

On the Representation of Integer Numbers by the sum of Cube roots

Prof. Dr. Raja Rama Gandhi and Edigles Guedes

¹Resource person in Math for Oxford University Press, Professor in Math, BITS-Vizag.

²World order Number Theorist, Pernambuco, Brazil.

e-mail:edigles.guedes@gmail.com

ABSTRACT. We developed some formulas to represent integer numbers as the sum of cube roots.

Lemma 1. If $c = \frac{a}{b}$ and $b \neq 0$, then

$$ab^2 + ab^2c = b^3c + a^2b.$$

Proof. Suppose that

$$y = 1 + zy,$$

then, we can do

$$z = \frac{t-1}{t}$$

and

$$y = t.$$

Let $t = \frac{a}{b}$ in numerator and $t = c$ in denominator, such that $c = \frac{a}{b}$. Therefore,

$$\frac{a}{b} = 1 + \frac{\frac{a}{b} - 1}{c} \left(\frac{a}{b}\right),$$

$$\frac{a}{b} = 1 + \frac{(a-b)a}{b^2c},$$

$$\frac{a}{b} = \frac{b^2c + (a-b)a}{b^2c},$$

$$ab^2c = b^3c + a^2b - ab^2,$$

$$ab^2 + ab^2c = b^3c + a^2b. \square$$

Corollary 1. If $b = \frac{a}{c}$ or $b = a$ or $b = 0$ and $c \neq 0$, then b satisfies the equation

$$cx^3 - a(c+1)x^2 + a^2x = 0.$$

Proof. We substitute $x = b$ in Lemma1 and use a bit of algebraic manipulation. \square

Lemma 2. If $c = \frac{a}{b}$ and $b \neq 0$, then

$$a^2b + ab^2c^2 = a^3 + b^3c^2.$$

Proof. Suppose that

$$y = 1 + zy^2,$$

then, we can do

$$z = \frac{t - 1}{t^2}$$

and

$$y = t.$$

Let $t = \frac{a}{b}$ and $t^2 = c^2$, such that $c = \frac{a}{b}$. Therefore,

$$\frac{a}{b} = 1 + \frac{\frac{a}{b} - 1}{c^2} \left(\frac{a}{b}\right)^2,$$

$$\frac{a}{b} = 1 + \frac{(a - b)a^2}{b^3c^2},$$

$$\frac{a}{b} = \frac{b^3c^2 + (a - b)a^2}{b^3c^2},$$

$$a = \frac{b^3c^2 + (a - b)a^2}{b^2c^2},$$

$$ab^2c^2 = b^3c^2 + a^3 - a^2b,$$

$$a^2b + ab^2c^2 = a^3 + b^3c^2. \square$$

Corollary 2. If $b = \frac{a}{c}$ or $b = -\frac{a}{c}$ or $b = a$ and $c \neq 0$, then b satisfies the equation

$$c^2x^3 - ac^2x^2 - a^2x + a^3 = 0.$$

Proof. Substitute $b = x$ in Lemma 2 and use a bit of algebraic manipulation. \square

Theorem 1. For $c \in \mathbb{Z}_{>1}$, then any integer a has the following representation of the two cube roots

$$a = \sqrt[3]{\frac{1}{2} \left(-\frac{a^3(c+1)}{c^2} + \sqrt{\frac{a^6(c+1)^2}{c^4} - \frac{4a^6(c^2+c+1)^3}{27c^6}} \right)} + \sqrt[3]{\frac{1}{2} \left(-\frac{a^3(c+1)}{c^2} - \sqrt{\frac{a^6(c+1)^2}{c^4} - \frac{4a^6(c^2+c+1)^3}{27c^6}} \right)}.$$

Proof. By Cardano's formula, a root of $x^3 + px + q = 0$ is given by

$$x_1 = \sqrt[3]{\frac{1}{2} \left(-q + \sqrt{q^2 + \frac{4p^3}{27}} \right)} + \sqrt[3]{\frac{1}{2} \left(-q - \sqrt{q^2 + \frac{4p^3}{27}} \right)}. \tag{1}$$

