Bulletin of Mathematical Sciences and Applications *Online: 2012-05-01*
ISSN: 2278-9634, Vol. 4, pp 38-41
doi:10.18052/www.scipress.com/BMSA.4.38

On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions IV: A Simple Proof for Cramér's Conjecture

Prof. Dr. Raja Rama Gandhi and Edigles Guedes

[1]Resource person in Math for Oxford University Press, Professor in Math, BITS-Vizag.
[2]World order Number Theorist, Pernambuco, Brazil.

RETRACTED

1. INTRODUCTION

In *On the Order of Magnitude of the Difference between Consecutive Prime Numbers* [1, p. 27], 1937, Harald Cramér conjectured, using a heuristic method founded on probabilistic arguments, that

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^2} = 1, \tag{1.a}$$

to see also Richard K. Guy's book: *Unsolved Problems in Number Theory* [2, p. 11].

In this paper, we prove that

$$\lim_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^2} < 1, \tag{1.b}$$

result which is stronger than the Cramér's conjecture.

2. PRELIMINARES

The Rosser's theorem [3] states that p_n is larger than $n\log n$. This can be improved by the following pair of bounds:

$$\log n + \log\log n - 1 < \frac{p_n}{n} < \log n + \log\log n, \tag{1}$$

for $n \geq 6$.

3. LEMMA AND THEOREMS

THEOREM 1. *For $n \in \mathbb{N}_{\geq 6}$, we have*

$$\frac{p_{n+1}-p_n}{\sqrt{2p_n}} < \sqrt{2}\left\{2n^2 - \left[2\sqrt{n(n+1)}+1\right]n + 2\sqrt{n(n+1)}\right\}.$$

Proof. In previous paper [4, p.__], we discover that

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}, \tag{2}$$

for $n \in \mathbb{N}_{\geq 6}$. Squaring the inequality (1), we have

$$p_{n+1} + p_n - 2\sqrt{p_{n+1}p_n} < 2n\left(2n+1-2\sqrt{n(n+1)}\right) \tag{3}$$

$$\Rightarrow p_{n+1} + p_n < 2\sqrt{p_{n+1}p_n} + 2n\left(2n+1-2\sqrt{n(n+1)}\right).$$

Multiplying (2) by $2\sqrt{p_n}$, we find

$$2\sqrt{p_{n+1}p_n} - 2p_n < 2\sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n} \tag{4}$$

$$\Rightarrow 2\sqrt{p_{n+1}p_n} < 2p_n + 2\sqrt{2}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\sqrt{p_n}.$$

From (3) and (4), we obtain

$$p_{n+1} + p_n < 2p_n + 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n} + 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right) \tag{5}$$

$$\Rightarrow p_{n+1} - p_n < 2\sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n} + 2n\left(2n + 1 - 2\sqrt{n(n+1)}\right).$$

Dividing both members of (5) by $\sqrt{p_n}$, we encounter

$$\frac{p_{n+1} - p_n}{\sqrt{p_n}} < 2\sqrt{2}\left(\sqrt{(n+1)} - \sqrt{n}\right)\sqrt{n} + \frac{2n}{\sqrt{p_n}}\left(2n + 1 - 2\sqrt{n(n+1)}\right).$$

On the other hand, we have that $\max_{n\in\mathbb{N}_{\geq 6}} \frac{1}{\sqrt{p_n}} = \frac{1}{\sqrt{13}} < \max_{n\in\mathbb{N}_{\geq 1}} \frac{1}{\sqrt{p_n}} = \frac{1}{\sqrt{2}}$, so

$$\begin{aligned}
\frac{p_{n+1} - p_n}{\sqrt{p_n}} &< 2\sqrt{2}\left(\sqrt{(n+1)} - \sqrt{n}\right)\sqrt{n} + \frac{2n}{\sqrt{p_n}}\left(2n + 1 - 2\sqrt{n(n+1)}\right) \\
&< 2\sqrt{2}\left(\sqrt{(n+1)} - \sqrt{n}\right)\sqrt{n} + 2n \max_{n\in\mathbb{N}_{\geq 6}} \frac{1}{\sqrt{p_n}}\left(2n + 1 - 2\sqrt{n(n+1)}\right) \\
&< 2\sqrt{2}\left(\sqrt{(n+1)} - \sqrt{n}\right)\sqrt{n} + 2n \max_{n\in\mathbb{N}_{\geq 1}} \frac{1}{\sqrt{p_n}}\left(2n + 1 - 2\sqrt{n(n+1)}\right) \\
&= 2\sqrt{2}\left(\sqrt{(n+1)} - \sqrt{n}\right)\sqrt{n} + \frac{2n}{\sqrt{2}}\left(2n + 1 - 2\sqrt{n(n+1)}\right) \\
&= 2\sqrt{2}\left(\sqrt{(n+1)} - \sqrt{n}\right)\sqrt{n} + \sqrt{2}\left(2n + 1 - 2\sqrt{n(n+1)}\right) \\
&= \sqrt{2}\left\{2n^2 - \left[2\sqrt{n(n+1)} + 1\right]n + 2\sqrt{n(n+1)}\right\}. \square
\end{aligned}$$

THEOREM 2. *For $n \in \mathbb{N}_{\geq 6}$, then*

$$p_{n+1} - p_n < 4n(\sqrt{n+1} - \sqrt{n})\sqrt{\log n}.$$

Proof. Multiplying (2) by $\sqrt{p_{n+1}}$ and $\sqrt{p_n}$, separately, we obtain

$$p_{n+1} - \sqrt{p_{n+1}p_n} < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_{n+1}}, \tag{6}$$

and

$$\sqrt{p_{n+1}p_n} - p_n < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\sqrt{p_n}. \tag{7}$$

