

About some transcendental numbers

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Abstract. Everyone knows that π , e or the Champernowne number are transcendentals, but what about $\pi + e$ or πe ? In this paper, we demonstrate a method in order to know if they also are.

The approach

A number is transcendental if it is not the root of a polynomial equation

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$ where the coefficients are rationals and different of zero. Otherwise it would be algebraic. We know that π , e and the Champernowne number are transcendentals but we do not know anything about their sum or their product. Effectively if A is algebraic B is transcendental, $A+B$ and AB are transcendentals, but if A is also transcendental we still do not know the nature of AB or $A+B$. Let us try to solve this problem.

Definition

A real number is composed if it can be written as $\prod_j p_j^{n_j}$ where p_j are integer primes and n_j are rationals. A prime real number can be written only as $p=p.1$. Thus we define other real prime numbers like π , e , $\ln(2)$. Thus $\sqrt[q]{p} = p^{\frac{1}{q}}$ is composed. Also $\sqrt[q]{p} + 1 = p^{\frac{1}{q}} + 1$ is prime and we have $\sqrt[q]{p} - 1 = (p-1)(\sqrt[q]{p} + 1)^{-1} (\sqrt[q]{p^2} + 1)^{-1} \dots (\sqrt[q]{p} + 1)^{-1}$ composed, for example.

Theorem

If T and T' are transcendental prime numbers then $T+T'$ and TT' are transcendentals.

Proof of the theorem

Let C and C' two composed transcendental numbers. We have 4 possibilities.

- 1) $CC^m, C+mC'$ are algebraics.
- 2) CC^m is algebraic and $C+mC'$ is transcendental.
- 3) CC^m is transcendental and $C+mC'$ is algebraic.
- 4) $CC^m, C+mC'$ are transcendentals.

Thus

- 1) $CC^m + mC^{m+1} = (C+mC')C^m = A+mC^{m+1} = A'C^m$ where A and A' are algebraics and C' is solution of an algebraic equation, it is impossible !
- 2) $CC^m, CC^{m'}$ are algebraics and $CC^m (CC^{m'})^{-1} = C^{m-n'}$ is algebraic if $n=n'$. There is only one $n=M$ for which CC^m is algebraic, all the others are transcendentals. If M is not unique, there are three possibilities : $CC^n (C'C^m)^n$ is transcendental for all n or $CC^{n-1} (C'C^{m-1})^n$ is transcendental for all n or there exists L, L' for which

$CC^{LM} = A, C^n C^{mM'} = A', CC^n (C'C^m)^L = A'', CC^{n-1} (C'C^{m-1})^{L'} = A'''$ are algebraics.

$$A^2 C^{L+L'-2M} C^{mL-L'} = A'' A'''$$

$$A'^2 C^{L-L'} C^{mL+L'-2M'} = A'' A'''^{-1}$$

$$C^{X'} = BC^{mY}$$

$$C^{Y'} = DC^{mZ}$$

$$X = L' + L - 2M$$

$$Y = L' - L$$

$$Z = -(L + L' - 2M')$$

$$B = A'' A''' A^{-2}$$

$$D = A^{12} A'^{-1} A'''$$

And

$$AA' C'^{L-M} C^{mL-M'} = A''$$

$$AA'^{-1} C'^{L'-M} C^{mM'-L'} = A'''$$

$$C^{U'} = EC^{mV'}$$

$$C^{U''} = FC^{mV''}$$

$$U = L' - M$$

$$V = M' - L$$

$$U' = L' - M'$$

$$V' = L' - M'$$

$$E = A'' A^{-1} A'^{-1}$$

$$F = A''' A^{-1} A'$$

$$C'^{-XY} = B^{-Y} C^{m-Y^2} = D^X C^{mXZ} \Rightarrow -Y^2 = -(L-L')^2 = XZ = -(L+L'-2M)(L+L'-2M')$$

$$C'^{UV'} = E^{V'} C^{mV''} = F^{-V} E^{V'} C^{U''V} \Rightarrow U'V = UV' = (L-M)(M'-L) = (L'-M)(L'-M')$$

$$C'^{YU} = E^Y C^{mY} = E^Y B^{-V} C'^{XV} \Rightarrow YU = XV = (L-M)(L'-L) = (M'-L)(L'+L-2M)$$

$$C'^{ZU} = E^Z C^{mZV} = E^Z D^{-V} C'^{VY} \Rightarrow ZU = -VY = -(L+L'-2M')(L'-M) = -(L'-L)(L'-M')$$