Suppose that $x = a$, and

$$x^3 = -px - q. \quad (2)$$

By Corollary 1, we obtain

$$\begin{aligned} c(-px - q) - a(c + 1)x^2 + a^2x &= 0, \\ -a(c + 1)x^2 + (a^2 - cp)x - cq &= 0, \\ a(c + 1)x^2 - (a^2 - cp)x + cq &= 0. \end{aligned}$$

By Bhaskara's formula, we find

$$x = \frac{(a^2 - cp) \pm \sqrt{(a^2 - cp)^2 - 4acq(c + 1)}}{2a(c + 1)}. \quad (3)$$

On the other hand, by Corollary 2 and (2), we obtain

$$\begin{aligned} c^2(-px - q) - ac^2x^2 - a^2x + a^3 &= 0, \\ -ac^2x^2 - (a^2 + c^2p)x + a^3 - c^2q &= 0. \end{aligned}$$

By Bhaskara's formula, we put

$$x = -\frac{a^2 + c^2p \pm \sqrt{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}}{2ac^2}. \quad (4)$$

We compare (3) with (4), and find

$$-\frac{a^2 + c^2p}{2ac^2} = \frac{a^2 - cp}{2a(c + 1)}$$

and

$$\frac{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}{4a^2c^4} = \frac{(a^2 - cp)^2 - 4acq(c + 1)}{4a^2(c + 1)^2}.$$

Solving the system of equations above, we get

$$p = -\frac{a^2(c^2 + c + 1)}{c^2} \quad (5)$$

and

$$q = \frac{a^3(c + 1)}{c^2}. \quad (6)$$

Therefrom, we substitute (5) and (6) in (1) and let $x_1 = a$, so we complete the proof. \square

1. Examples

For $a = 3$ and $c = 2$, then

$$3 = \sqrt[3]{\frac{1}{2}\left(-\frac{81}{4} + \frac{15i\sqrt{3}}{2}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{81}{4} - \frac{15i\sqrt{3}}{2}\right)};$$

for $a = 3$ and $c = 3$, then

$$3 = \sqrt[3]{\frac{1}{2}\left(-12 + \frac{70i}{3\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-12 - \frac{70i}{3\sqrt{3}}\right)};$$

for $a = 4$ and $c = 2$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(-48 + \frac{160i}{3\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-48 - \frac{160i}{3\sqrt{3}}\right)};$$

for $a = 4$ and $c = 3$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(-\frac{256}{9} + \frac{4480i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{256}{9} - \frac{4480i}{81\sqrt{3}}\right)};$$

for $a = 5$ and $c = 2$, then

$$5 = \sqrt[3]{\frac{1}{2}\left(-\frac{375}{4} + \frac{625i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{375}{4} - \frac{625i}{6\sqrt{3}}\right)};$$

for $a = 5$ and $c = 3$, then

$$5 = \sqrt[3]{\frac{1}{2}\left(-\frac{500}{9} + \frac{8750i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{500}{9} - \frac{8750i}{81\sqrt{3}}\right)}.$$

Corollary 3. For $c \in \mathbb{Z}_{>1}$, then any rational number $\frac{a}{b}$ and $b \neq 0$, has the following representation by a sum of the two cube roots

$$\frac{a}{b} = \sqrt[3]{-\frac{1}{2}\left[\sqrt{\frac{a^6(c+1)^2}{b^6c^4} - \frac{4a^6(c^2+c+1)^3}{27b^6c^6}} + \frac{a^3(c+1)}{b^3c^2}\right]} + \sqrt[3]{\frac{1}{2}\left[\sqrt{\frac{a^6(c+1)^2}{b^6c^4} - \frac{4a^6(c^2+c+1)^3}{27b^6c^6}} - \frac{a^3(c+1)}{b^3c^2}\right]}.$$

