

On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions IV: A Simple Proof for Cramér's Conjecture

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1. INTRODUCTION

In *On the Order of Magnitude of the Difference between Consecutive Prime Numbers* [1, p. 27], 1937, Harald Cramér conjectured, using a heuristic method founded on probabilistic arguments, that

$$\limsup_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1, \quad (1.a)$$

to see also Richard K. Guy's book: *Unsolved Problems in Number Theory* [2, p. 114].

In this paper, we prove that

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < 1, \quad (1.b)$$

result which is stronger than the Cramér's conjecture.

2. PRELIMINARES

The Rosser's theorem [3] states that p_n larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + 1 - \log n - \frac{1}{n} < \frac{p_n}{n} < \log n + \log \log n, \quad (1)$$

for $n \geq 6$.

3. LEMMA AND THEOREMS

THEOREM 1. For $n \in \mathbb{N}_{\geq 6}$, it holds that

$$\frac{p_{n+1} - p_n}{\sqrt{p_n}} < \sqrt{2} \left\{ 2n^2 - \left[2\sqrt{n(n+1)} + 1 \right] n + 2\sqrt{n(n+1)} \right\}.$$

Proof. In our previous paper [4, p. ...], we discover that

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}, \quad (2)$$

for $n \in \mathbb{N}_{\geq 6}$. Squaring the inequality (1), we have

$$\begin{aligned} p_{n+1} + p_n - 2\sqrt{p_{n+1}p_n} &< 2n \left(2n + 1 - 2\sqrt{n(n+1)} \right) \\ \Rightarrow p_{n+1} + p_n &< 2\sqrt{p_{n+1}p_n} + 2n \left(2n + 1 - 2\sqrt{n(n+1)} \right). \end{aligned} \quad (3)$$

Multiplying (2) by $2\sqrt{p_n}$, we find

$$\begin{aligned} 2\sqrt{p_{n+1}p_n} - 2p_n &< 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n} \\ \Rightarrow 2\sqrt{p_{n+1}p_n} &< 2p_n + 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n}. \end{aligned} \quad (4)$$

From (3) and (4), we obtain

$$p_{n+1} + p_n < 2p_n + 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n} + 2n(2n+1 - 2\sqrt{n(n+1)}) \tag{5}$$

$$\Rightarrow p_{n+1} - p_n < 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n} + 2n(2n+1 - 2\sqrt{n(n+1)}).$$

Dividing both members of (5) by $\sqrt{p_n}$, we encounter

$$\frac{p_{n+1} - p_n}{\sqrt{p_n}} < 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n} + \frac{2n}{\sqrt{p_n}}(2n+1 - 2\sqrt{n(n+1)}).$$

On the other hand, we have that $\max_{n \in \mathbb{N}_{\geq 6}} \frac{1}{\sqrt{p_n}} = \frac{1}{\sqrt{13}} < \max_{n \in \mathbb{N}_{\geq 1}} \frac{1}{\sqrt{p_n}} = \frac{1}{\sqrt{2}}$ so

$$\begin{aligned} \frac{p_{n+1} - p_n}{\sqrt{p_n}} &< 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n} + \frac{2n}{\sqrt{p_n}}(2n+1 - 2\sqrt{n(n+1)}) \\ &< 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n} + 2n \max_{n \in \mathbb{N}_{\geq 6}} \frac{1}{\sqrt{p_n}}(2n+1 - 2\sqrt{n(n+1)}) \\ &< 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n} + 2n \max_{n \in \mathbb{N}_{\geq 1}} \frac{1}{\sqrt{p_n}}(2n+1 - 2\sqrt{n(n+1)}) \\ &= 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n} + \frac{2n}{\sqrt{2}}(2n+1 - 2\sqrt{n(n+1)}) \\ &= 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n} + \sqrt{2}(2n+1 - 2\sqrt{n(n+1)}) \\ &= \sqrt{2}\{2n^2 - [2\sqrt{n(n+1)} + 1]n + 2\sqrt{n(n+1)}\}. \square \end{aligned}$$

