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On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions III: A Simple Proof for Andrica's Conjecture

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1. NOTATION

We will use the notation of Armitage and Eberlein, [see 1, p. 103]:

k is a real number such that $0 < k < 1$; $k' = (1 - k^2)^{1/2}$ is the complementary modulus,

$$K = K(k) = \int_0^{2\pi} \frac{d\psi}{(1-k^2 \sin^2 \psi)'}, \qquad K' = K(k'). \tag{1}$$

$$\tau = i\,K'/K\,, \qquad q = \exp(\pi i \tau). \tag{2}$$

2. PRELIMINARES

The Rosser's theorem [2] states that p_n is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + \log\log n - 1 < \frac{p_n}{n} < \log n + \log\log n, \tag{3}$$

for $n \geq 6$.

3. THEOREM

THEOREM 1 (Andrica's Conjecture). *For* $n \in \mathbb{N}_{\geq 1}$, *then*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1. \tag{3}$$

Proof 1. Step 1. In previous paper [3, *On Andrica's Conjecture, gaps Between Primes and Jacobi Theta Functions*], Theorem 1, we discover that

$$\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n}, \tag{4}$$

for $k := \frac{p_n}{p_{n+1}}$ to be a k modulus. Ergo, dividindo ambos os membros of (14) by $\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n}$, we have

$$\sqrt{\frac{n\log n + n\log\log n - n}{n\log n + n\log\log n}} < \left(\frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{n\log n + n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3 - \theta_2}\right) < 1,$$

$$\sqrt{1 - \frac{1}{\log n + \log\log n}} < \left(\frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{n\log n + n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3 - \theta_2}\right) < 1, \tag{5}$$

But, by the Rosser's theorem, we find

$$\sqrt{(n+1)\log(n+1) + (n+1)\log\log(n+1) - (n+1)} - \sqrt{n\log n + n\log\log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \sqrt{(n+1)\log(n+1) + (n+1)\log\log(n+1)} - \sqrt{n\log n + n\log\log n}. \tag{6}$$

Dividing (6) by $\sqrt{n\log n + n\log\log n}$, we obtain

$$\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n}} - \sqrt{1-\frac{1}{\log n+\log\log n}} < \frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}} < \sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)}{\log n+\log\log n}} - 1. \tag{7}$$

Dividing (5) by (7), we encounter

$$\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n}}-\sqrt{1-\frac{1}{\log n+\log\log n}}} < \frac{\theta_2}{\theta_3-\theta_2} < \frac{1}{\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)}{\log n+\log\log n}}-1}. \tag{8}$$

Substituting the left-hand side of (8) in (5), we have

$$\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n}}-\sqrt{1-\frac{1}{\log n+\log\log n}}}\left(\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}\right) < \left(\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3-\theta_2}\right) < 1, \tag{9}$$

consequently,

$$\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{(n+1)[\log(n+1)+\log\log(n+1)-1]}-\sqrt{n(\log n+\log\log n-1)}}\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right) < \left(\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3-\theta_2}\right) < 1. \tag{10}$$

Simplifying (10), we find

$$\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{(n+1)[\log(n+1)+\log\log(n+1)-1]}-\sqrt{n(\log n+\log\log n-1)}}\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right) < 1 \Rightarrow \sqrt{p_{n+1}}-\sqrt{p_n} < \frac{\sqrt{(n+1)[\log(n+1)+\log\log(n+1)-1]}-\sqrt{n(\log n+\log\log n-1)}}{\sqrt{\frac{\log n+\log\log n-1}{\log n+\log\log n}}}. \tag{11}$$

But, we observe that $\sqrt{\log(n+1)+\log\log(n+1)-1} \cong \sqrt{\log n+\log\log n-1}$; for example: 1) for $n=11$, then $\sqrt{\log(12)+\log\log(12)-1} = 1.54762454851\ldots$ and $\sqrt{\log(11)+\log\log(11)-1} = 1.50747691714\ldots$; 2) for $n=110$, then $\sqrt{\log(111)+\log\log(111)-1} = 2.2932767734\ldots$ and $\sqrt{\log(110)+\log\log(110)-1} = 2.29088303384\ldots$; 3) for $n=1100$, then $\sqrt{\log(1101)+\log\log(1101)-1} = 2.81965456368\ldots$ and $\sqrt{\log(1100)+\log\log(1100)-1} = 2.81947041741\ldots$

Therefore, the equation (11) can be written as

$$\sqrt{p_{n+1}} - \sqrt{p_n} \lesssim \frac{\sqrt{(n+1)(\log n+\log\log n-1)}-\sqrt{n(\log n+\log\log n-1)}}{\sqrt{\frac{\log n+\log\log n-1}{\log n+\log\log n}}} \tag{12}$$

$$= \frac{\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n+\log\log n-1}}{\sqrt{\frac{\log n+\log\log n-1}{\log n+\log\log n}}}$$

$$= \frac{\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n+\log\log n-1}}{\sqrt{\log n+\log\log n-1}}\sqrt{\log n+\log\log n}$$

$$= \left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n+\log\log n} < \sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n}$$

$$< \sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}.$$

On the other hand, for $f(n)=\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}$, the maximum occurs when n tends at infinity, and $f(n)\downarrow\frac{1}{\sqrt{2}}$; we have, for $\lim_{n\to\infty}\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}=\frac{1}{\sqrt{2}}=0.707106781187\ldots$ For $n\in\mathbb{N}_{\geq 6}$, because of condition for Rosser's theorem, then

$$\sqrt{p_{n+1}}-\sqrt{p_n}<\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}\leq\lim_{n\to\infty}\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}= 0.707106781187\ldots<1. \tag{13}$$