$$\Rightarrow L^2 + L'^2 - 2LL' = L^2 + L'^2 + 2LL' - 2(M'+M)(L+L') + 4MM'$$

$$\Rightarrow M^2 + M'^2 - 2MM' = M^2 + M'^2 + 2MM' - 2(M'+M)(L+L') + 4LL'$$

$$\Rightarrow (M-M')^2 = (M+M'-2L)(M+M'-2L')$$

Thus

$$(L-M)(L'-M') = (M'-L)(L'+L-2M) + (M-L)(L'-M')$$

$$(L'-M)(L'-M') = (M'-L)(L'-M) + (M'-L)(L-M)$$

$$(L'-M)(L+L'-M-M') = (M'-L)(L-M) = (L'-M)(L'-M')$$

$$\Rightarrow (L'-M)(L-M) = 0 \Rightarrow (M'-L)(M'-L') = 0$$

We deduce

$M=M'=L=L'$ is unique !

3) $C+mC'-(C+m'C')=(m-m')C' \Rightarrow m=m'$ there is only one $m=N$ for which $C+mC'$ is

algebraic, all the others are transcendentals. If N is not unique there are three possibilities :

$C+C''+m(C'+C''')$ is transcendental for all m , $C-C'''+m(C'-C''')$ is transcendental for all m or there exists L, L' for which $C+NC'=A, C'''+N'C'''=A', C+C'''+L(C'+C''')=A'', C-C'''+L'(C'-C''')=A'''$ are algebraics, thus

$$A'''+A''''=2A+(L+L'-2N)C'+(L-L')C''''$$

$$A''-A''''=2A'+(L-L')C'+(L+L'-2N')C''''$$

$$A''=A+A'+(L-N)C'+(L-N')C''$$

$$A''''=A-A'+(L'-N)C'+(N'-L')C''$$

$$\text{Thus } (L-L')^2 = (L+L'-2N)(L+L'-2N') \quad \text{and} \quad (L-N)(N'-L) = (L'-N)(L'-N') \quad \text{and}$$

$(L-N)(L'-L) = (N'-L)(L'+L-2N)$ and $-(L+L'-2N')(L'-N) = -(L'-L)(L'-N')$ by the same calculus than higher $N=N'=L=L'$ is unique !

Thus, there are finally two possibilities

- I) There are three subpossibilities : $CC^n(C'C^m)^n$ is transcendental for all n , for all T, T' prime transcendental numbers, there exist $C = TC^{n-1}, C' = T'C^{m-1} \Rightarrow TT^m$ is transcendental for all n , particularly $n=1$ and TT' and $T+T'$ are transcendentals. Or $CC^{n-1}(C'C^{m-1})^n$ is transcendental for all n , for all T, T' prime transcendental numbers, there exist $C = TC^{n-1}, C'^{-1} = T'C^{m-1} \Rightarrow TT^m$ is transcendental for all n , particularly $n=1$ and TT^4 and $T+T'$ are transcendentals.

Third subpossibility : CC^m for all $n \neq M$ for all T, T' prime transcendental numbers there exist $C = T^{\frac{1}{2}}, C' = T'^{\frac{1}{n}} \Rightarrow TT'$ transcendental and $T+T'$ transcendental.

- II) There are three subpossibilities $C+C''+m(C'+C''')$ is transcendental for all n , for all T, T' prime transcendental numbers, there exist $C=T-C'', C'=T'-C'''$ and $T+mT'$ is transcendental for all m , particularly $m=1$ and $T+T'$ and TT' are transcendentals. Or $C-C''+m(C'-C''')$ is transcendental for all m , for all T, T' prime transcendental numbers, there exist $C=T+C'', C'=T'+C'''$ and $T+mT'$ is transcendental for all m , particularly $m=1$ and $T+T'$ and TT' are transcendentals. Third subpossibility : $C+mC'$ is transcendental for all $m \neq N$. For all T, T' prime transcendental numbers, there exist $2C=T, mC'=T'$ and $T+T'$ and TT^4 are transcendentals.

The theorem application

$T = \pi, T' = e$ imply that the sum and the product of π and e are transcendentals.

Conclusion

Through this exposé, we have given a method to find the nature of several numbers, we have shown the nature of some of them.

Bibliography

- [1] Alan Baker, Transcendental number theory, Cambridge university press, 1975