THEOREM 2. For $n \in \mathbb{N}_{\geq 6}$, the

$$p_{n+1} - p_n < 4n(\sqrt{n+1} - \sqrt{n})\sqrt{\log n}.$$

Proof. Multiplying (2) by $\sqrt{p_{n+1}}$ and $\sqrt{p_n}$, separately, we obtain

$$p_{n+1}\sqrt{p_{n+1}p_n} < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_{n+1}}, \tag{6}$$

and

$$\sqrt{p_{n+1}p_n} - p_n < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n}. \tag{7}$$

Combining (6) with (7), member by member, we have

$$p_{n+1} - p_n < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}). \tag{8}$$

From (1) and (8), we find

$$\begin{aligned} p_{n+1} - p_n &< \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}) \\ &< \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{(n+1)\log(n+1)} + \sqrt{(n+1)\log\log(n+1)} \\ &\quad + \sqrt{n\log n + n\log\log n}) \\ &< \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{2(n+1)\log(n+1)} + \sqrt{2n\log n}) \end{aligned} \tag{9}$$

$$= 2(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{(n+1)\log(n+1)} + \sqrt{n\log n}).$$

Since $\sqrt{(n+1)\log(n+1)} \cong \sqrt{n\log n}$, we find

$$p_{n+1} - p_n < 2(\sqrt{n+1} - \sqrt{n})\sqrt{n}(2\sqrt{n\log n}) = 4n(\sqrt{n+1} - \sqrt{n})\sqrt{\log n}. \square$$

THEOREM 3 (Stronger Cramér’s conjecture).

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < 1.$$

Proof. In Theorem 2, see [6], we have

$$\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n - n) < g(p_n) < \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n), \tag{10}$$

where $k = \frac{p_n}{p_{n+1}}$ to be a k modulus. In other words,

$$\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n - n) < p_{n+1} - p_n < \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n), \tag{11}$$

Dividing (11) by $\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n \log n + n \log \log n)$, we encounter

$$\frac{n \log n + n \log \log n - n}{n \log n + n \log \log n} < \left(\frac{\theta_2^2}{\theta_3^2 - \theta_2^2}\right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1, \tag{12}$$

ergo,

$$\frac{\log n + \log \log n - 1}{\log n + \log \log n} < \left(\frac{\theta_2^2}{\theta_3^2 - \theta_2^2}\right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1, \tag{13}$$

But, by the Rosser’s theorem, we find

$$\frac{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n - 1 - n \log n - n \log \log n + n}{n \log n + n \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n \log n - n \log \log n}{n \log n + n \log \log n}, \tag{14}$$

wherefore,

$$\frac{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n \log n - n \log \log n - 1}{n \log n + n \log \log n} < \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n \log n - n \log \log n}{n \log n + n \log \log n}. \tag{15}$$

Dividing (15) by (13), we find

$$\frac{\log n + \log \log n - 1}{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n \log n - n \log \log n - 1} < \frac{\theta_2^2}{\theta_3^2 - \theta_2^2} < \frac{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n \log n - n \log \log n}{(n+1)\log(n+1) + (n+1)\log \log(n+1) - n \log n - n \log \log n}, \tag{16}$$

From (12) and (16), we have

$$\left(\frac{\log n + \log \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \left(\frac{\theta_2^2}{\theta_3^2 - \theta_2^2} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1, \tag{17}$$

therefore,

$$\left(\frac{\log n + \log \log n - 1}{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1} \right) \frac{p_{n+1} - p_n}{n \log n + n \log \log n} < 1, \tag{18}$$

consequently,

$$\frac{p_{n+1} - p_n}{n \log n + n \log \log n} < \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{\log n + \log \log n - 1}. \tag{19}$$

On the other hand, applying the Rosser's theorem, we obtain

$$\frac{(\log p_n)^2}{n \log n + n \log \log n} \ll \frac{p_n}{n \log n + n \log \log n} < \frac{n \log n + n \log \log n}{n \log n + n \log \log n} = 1 \tag{20}$$

Dividing (19) by (20), we encounter

$$\frac{p_{n+1} - p_n}{(\log p_n)^2} < \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{\log n + \log \log n - 1}. \tag{21}$$

Applying the limit as $n \rightarrow \infty$ in both members of the equality above, we obtain

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} < \lim_{n \rightarrow \infty} \frac{(n+1) \log(n+1) + (n+1) \log \log(n+1) - n \log n - n \log \log n - 1}{\log n + \log \log n - 1} = 1. \square$$

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