

## ABOUT SOME TRANSCENDENTAL NUMBERS

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**Abstract.** Mathematicians of the XIX century have proved that  $\pi$ ,  $e$  and the Champernowne number are transcendentals, but what about  $\pi+e$  or  $\pi e$ ? The Riemann hypothesis can it be generalized? In this paper, we demonstrate a method in order to know if it is possible to solve these two problems.

### The approach of the first problem

A number is transcendental if it is not the root of a polynomial equation :

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$  where the coefficients are rationals and different of zero. Otherwise it would be algebraic. We know since the XIX century that  $\pi$ ,  $e$  and the Champernowne number are transcendentals but we do not know anything about their sum, their difference or their product. Effectively if  $A$  is algebraic and  $B$  is transcendental,  $A+B$  and  $AB$  are transcendentals, but if  $A$  is also transcendental we still do not know the nature of  $AB$ ,  $A-B$  or  $A+B$ . Let us try to solve this problem.

### Definition

A real number is compound if it can be written as  $\prod_j p_j^{n_j}$  where  $p_j$  are integer primes and  $n_j$  are rationals. This decomposition in prime factors is unique. A prime real number can be written

only as  $p = p \cdot 1$ . Thus we define other real prime numbers like  $\pi$ ,  $e$ ,  $\ln(2)$ . Thus  $\sqrt[q]{p} = p^{\frac{1}{q}}$  is

compound. Also  $\sqrt[q]{p} + 1 = p^{\frac{1}{q}} + 1$  is prime and we have

$\sqrt[q]{p} - 1 = (p-1)(\sqrt[q]{p} + 1)^{-1} (\sqrt[q]{p^2} + 1)^{-1} \dots (\sqrt{p} + 1)^{-1}$  compound, for example.

### Theorem

If  $T$  and  $T'$  are transcendental prime numbers then  $T+T'$ ,  $T-T'$  and  $TT'$  are transcendentals.

### Proof of the theorem

Let  $C$  and  $C'$  two compound transcendental numbers. We have 4 possibilities.

- 1)  $CC^m, C + mC'$  are algebraics.
- 2)  $CC^m$  is algebraic and  $C + mC'$  is transcendental.
- 3)  $CC^m$  is transcendental and  $C + mC'$  is algebraic.
- 4)  $CC^m, C + mC'$  are transcendentals.

Thus

- 1)  $CC^m + mC^{m+1} = (C + mC')C^m = A + mC^{m+1} = A'C^m$  where  $A$  and  $A'$  are algebraics and  $C'$  is solution of an algebraic equation, it is impossible !
- 2)  $CC^{n'}, CC^{n''}$  are algebraics and  $CC^{n'}(CC^{n''})^{-1} = C^{n'-n''}$  is algebraic if  $n=n''$ . There is only one  $n=M$  for which  $CC^{n'}$  is algebraic, all the others are transcendentals. If  $M$  is not unique, there are three possibilities :  $CC^{n'}(C'C^{m'})^n$  is transcendental for all  $n$  or  $CC^{n'-1}(C'C^{m'-1})^n$  is transcendental for all  $n$  or there exists  $L, L'$  for which  $CC^{L'M} = A, C^{L''M} = A', CC^{L'}(C'C^{L''})^{L'} = A''$ ,  $CC^{L'-1}(C'C^{L''-1})^{L'} = A'''$  are algebraics.

$$A^2 C^{L+L'-2M} C^{mL-L'} A^n A^m$$

$$A^{l^2} C^{L-L'} C^{mL+L'-2M'} A^n A^{m-1}$$

$$C^{l^X} BC^{mY}$$

$$C^{l^{-Y}} DC^{mZ}$$

$$X = L' + L - 2M$$

$$Y = L' - L$$

$$Z = -(L + L' - 2M')$$

$$B = A^n A^m A^{-2}$$

$$D = A^{l^2} A^{m-1} A^m$$

And

$$AA' C^{L-M} C^{mL-M'} A^n$$

$$AA'^{-1} C^{L'-M} C^{mM'-L'} A^m$$

$$C^{l^U} EC^{mV}$$

$$C^{l^{U'}} FC^{mV'}$$

$$U = L' - M$$

$$V = M' - L$$

$$U' = L' - M'$$

$$V' = L' - M'$$

$$E = A^n A^{-1} A'^{-1}$$

$$F = A^m A^{-1} A'$$

$$C^{l^{-XY}} = B^{-Y} C^{m^{-Y^2}} = D^X C^{mXZ} \Rightarrow -Y^2 = -(L-L')^2 = XZ = -(L+L'-2M)(L+L'-2M')$$

