)n}. \quad (2.2)$$

Proof. Let $\alpha = n$, in Lemma 1. \square

THEOREM 1. Let $\text{Re}(s) > 0$ and $s \neq 1$, then

$$\zeta(s) = \frac{2^s}{2^{s-1}} + \frac{\zeta(s, \frac{3}{2})}{2^{s-1}}, \quad (2.3)$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta(s, a)$ is the Hurwitz zeta function.

Proof. For $\text{Re}(s) > 0$, then [see 1]

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \Rightarrow \zeta(s) - \frac{1}{1-2^{1-s}} = \frac{1}{1-2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^{s-1}} \cdot \frac{1}{n}. \quad (2.4)$$

Substituting (2.2) in (2.4), we obtain

$$\begin{aligned} \zeta(s) - \frac{1}{1-2^{1-s}} &= \frac{1}{1-2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^{s-1}} - \frac{1}{1-2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^s} \\ \zeta(s) + \frac{1}{1-2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^s} &= \frac{1}{1-2^{1-s}} + \frac{1}{1-2^{1-s}} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n^{s-1}}. \end{aligned} \quad (2.5)$$

On the other hand, in [2, p. 9], we encounter

$$\int_0^{\infty} e^{-nx} x^{s-1} dx = \frac{\Gamma(s)}{n^s}, \quad (2.6)$$

for $\text{Re}(s) > 0$ and $n > 0$.

We set (2.6) in (2.5)

$$\begin{aligned} \zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} \int_0^{\infty} e^{-nx} x^{s-1} dx \\ &= \frac{1}{1-2^{1-s}} + \frac{1}{(1-2^{1-s})\Gamma(s)} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}n}{n-1} \int_0^{\infty} e^{-nx} x^{s-1} dx, \\ \zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} e^{-nx} \right) x^{s-1} dx \\ &= \frac{1}{1-2^{1-s}} + \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}n}{n-1} e^{-nx} \right) x^{s-1} dx, \\ \zeta(s) - \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \log(e^{-x} + 1) e^{-x} x^{s-1} dx \\ &= \frac{1}{1-2^{1-s}} - \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \left[\frac{(e^{-x} + 1) \log(e^{-x} + 1) + e^{-x}}{e^x + 1} \right] x^{s-1} dx, \\ \zeta(s) - \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \log(e^{-x} + 1) e^{-x} x^{s-1} dx \\ &= \frac{1}{1-2^{1-s}} - \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \left[\frac{(e^{-x} + 1) \log(e^{-x} + 1)}{e^x + 1} \right] x^{s-1} dx \\ &\quad - \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{e^{-x} x^{s-1}}{e^x + 1} dx, \end{aligned}$$

$$\begin{aligned}
 \zeta(s) + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty \left[\frac{(e^{-x} + 1) \log(e^{-x} + 1)}{e^x + 1} \right] x^{s-1} dx \\
 - \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty \log(e^{-x} + 1) e^{-x} x^{s-1} dx \\
 = \frac{1}{1 - 2^{1-s}} - \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty \frac{e^{-x} x^{s-1}}{e^x + 1} dx, \\
 \zeta(s) + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty \left[\frac{(e^{-x} + 1) \log(e^{-x} + 1)}{e^x + 1} - \log(e^{-x} + 1) e^{-x} \right] x^{s-1} dx \\
 = \frac{1}{1 - 2^{1-s}} - \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty \frac{e^{-x} x^{s-1}}{e^x + 1} dx, \\
 \zeta(s) = \frac{1}{1 - 2^{1-s}} - \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty \frac{e^{-x} x^{s-1}}{e^x + 1} dx, \\
 \zeta(s) + \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{e^{-x} x^{s-1}}{e^x+1} dx = \frac{1}{1-2^{1-s}}. \tag{2.7}
 \end{aligned}$$

We calculate

$$\int_0^\infty \frac{e^{-x} x^{s-1}}{e^x+1} dx = 2^{-s} \Gamma(s) \left(\zeta(s) - \zeta\left(s, \frac{3}{2}\right) \right), \tag{2.8}$$

for $\text{Re}(s) > 0$.

Substituting (2.8) in (2.7), we have

$$\begin{aligned}
 \zeta(s) + \frac{1}{(1 - 2^{1-s})\Gamma(s)} \left\{ 2^{-s} \Gamma(s) \left[\zeta(s) - \zeta\left(s, \frac{3}{2}\right) \right] \right\} &= \frac{1}{1 - 2^{1-s}}, \\
 \zeta(s) + \frac{\zeta(s) - \zeta\left(s, \frac{3}{2}\right)}{2^s - 2} &= \frac{1}{1 - 2^{1-s}}, \\
 \left(\frac{2^s - 1}{2^s - 2}\right) \zeta(s) &= \frac{1}{1 - 2^{1-s}} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 2}, \\
 \left(\frac{2^s - 1}{2^s - 2}\right) \zeta(s) &= \frac{2^s}{2^s - 2} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 2}, \\
 (2^s - 1) \zeta(s) &= 2^s + \zeta\left(s, \frac{3}{2}\right), \\
 \zeta(s) &= \frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1}. \square
 \end{aligned}$$

COROLLARY 1. Let $\zeta(s) = 0$ and $\text{Re}(s) > 0$, then

$$2^s + \zeta\left(s, \frac{3}{2}\right) = 0 \tag{2.9}$$

holds.