Proof. Let $a \rightarrow \frac{a}{b}$ in Theorem 1, this completes the proof. \square

2. Examples

For $a = 2, b = 3$ and $c = 2$, then

$$\frac{2}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{2}{9} + \frac{20i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{2}{9} - \frac{20i}{81\sqrt{3}}\right)};$$

for $a = 2, b = 3$ and $c = 3$, then

$$\frac{2}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{32}{243} + \frac{560i}{2187\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{32}{243} - \frac{560i}{2187\sqrt{3}}\right)};$$

for $a = 1, b = 3$ and $c = 2$, then

$$\frac{1}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{1}{36} + \frac{5i}{162\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{1}{36} - \frac{5i}{162\sqrt{3}}\right)};$$

for $a = 1, b = 3$ and $c = 3$, then

$$\frac{1}{3} = \sqrt[3]{\frac{1}{2}\left(-\frac{4}{243} + \frac{70i}{2187\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{243} - \frac{70i}{2187\sqrt{3}}\right)}.$$

Theorem 2. For $c \in \mathbb{Z}_{\geq 1}$, then any integer a has the following representation by the sum of the two cube roots

$$a = \sqrt[3]{\frac{1}{2}\left(\frac{a^3}{c} + \sqrt{\frac{a^6}{c^2} - \frac{4a^6(c-1)^3}{27c^3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{a^3}{c} - \sqrt{\frac{a^6}{c^2} - \frac{4a^6(c-1)^3}{27c^3}}\right)}.$$

Proof. By Cardano's formula, a root of $x^3 + px + q = 0$ is given by

$$x_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}. \quad (7)$$

Suppose that $x = \frac{a}{c}$, thus,

$$x^3 = -px - q. \quad (8)$$

By Corollary 2, we obtain

$$-ac^2x^2 - (a^2 + c^2p)x + a^3 - c^2q = 0,$$

by Baskara's formula

$$x = -\frac{a^2 + c^2p \pm \sqrt{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}}{2ac^2}. \quad (9)$$

By Corollary 3, we encounter

$$\begin{aligned} c^3x(-px - q) - ac^3(-px - q) - a^3x + a^4 &= 0, \\ -c^3px^2 + (ac^3p - c^3q - a^3)x + a^4 + ac^3q &= 0. \end{aligned}$$

Again, by Baskara's formula

$$x = -\frac{a^3 + c^3q - ac^3p \pm \sqrt{(ac^3p - c^3q - a^3)^2 + 4c^3p(a^4 + ac^3q)}}{2c^3p}. \quad (10)$$

Compare (7) with (8), and we find

$$\frac{a^2 + c^2p}{2ac^2} = \frac{a^3 + c^3q - ac^3p}{2c^3p}$$

and

$$\frac{\sqrt{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}}{2ac^2} = \frac{\sqrt{(ac^3p - c^3q - a^3)^2 + 4c^3p(a^4 + ac^3q)}}{2c^3p} \therefore$$

$$\frac{(a^2 + c^2p)^2 + 4ac^2(a^3 - c^2q)}{4a^2c^4} = \frac{(ac^3p - c^3q - a^3)^2 + 4c^3p(a^4 + ac^3q)}{4c^6p^2}.$$

Solving the system of equations above, we get

$$p = -\frac{a^2(c-1)}{c} \tag{10}$$

and

$$q = -\frac{a^3}{c} \tag{11}$$

or

$$p = -\frac{a^2(c^2+c+1)}{c^2} \tag{12}$$

and

$$q = \frac{a^3(c+1)}{c^2}. \tag{13}$$

The solutions (12) and (13) are equals to solutions (5) and (6); thereof, we replace (10) and (11) in (7), and let $x_1 = a$, this completes the proof. \square