$$C^{l^{UV'}} = E^{V'} C^{m^{V'V'}} = F^{-V'} E^{V'} C^{l^{UV'}} \Rightarrow U'V = UV' = (L-M)(M'-L) = (L'-M)(L'-M')$$

$$C^{l^{XU}} = E^Y C^{m^{Y^2}} = E^Y B^{-Y} C^{l^{XY}} \Rightarrow YU = XV = (L-M)(L'-L) = (M'-L)(L'+L-2M)$$

$$C^{l^{ZU}} = E^Z C^{m^{Z^2}} = E^Z D^{-Z} C^{l^{-ZY}} \Rightarrow ZU = -VY = -(L+L'-2M')(L'-M) = -(L'-L)(L'-M')$$

$$\Rightarrow L^2 + L'^2 - 2LL' = L^2 + L'^2 + 2LL' - 2(M'+M)(L+L') + 4MM'$$

$$\Rightarrow M^2 + M'^2 - 2MM' = M^2 + M'^2 + 2MM' - 2(M'+M)(L+L') + 4LL'$$

$$\Rightarrow (M-M')^2 = (M+M'-2L)(M+M'-2L')$$

Thus

$$(L-M)(L'-M') = (M'-L)(L'+L-2M) + (M-L')(L'-M')$$

$$(L'-M)(L'-M') = (M'-L)(L'-M) + (M'-L)(L-M)$$

$$(L'-M)(L+L'-M-M') = (M'-L)(L-M) = (L'-M)(L'-M')$$

$$\Rightarrow (L'-M)(L-M) = 0 \Rightarrow (M'-L)(M'-L') = 0$$

We deduce

$M=M'=L=L'$  is unique !

3)  $C + mC' - (C + m'C') = (m-m')C' \Rightarrow m = m'$  because this number is algebraic at this condition. There is only one  $m=N$  for which  $C+mC'$  is algebraic, all the others are transcendentals. If  $N$  is not unique there are three possibilities :

$C+C''+m(C'+C''')$  is transcendental for all  $m$ ,  $C-C''+m(C'-C''')$  is transcendental for all  $m$  or there exists  $L, L'$  for which  $C+NC'=A$ ,  $C''+N'C'''=A'$ ,  $C+C''+L(C'+C''')=A''$ ,  $C-C''+L'(C'-C''')=A'''$  are algebraics, thus

$$A''+A'''=2A+(L+L'-2N)C'+(L-L')C'''$$

$$A'' - A''' = 2A' + (L - L')C' + (L + L' - 2N')C'''$$

$$A'' = A + A' + (L - N)C' + (L - N')C''$$

$$A''' = A - A' + (L' - N)C' + (N' - L')C''$$

Thus  $(L - L')^2 = (L + L' - 2N)(L + L' - 2N')$  and  $(L - N)(N' - L) = (L' - N)(L' - N')$  and  $(L - N)(L' - L) = (N' - L)(L' + L - 2N)$  and  $-(L + L' - 2N')(L' - N) = -(L' - L)(L' - N')$  by the same calculus than higher  $N = N' = L = L'$  is unique !

Thus, there are finally two possibilities

I) There are three subpossibilities :  $CC''(C''C'''' )^n$  is transcendental for all n, for all T, T' prime transcendental numbers, there exist  $C = TC''^{-1}, C' = T' C''''^{-1} \Rightarrow TT''^n$  is transcendental for all n, particularly n=1 and TT' and T+T' are transcendentals. Or  $CC''^{-1}(C' C''''^{-1})^n$  is transcendental for all n, for all T, T' prime transcendental numbers, there exist  $C = TC''^{-1}, C''^{-1} = T' C''''^{-1} \Rightarrow TT''^n$  is transcendental for all n, particularly n=1 and TT' and T+T' are transcendentals.

Third subpossibility :  $CC''^n$  for all  $n \neq M$  for all T, T' prime transcendental numbers there exist  $1 \leq C = T^{1/2}, C' = T'^{1/n} \Rightarrow TT'$  transcendental and T+T' transcendental.