Summing (6) with (7), member by member, we have

$$p_{n+1} - p_n < \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}). \tag{8}$$

From (1) and (8), we find

$$\begin{aligned}
p_{n+1} - p_n &< \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}(\sqrt{p_{n+1}} + \sqrt{p_n}) \qquad (9) \\
&< \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\Big(\sqrt{(n+1)\log(n+1) + (n+1)\log\log(n+1)} \\
&\qquad + \sqrt{n\log n + n\log\log n}\Big) \\
&< \sqrt{2}(\sqrt{n+1} - \sqrt{n})\sqrt{n}\left(\sqrt{2(n+1)\log(n+1)} + \sqrt{2n\log n}\right)
\end{aligned}$$

$$= 2\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(\sqrt{(n+1)\log(n+1)}+\sqrt{n\log n}\right).$$

Since $\sqrt{(n+1)\log(n+1)} \cong \sqrt{n\log n}$, we find

$$p_{n+1}-p_n < 2\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}\left(2\sqrt{n\log n}\right) = 4n\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{\log n}.\square$$

THEOREM 3 (Stronger Cramér's conjecture).

$$\lim_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^2} < 1.$$

Proof. In Theorem 2, see [6], we have

$$\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n - n) < g(p_n) < \left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n), \tag{10}$$

where $k = \frac{p_n}{p_{n+1}}$ to be a k modulus. In other words,

$$\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n - n) < p_{n+1}-p_n < \left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n), \tag{11}$$

Dividing (11) by $\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n)$, we encounter

$$\frac{n\log n+n\log\log n-n}{n\log n+n\log\log n} < \left(\frac{\theta_2^2}{\theta_3^2-\theta_2^2}\right)\frac{p_{n+1}-p_n}{n\log n+n\log\log n} < 1, \tag{12}$$

ergo,

$$\frac{\log n+\log\log n-1}{\log n+\log\log n} < \left(\frac{\theta_2^2}{\theta_3^2-\theta_2^2}\right)\frac{p_{n+1}-p_n}{n\log n+n\log\log n} < 1, \tag{13}$$

But, by the Rosser's theorem we find

$$\frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n-1-n\log n-n\log\log n+n}{n\log n+n\log\log n} < \frac{p_{n+1}-p_n}{n\log n+n\log\log n} < \frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n}{n\log n+n\log\log n}, \tag{14}$$

wherefore,

$$\frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1}{n\log n+n\log\log n} < \frac{p_{n+1}-p_n}{n\log n+n\log\log n} < \frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n}{n\log n+n\log\log n}. \tag{15}$$

Dividing (15) by (13), we find

$$\frac{\log n+\log\log n-1}{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1} < \frac{\theta_2^2}{\theta_3^2-\theta_2^2} < \frac{n\log n+n\log\log n}{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n}, \tag{16}$$

From (12) and (16), we have

$$\left(\frac{\log n+\log\log n-1}{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1}\right)\frac{p_{n+1}-p_n}{n\log n+n\log\log n}<$$
$$\left(\frac{\theta_2^2}{\theta_3^2-\theta_2^2}\right)\frac{p_{n+1}-p_n}{n\log n+n\log\log n}<1, \quad (17)$$

therefore,

$$\left(\frac{\log n+\log\log n-1}{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1}\right)\frac{p_{n+1}-p_n}{n\log n+n\log\log n}<1, \quad (18)$$

consequently,

$$\frac{p_{n+1}-p_n}{n\log n+n\log\log n}<\frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1}{\log n+\log\log n-1}. \quad (19)$$

On the other hand, applying the Rosser's theorem, we obtain

$$\frac{(\log p_n)^2}{n\log n+n\log\log n}\ll\frac{p_n}{n\log n+n\log\log n}<\frac{n\log n+n\log\log n}{n\log n+n\log\log n}=1 \quad (20)$$

Dividing (19) by (20), we encounter

$$\frac{p_{n+1}-p_n}{(\log p_n)^2}<\frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1}{\log n+\log\log n-1}. \quad (21)$$

Applying the limit as $n\to\infty$ in both members of inequality above, we obtain

$$\lim_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^2}<\lim_{n\to\infty}\frac{(n+1)\log(n+1)+(n+1)\log\log(n+1)-n\log n-n\log\log n-1}{\log n+\log\log n-1}$$
$$=1.\square$$

RETRACTED

ACKNOWLEDGMENTS

I thank Prof. Dr. K. Raja Rama Gandhi and your society for their encouragement and support during the development of this paper.

REFERENCES

[1] H. Cramér, *On the Order of Magnitude of the Difference between Consecutive Prime Numbers*, Acta Arith. 2 (1937) 23-46.

[2] Guy Richard K., *Unsolved Problems in Number Theory*, Springer, 2004.

[3] http://en.wikipedia.org/wiki/Prime_number_theorem, available in April 30, 2013.

[4] Prof. Dr. K. Raja Rama Gandhi and Edigles Guedes, *On Andrica's Conjecture, gaps Between Primes and Jacobi Theta Functions III: A Simple Proof for Andrica's Conjecture*, ____.

[5] J. B. Rosser & L. Schoenfeld, *Approximate Formulas for Some Functions of Prime Numbers*, Illinois J. Math. 6 (1962):64-94.

[6] Prof. Dr. Raja Rama Gandhi and Edigles Guedes, *On Andrica's Conjecture, gaps Between Primes and Jacobi Theta Functions,* April 22, 2013, ____.