On Andrica’s Conjecture, Cramér’s Conjecture, gaps Between Primes and Jacobi Theta Functions IV: A Simple Proof for Cramér’s Conjecture

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1. INTRODUCTION

In On the Order of Magnitude of the Difference between Consecutive Prime Numbers [1, p. 27], 1937, Harald Cramér conjectured, using a heuristic method founded on probabilistic arguments, that

$$\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1,$$

(1.a)


In this paper, we prove that

$$\lim_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < 1,$$

(1.b)

result which is stronger than the Cramér’s conjecture.

2. PRELIMINARES

The Rosser’s theorem [3] states that $p_n$ is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + 1 < \log p_n < \log n + \log \log n,$$

(1)

for $n \geq 6$.

3. LEMMA AND THEOREMS

THEOREM 1. For $n \in \mathbb{N}_{\geq 6}$, let

$$\frac{p_{n+1} - p_n}{p_n} < \sqrt{2} \left(2n^2 - \left[2\sqrt{n(n+1)} + 1\right]n + 2\sqrt{n(n+1)}\right).$$

Proof. In previous paper [4, p. __], we discover that

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \sqrt{2} (\sqrt{n + 1} - \sqrt{n})\sqrt{n},$$

(2)

for $n \in \mathbb{N}_{\geq 6}$. Squaring the inequality (1), we have

$$p_{n+1} + p_n - 2\sqrt{p_{n+1}p_n} < 2n \left(2n + 1 - 2\sqrt{n(n+1)}\right)$$

(3)

$$\Rightarrow p_{n+1} + p_n < 2\sqrt{p_{n+1}p_n} + 2n \left(2n + 1 - 2\sqrt{n(n+1)}\right).$$

Multiplying (2) by $2\sqrt{p_n}$, we find

$$2\sqrt{p_{n+1}p_n} - 2p_n < 2\sqrt{2} (\sqrt{n + 1} - \sqrt{n})\sqrt{n}\sqrt{p_n}$$

(4)

$$\Rightarrow 2\sqrt{p_{n+1}p_n} < 2p_n + 2\sqrt{2} (\sqrt{n + 1} - \sqrt{n})\sqrt{n}\sqrt{p_n}.$$
From (3) and (4), we obtain
\[
p_{n+1} + p_n < 2p_n + 2\sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n}\sqrt{p_n} + 2n \left( 2n + 1 - 2\sqrt{n(n+1)} \right) \tag{5}\]
\[
\Rightarrow p_{n+1} - p_n < 2\sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n}\sqrt{p_n} + 2n \left( 2n + 1 - 2\sqrt{n(n+1)} \right).
\]
Dividing both members of (5) by \(\sqrt{p_n}\), we encounter
\[
\frac{p_{n+1} - p_n}{\sqrt{p_n}} < 2\sqrt{2} \left( \sqrt{(n + 1) - \sqrt{n}} \right) \sqrt{n} + \frac{2n}{\sqrt{p_n}} \left( 2n + 1 - 2\sqrt{n(n+1)} \right).
\]
On the other hand, we have that \(\max_{n \in \mathbb{N}_{\geq 6}} \frac{1}{\sqrt{p_n}} = \frac{1}{\sqrt{13}} < \max_{n \in \mathbb{N}_{\geq 21}} \frac{1}{\sqrt{p_n}} = \frac{1}{\sqrt{27}}\)
\[
\frac{p_{n+1} - p_n}{\sqrt{p_n}} < 2\sqrt{2} \left( \sqrt{(n + 1) - \sqrt{n}} \right) \sqrt{n} + 2n \max_{n \in \mathbb{N}_{\geq 26}} \frac{1}{\sqrt{p_n}} \left( 2n + 1 - 2\sqrt{n(n+1)} \right).
\]
\[
< 2\sqrt{2} \left( \sqrt{(n + 1) - \sqrt{n}} \right) \sqrt{n} + 2n \max_{n \in \mathbb{N}_{\geq 21}} \frac{1}{\sqrt{p_n}} \left( 2n + 1 - 2\sqrt{n(n+1)} \right)
\]
\[
= 2\sqrt{2} \left( \sqrt{(n + 1) - \sqrt{n}} \right) \sqrt{n} + \frac{2n}{\sqrt{2}} \sqrt{n + 1 - 2\sqrt{n(n+1)}},
\]
\[
= \sqrt{2} \left( 2n^2 - \left[ 2\sqrt{n(n+1)} + 1 \right] n + 2\sqrt{n(n+1)} \right). \tag{6}
\]