Step 2. For $n=1$, then $\sqrt{p_2}-\sqrt{p_1}=\sqrt{3}-\sqrt{2}=0.317837245196\ldots<1$; for $n=2$, then $\sqrt{p_3}-\sqrt{p_2}=\sqrt{5}-\sqrt{3}=0.504017169931\ldots<1$; for $n=3$, then $\sqrt{p_4}-\sqrt{p_3}=\sqrt{7}-\sqrt{5}=0.409683333565\ldots<1$; for $n=4$, then $\sqrt{p_5}-\sqrt{p_4}=\sqrt{11}-\sqrt{7}=0.670873479291\ldots<1$; for $n=5$, then $\sqrt{p_6}-\sqrt{p_5}=\sqrt{13}-\sqrt{11}=0.267949192431\ldots<1$. □

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Table 1

The order of table: 1.ª column: n; 2.ª column: $\sqrt{p_{n+1}} - \sqrt{p_n}$; 3.ª column: $(\sqrt{n+1} - \sqrt{n})\sqrt{2n}$; 4.ª column: $(\sqrt{n+1} - \sqrt{n})\sqrt{2n} - (\sqrt{p_{n+1}} - \sqrt{p_n})$.

1	0.317837245196	0.585786437627	0.267949192431
2	0.504017169931	0.635674490392	0.13165732046
3	0.409683333565	0.656338798447	0.24665546488
4	0.670873479291	0.667701070844	−0.00317240844
5	0.288926485108	0.674898880549	0.385972395441
6	0.517554350154	0.679870015673	0.16231566552
7	0.235793317923	0.683510307647	0.447716989724
8	0.436932579772	0.686291501015	0.24935892124
9	0.589333283822	0.688485803641	0.09915251982
10	0.182599555695	0.69026135046	0.507661794765
11	0.514998167468	0.691727623168	0.1767294557
12	0.320361707134	0.692958984179	0.37259727704
13	0.154314286869	0.694007717489	0.53969343062[illegible]
14	0.298216076099	0.694911658696	0.396695582[illegible]97
15	0.424455288879	0.69569886461	0.2712435[illegible]3
16	0.401035858588	0.696390581412	0.29535[illegible]722[illegible]
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22	0.222239161828	0.69924906[illegible]966	0.47700990[illegible]137
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29	0.18983930[illegible]	0.701[illegible]13920409	0.511274616585
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54	0.188240024126	0.703863103844	0.515623079717

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56	0.18394472663	0.703977928859	0.52003320223
57	0.0608581662976	0.704032345758	0.643174179461
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59	0.119737637147	0.704135691838	0.584398054691
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153	0.0672293074636	0.705955138423	0.638725830959
154	0.33389546908	0.705962592461	0.37206712338
155	0.0663358528219	0.705969950626	0.639634097804
156	0.132236236875	0.705977214756	0.573740977881
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177	0.15386479491	0.706110854167	0.5522460[illegible]57
178	0.0306858239975	0.706116433612	0.67543[illegible]609[illegible]
179	0.0918846747414	0.70612195089	0.6142372761[illegible]
180	0.274117595125	0.706127407035	0.4[illegible]09811909
181	0.0606060861584	0.706132803056	[illegible].645[illegible]716898
182	0.0302613798056	0.706138139943	0.67587[illegible]0137
183	0.0604398137253	0.706143418[illegible]61	0.6457036[illegible]6
184	0.0904534874729	0.70614864[illegible]157	0.6156951[illegible]2685
185	0.0902077998622	0.7061538[illegible]5357	0.615[illegible]46005495
186	0.119898323443	0.7061589[illegible]65	0.586[illegible]60591722
187	0.0896422260953	0.706163970[illegible]	0.61[illegible]5521744374
188	0.0894030725908	[illegible]168972133	[illegible]616765899544
189	0.325795674552	0.[illegible]061[illegible]1013	0.38037824646
190	0.0294627853245	0.7[illegible]617881[illegible]3	0.676716032609
191	0.1469321[illegible]26	0.70[illegible]183663711	0.55925154645
192	0.117091[illegible]545[illegible]	0.706[illegible]88459143	0.589097004565
193	0.1458[illegible]29034	0.7061[illegible]3205009	0.560389914667
194	0.087185[illegible]02[illegible]	[illegible]197902074	0.619012103049
195	0.[illegible]8696572[illegible]67	0.706202551087	0.619236826651
196	[illegible]156148239[illegible]	0.706207152782	0.590592328797
19[illegible]	0.[illegible]70263220[illegible]	0.706211707877	0.533509075668
198	0.057[illegible]75504888	0.706216217077	0.648838666588
199	0.08588[illegible]50666	0.706220681072	0.620330866005
200	0.08567[illegible]3858248	0.706225100537	0.620545714713

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