Proof. If we assume that $\zeta(s) = 0$, in Theorem 1, then

$$\frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1} = 0 \Rightarrow 2^s + \zeta\left(s, \frac{3}{2}\right) = 0. \square$$

The Theorem 2 can be rewritten as follows:

COROLLARY 1.a. For $\text{Re}(s) > 0$, all the roots of $\zeta(s)$ satisfies the equation

$$2^s + \zeta\left(s, \frac{3}{2}\right) = 0. \quad (2.10)$$

COROLLARY 2. For $\text{Re}(s) > 0$, then

$$(2^s - 1)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\left[2^s + \zeta\left(s, \frac{3}{2}\right)\right], \quad (2.11)$$

and, for $0 < \text{Re}(s) < 1$, then

$$(2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1-s, \frac{3}{2}\right)\right], \quad (2.12)$$

where $\Gamma(s)$ is the gamma function, $\zeta(s)$ is the Riemann zeta function and $\zeta(s, a)$ is the Hurwitz zeta function.

Proof. In [1] and [3, p. 136-144], we have

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s) \Rightarrow \zeta(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\pi^{s-1/2}\zeta(1-s), \quad (2.12)$$

for all complex s , proved by Riemann (1859).

Substituting (2.12) in (2.3)

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\pi^{s-1/2}\zeta(1-s) = \frac{2^s}{2^s-1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s-1} \Rightarrow (2^s - 1)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\left[2^s + \zeta\left(s, \frac{3}{2}\right)\right]. \quad (2.13)$$

Let $1-s \rightarrow s$ in (2.13)

$$(2^{1-s} - 1)\Gamma\left(\frac{s}{2}\right)\pi^{-s+1/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\left[2^{1-s} + \zeta\left(1-s, \frac{3}{2}\right)\right]. \square$$

THEOREM 2. Let $\text{Re}(s) > 1$, then

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{3^{1-s}}{(2-2^{1-s})(s-1)} + \frac{2^{s-1}}{3^s(2^s-1)} + \frac{2}{2^s-1} \int_0^\infty \frac{\sin\left[s \tan^{-1}\left(\frac{2t}{3}\right)\right]}{\left[\left(\frac{3}{2}\right)^2 + t^2\right]^{s/2}} \frac{dt}{e^{2\pi t} - 1}, \quad (2.14)$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. In [4], we encounter the Abel-Plana formula for Hurwitz zeta function, that is,

$$\zeta(s, \alpha) = \sum_{n=0}^\infty \frac{1}{(n+\alpha)^s} = \frac{\alpha^{1-s}}{s-1} + \frac{1}{2\alpha^s} + 2 \int_0^\infty \frac{\sin\left(s \tan^{-1}\left(\frac{t}{\alpha}\right)\right)}{(\alpha^2+t^2)^{s/2}} \frac{dt}{e^{2\pi t} - 1}. \quad (2.15)$$

Let $\alpha = \frac{3}{2}$ in (2.15)

$$\zeta\left(s, \frac{3}{2}\right) = \frac{3^{1-s}}{2^{1-s}(s-1)} + \frac{2^{s-1}}{3^s} + 2 \int_0^\infty \frac{\sin\left[s \tan^{-1}\left(\frac{2t}{3}\right)\right]}{\left[\left(\frac{3}{2}\right)^2 + t^2\right]^{s/2}} \frac{dt}{e^{2\pi t} - 1}. \quad (2.16)$$

Substituting (2.16) in (2.3), we complete the proof. \square

THEOREM 3. Let $\text{Re}(s) > 1$, then

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{1}{(s-1)(2^s-1)} + \frac{1}{2^s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n \left(\frac{3}{2}\right) (s-1)^n, \quad (2.17)$$

where $\zeta(s)$ is the Riemann zeta function and $\gamma_n(\alpha)$ are the Stieltjes constants.

Proof. In [5], we find the Laurent series expansion of the Hurwitz zeta function:

$$\zeta(s, \alpha) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(\alpha) (s-1)^n. \quad (2.18)$$

Let $\alpha = \frac{3}{2}$ in (2.18)

$$\zeta\left(s, \frac{3}{2}\right) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n\left(\frac{3}{2}\right) (s-1)^n. \quad (2.19)$$

Substituting (2.19) in (2.3), we complete the proof. \square

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