Bulletin of Mathematical Sciences and Applications *Online: 2012-05-01*
ISSN: 2278-9634, Vol. 4, pp 6-8
doi:10.18052/www.scipress.com/BMSA.4.6
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About some transcendental numbers

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Abstract. Everyone knows that π, e or the Champernowne number are transcendentals, but what about $\pi + e$ or πe? In this paper, we demonstrate a method in order to know if they also are.

The approach

A number is transcendental if it is not the root of a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$ where the coefficients are rationals and different of zero. Otherwise it would be algebraic. We know that π, a and the Champenowne number are transcendentals but we do not know anything about their sum ot their product. Effectively if A is algebraic B is transcendental, A+B and AB are transcendentals, but if A is also transcendental we still do not the nature of AB or A+B. Let us trie to solve this problem.

Definition

A real number is composed if it can be written as $\prod_j p_j^{n_j}$ where p_j are integer primes and n_j are rationals. A prime real number can be written only as p=p.1 . Thus we define other real prime nubers like π, e, $\ln(2)$. Thus $\sqrt[q]{p} = p^{\frac{1}{q}}$ is composed. Also $\sqrt[q]{p} + 1 = p^{\frac{1}{q}} + 1$ is prime and we have $\sqrt[2^i]{p} - 1 = (p-1)(\sqrt[2^i]{p} + 1)^{-1}(\sqrt[2^{i-1}]{p} + 1)^{-1}...(\sqrt{p} + 1)^{-1}$ composed, for example.

Theorem

If T and T' are transcendental prime numbers then T+T' and TT' are transcendentals.

Proof of the theorem

Let C and C' two composed transcendental numbers. We have 4 possibilities.

1) $CC'^n, C + mC'$ are algebraics.
2) CC'^n is algebraic and $C + mC'$ is transcendental.
3) CC'^n is transcendental and $C + mC'$ is algebraic.
4) $CC'^n, C + mC'$ are transcendentals.

Thus

1) $CC'^n + mC'^{n+1} = (C + mC')C'^n = A + mC'^{n+1} = A'C'^n$ where A and A' are algebraics and C' is solution of an algebraic equation, it is impossible !
2) $CC'^n, CC'^{n'}$ are algebraics and $CC'^n(CC'^{n'})^{-1} = C'^{n-n'}$ is algebraic if n=n'. There is only one n=M for which CC'^n is algebraic, all the others are transcendentals. If M is not unique, there are three possibilities : $CC''(C'C''')^n$ is transcendental for all n or $CC''^{-1}(C'C'''^{-1})^n$ is transcendental for all n or there exists L, L' for which

$CC'^M = A, C''C'''^{M'} = A', CC''(C'C''')^L = A'', CC''^{-1}(C'C'''^{-1})^{L'} = A'''$ are algebraics.

$$A^2 C'^{L+L'-2M} C'''^{L-L'} = A''A'''$$

$$A'^2 C'^{L-L'} C'''^{L+L'-2M'} = A''A'''^{-1}$$

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$C'^{X} = BC'''^{Y}$

$C'^{-Y} = DC'''^{Z}$

$X = L' + L - 2M$

$Y = L' - L$

$Z = -(L + L' - 2M')$

$B = A'' A''' A^{-2}$

$D = A'^{2} A''^{-1} A'''$

And

$AA' C'^{L-M} C'''^{L-M'} = A''$

$AA'^{-1} C'^{L'-M} C'''^{M'-L'} = A'''$

$C'^{U} = EC'''^{V}$

$C'^{U'} = FC'''^{V'}$

$U = L' - M$

$V = M' - L$

$U' = L' - M$

$V' = L' - M'$

$E = A'' A^{-1} A'^{-1}$

$F = A''' A^{-1} A'$

$C'^{-XY} = B^{-Y} C'''^{-Y^2} = D^{X} C'''^{XZ} \Rightarrow -Y^2 = -(L-L')^2 = XZ = -(L+L'-2M)(L+L'-2M')$

$C'^{UV'} = E^{V'} C'''^{VV'} = F^{-V} E^{V'} C'^{U'V} \Rightarrow U'V = UV' = (L-M)(M'-L) = (L'-M)(L'-M')$