II) There are three subpossibilities  $C + C''^{+m}(C' + C'''' )$  is transcendental for all n, for all T, T' prime transcendental numbers, there exist  $C = T - C'', C' = T' - C''''$  and T+mT' is transcendental for all m, particularly m=1 and T+T' and TT' are transcendentals. Or  $CC''^m + m(C' - C'''' )$  is transcendental for all m, for all T, T' prime transcendental numbers, there exist  $C = T + C'', C' = T' + C''''$  and T+mT' is transcendental for all m, particularly m=1 and T+T' and TT' are transcendentals. Third subpossibility :  $C + mC'$  is transcendental for all  $m \neq N$ . For all T, T' prime transcendental numbers, there exist  $2C = T, mC' = T'$  and T+T' and TT' are transcendentals.

**The theorem application**

$T = \pi, T' = e$  implice that the sum and the produt of p and e are transcendentals.

**The approach of Riemann hypothesis**

$$\zeta(z) = \sum_{t=1}^{\infty} \frac{1}{t^z}$$

The Riemann hypothesis states that the non trivial zeros of the Riemann zeta function

lie on the critical line  $\frac{1}{2} + iy$ . We have

$$\zeta(z) = \sum_{t=1}^{\infty} \frac{1}{t^z} = \prod_{primes} \frac{1}{1 - p^{-z}}$$

For t integer, Euler has proved that it is still true and it becomes

$\prod_{primes} \frac{1}{1 - p^{-z}} = \int_1^{\infty} \frac{dt}{t^z} = \frac{1}{1-z} [t^{1-z}]_1^{\infty}$  But there are the trivial zeros : we have

$\zeta(-2k) = 0, \forall k \in N$  and  $[t^{1+2k}]_1^{\infty} = 0$  but if  $[t^z]_{1}^{\infty}$  but if  $[t^z]_{1}^{\infty}$  is the limit in the

infinity,  $[t^{1+2k}]_{1}^{\infty} = 1, \forall k \in N$  and

$$\left[ t^{1-\frac{1}{2}+iy} \right]_{1}^{\infty} = \left[ t^{\frac{1}{2}-iy} \right]_{1}^{\infty} = \left[ t^{\frac{1}{2}+iy} \right]_{1}^{\infty} = \frac{1}{2} \left( \left[ t^{\frac{1}{2}-iy} \right]_{1}^{\infty} + \left[ t^{\frac{1}{2}+iy} \right]_{1}^{\infty} \right) = a$$

$$\Rightarrow \left[ t^{2(\frac{1}{2}+iy)} - 2at^{\frac{1}{2}+iy} + t \right]_{1}^{\infty} = \left[ t^{2(\frac{1}{2}+iy)} - 2at^{\frac{1}{2}+iy} + 1 \right]_{1}^{\infty} = 0 \Rightarrow \left[ t^{\frac{1}{2}+iy} \right]_{1}^{\infty} = a = a + \sqrt{a^2 - 1} = 1$$

$$\zeta\left(\frac{1}{2} + iy\right) = \left[ \frac{1}{1 - \frac{1}{2} - iy} t^{1 - \frac{1}{2} - iy} \right]_1^\infty = 0$$

it means that

$$\zeta(x + iy) = 0 = \left[ \frac{t^{1-x-iy}}{1-x-iy} \right]_1^\infty = \left[ \frac{t^{\frac{1-iy}{2}} t^{\frac{1-x}{2}}}{1-x-iy} \right]_1^\infty = \left[ \frac{t^{\frac{1-x}{2}}}{1-x-iy} \right]_1^\infty = 0 \Rightarrow x = \frac{1}{2}$$

Let now

We have proved that the non trivial zeros of the Riemann function for the reals lie in the critical line ! So the hypothesis for the real numbers is proved. The Riemann hypothesis is important because it gives information about the zeros of the Riemann function and the distribution of those zeros are related to real primes !

### Conclusion

Through this exposé, we have given a method to find the nature of several numbers, we have shown the nature of some of them. We have also generalized the Riemann hypothesis to the real numbers and proved it !

### References

- [1] Alan Baker, Transcendental number theory, Cambridge university press, 1975
- [2] Karl Sabbagh, The Riemann hypothesis, the greatest unsolved problem in mathematics, Farrar, Straus and Giroux, 2004.