**THEOREM 2.** For \(n \in \mathbb{N}_{\geq 6}\), the
\[
p_{n+1} - p_n < 4n(\sqrt{n + 1} - \sqrt{n})\sqrt{\log n}.
\]

**Proof.** Multiplying (2) by \(\sqrt{p_{n+1}}\) and \(\sqrt{p_n}\), separately, we obtain
\[
p_{n+1} \sqrt{p_{n+1}p_n} < \sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n}\sqrt{p_{n+1}}, \tag{6}
\]
and
\[
\sqrt{p_{n+1}p_n} - p_n < \sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n}\sqrt{p_n}. \tag{7}
\]
Summing, with (7), member by member, we have
\[
p_{n+1} - p_n < \sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}). \tag{8}
\]
From (1) and (8), we find
\[
p_{n+1} - p_n < \sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}) \tag{9}
\]
\[
< \sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n} \left( \sqrt{(n + 1)\log(n + 1)} + (n + 1)\log \log(n + 1) + \sqrt{n\log n + n\log \log n} \right)
\]
\[
< \sqrt{2}(\sqrt{n + 1} - \sqrt{n})\sqrt{n} \left( \sqrt{2(n + 1)\log(n + 1)} + 2n\log n \right).
\]
= 2(\sqrt{n + 1} - \sqrt{n})\sqrt{n} \left( \sqrt{(n + 1) \log(n + 1)} + \sqrt{n \log n} \right).

Since \(\sqrt{(n + 1) \log(n + 1)} \simeq \sqrt{n \log n}\), we find

\[ p_{n+1} - p_n < 2(\sqrt{n + 1} - \sqrt{n})\sqrt{n}(2\sqrt{n \log n}) = 4n(\sqrt{n + 1} - \sqrt{n})\sqrt{n \log n}. \]

**THEOREM 3** (Stronger Cramér’s conjecture).

\[
\lim_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^{\frac{3}{2}}} < 1.
\]

**Proof.** In Theorem 2, see [6], we have

\[
\left(\frac{\theta_2^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n - n) < g(p_n) < \left(\frac{\theta_2^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n),
\]

where \(k = \frac{p_n}{p_{n+1}}\) to be a \(k\) modulus. In other words,

\[
\left(\frac{\theta_2^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n - n) < p_{n+1} - p_n < \left(\frac{\theta_2^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n).
\]

Dividing (11) by \(\left(\frac{\theta_2^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n)\), we encounter

\[
\frac{n \log n + n \log \log n - n}{n \log n + n \log \log n} < \left(\frac{\theta_2^2}{\theta_2^2 - \theta_2^2}\right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1,
\]

ergo,

\[
\frac{\log n + \log \log n - 1}{\log n + \log \log n} < \left(\frac{\theta_2^2}{\theta_2^2 - \theta_2^2}\right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1.
\]

But, by the Rosser’s theorem, we find

\[
\frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n}{n \log n + n \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n}{n \log n + n \log \log n},
\]

wherefore,

\[
\frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n}.
\]

Dividing (15) by (13), we find

\[
\frac{\log n + \log \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n} < \frac{\theta_2^2}{\theta_2^2 - \theta_2^2} < \frac{\log n + \log \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n}.
\]

From (12) and (16), we have
therefore,

$$\left( \frac{\log n + \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1, \tag{17}$$

consequently,

$$\left( \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{\log \log n + \log n - 1} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1, \tag{18}$$

On the other hand, applying the Rosser’s theorem, we obtain

$$\left( \frac{(\log p_n)^2}{n \log n + n \log \log n} \right) \frac{p_n}{n \log n + n \log \log n} < \frac{n \log n + n \log \log n - 1}{n \log n + n \log \log n} = 1, \tag{20}$$

Dividing (19) by (20), we encounter

$$\frac{p_{n+1} - p_n}{(\log p_n)^2} < \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{\log \log n + \log n - 1}. \tag{21}$$

Applying the limit as \( n \to \infty \) in both members of the inequality above, we obtain

$$\lim_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1. \square$$

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REFERENCES