$C'^{YU} = E^{Y} C'''^{VY} = E^{Y} B^{-V} C'^{XV} \Rightarrow YU = XV = (L-M)(L'-L) = (M'-L)(L'+L-2M)$

$C'^{ZU} = E^{Z} C'''^{ZV} = E^{Z} D^{-V} C'^{-VY} \Rightarrow ZU = -VY = -(L+L'-2M')(L'-M) = -(L'-L)(L'-M')$

$\Rightarrow L^2 + L'^2 - 2LL' = L^2 + L'^2 + 2LL' - 2(M'+M)(L+L') + 4MM'$

$\Rightarrow M^2 + M'^2 - 2MM' = M^2 + M'^2 + 2MM' - 2(M'+M)(L+L') + 4LL'$

$\Rightarrow (M-M')^2 = (M+M'-2L)(M+M'-2L')$

Thus

$(L-M)(L'-M') = (M'-L)(L'+L-2M) + (M-L')(L'-M')$

$(L'-M)(L'-M') = (M'-L)(L'-M) + (M'-L)(L-M)$

$(L'-M)(L+L'-M-M') = (M'-L)(L-M) = (L'-M)(L'-M')$

$\Rightarrow (L'-M)(L-M) = 0 \Rightarrow (M'-L)(M'-L') = 0$

We deduce

M=M'=L=L' is unique !

3) $C + mC' - (C + m'C') = (m - m')C' \Rightarrow m = m'$ there is only one m=N for which C+mC' is algebraic, all the others are transcendentals. If N is not unique there are three possibilities : C+C''+m(C'+C''') is transcendental for all m, C-C''+m(C'-C''') is transcendental for all m or there exists L, L' for which C+NC'=A, C''+N'C'''=A', C+C''+L(C'+C''')=A'', C-C''+L'(C'-C''')=A''' are algebraics, thus

A''+A'''=2A+(L+L'-2N)C'+(L-L')C'''

A''-A'''=2A'+(L-L')C'+(L+L'-2N')C'''

A''=A+A'+(L-N)C'+(L-N')C''

A'''=A-A'+(L'-N)C'+(N'-L')C''

Thus $(L-L')^2 = (L+L'-2N)(L+L'-2N')$ and $(L-N)(N'-L) = (L'-N)(L'-N')$ and $(L-N)(L'-L) = (N'-L)(L'+L-2N)$ and $-(L+L'-2N')(L'-N) = -(L'-L)(L'-N')$ by the same calculus than higher N=N'=L=L' is unique !

Thus, there are finaly two possibilities

I) There are three subpossibilities : $CC''(C'C''')^n$ is transcendental for all n, for all T, T' prime transcendental numbers, there exist $C = TC''^{-1}, C' = T'C'''^{-1} \Rightarrow TT'^n$ is transcendental for all n, particularly n=1 and TT' and T+T' are transcendentals. Or $CC''^{-1}(C'C'''^{-1})^n$ is transcendental for all n, for all T, T' prime transcendental numbers, there exist $C = TC''^{-1}, C'^{-1} = T'C'''^{-1} \Rightarrow TT'^n$ is transcendental for all n, particularly n=1 and TT4 and T+T' are transcendentals.

Third subpossibility : CC'^n for all $n \neq M$ for all T, T' prime transcendental numbers there exist $C = T^{\frac{1}{2}}, C' = T'^{\frac{1}{n}} \Rightarrow TT'$ transcendental and T+T' transcendental.

II)There are three subpossibilities C+C''+m(C'+C''') is transcendental for all n, for all T, T' prime transcendental numbers, there exist C=T-C'', C'=T'-C''' and T+mT' is transcendental for all m, particularly m=1 and T+T' and TT' are transcendentals. Or C-C''+m(C'-C''') is transcendental for all m, for all T, T' prime transcendental numbers, there exist C=T+C'', C'=T'+C''' and T+mT' is transcendental for all m, particularly m=1 and T+T' and TT' are transcendentals. Third subpossibility : C+mC' is transcendental for all $m \neq N$. For all T, T' prime transcendental numbers, there exist 2C=T, mC'=T' and T+T' and TT4 are transcendentals.

The theorem application

$T = \pi, T' = e$ implie that the sum and the produt of π and e are transcendentals.

Conclusion

Through this exposé, we have given a method to find the nature of several numbers, we have shown the nature of some of them.

Bibliography

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