3. Examples

For $a = 3$ and $c = 2$, then

$$3 = \sqrt[3]{\frac{1}{2}\left(\frac{27}{2} + \frac{15\sqrt{3}}{2}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{27}{2} - \frac{15\sqrt{3}}{2}\right)};$$

for $a = 3$ and $c = 5$, then

$$3 = \sqrt[3]{\frac{1}{2}\left(\frac{27}{5} + \frac{33}{5}i\sqrt{\frac{3}{5}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{27}{5} - \frac{33}{5}i\sqrt{\frac{3}{5}}\right)};$$

for $a = 4$ and $c = 2$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(32 + \frac{160}{3\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(32 - \frac{160}{3\sqrt{3}}\right)};$$

for $a = 5$ and $c = 2$, then

$$5 = \sqrt[3]{\frac{1}{2}\left(\frac{125}{2} + \frac{625}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{125}{2} - \frac{625}{6\sqrt{3}}\right)}$$

for $a = 4$ and $c = 5$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(\frac{64}{5} + \frac{704i}{15\sqrt{15}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{64}{5} - \frac{704i}{15\sqrt{15}}\right)};$$

for $a = 4$ and $c = 3$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(-\frac{256}{9} + \frac{4480i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{256}{9} - \frac{4480i}{81\sqrt{3}}\right)};$$

for $a = 4$ and $c = 7$, then

$$4 = \sqrt[3]{\frac{1}{2}\left(\frac{64}{7} + \frac{320i}{7\sqrt{7}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{64}{7} - \frac{320i}{7\sqrt{7}}\right)};$$

for $a = 5$ and $c = 8$, then

$$5 = \sqrt[3]{\frac{1}{2}\left(\frac{125}{8} + \frac{2125i}{24\sqrt{6}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{125}{8} - \frac{2125i}{24\sqrt{6}}\right)}.$$

Corollary 4. For $c \in \mathbb{Z}_{\geq 1}$, then any rational number $\frac{a}{b}$ and $b \neq 0$, has the following representation by the sum of the two cube roots

$$\frac{a}{b} = \sqrt[3]{\frac{1}{2}\left[\frac{a^3}{b^3c} - \sqrt{\frac{a^6}{b^6c^2} - \frac{4a^6(c-1)^3}{27b^6c^3}}\right]} + \sqrt[3]{\frac{1}{2}\left[\frac{a^3}{b^3c} + \sqrt{\frac{a^6}{b^6c^2} - \frac{4a^6(c-1)^3}{27b^6c^3}}\right]}$$

Proof. Let $a \rightarrow \frac{a}{b}$ in Theorem 2, this completes the proof. \square

4. Examples

For $a = 1, b = 3$ and $c = 2$, then

$$\frac{1}{3} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{54} + \frac{5}{162\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{54} - \frac{5}{162\sqrt{3}}\right)};$$

for $a = 1, b = 4$ and $c = 2$, then

$$\frac{1}{4} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{128} + \frac{5}{384\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{128} - \frac{5}{384\sqrt{3}}\right)};$$

for $a = 1, b = 5$ and $c = 2$, then

$$\frac{1}{5} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{250} + \frac{1}{150\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{250} - \frac{1}{150\sqrt{3}}\right)};$$

for $a = 1, b = 6$ and $c = 2$, then

$$\frac{1}{6} = \sqrt[3]{\frac{1}{2}\left(\frac{1}{432} + \frac{5}{1296\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{432} - \frac{5}{1296\sqrt{3}}\right)}.$$

Corollary 5. For $c \in \mathbb{Z}_{\geq 1}$, then

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{c} + \sqrt{\frac{1}{c^2} - \frac{4(c-1)^3}{27c^3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{c} - \sqrt{\frac{1}{c^2} - \frac{4(c-1)^3}{27c^3}}\right)}.$$

Proof. Simplifying the right-hand side of Theorem 2 and dividing both members for a . \square

5. Examples

For $c = 2$,

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{2} + \frac{5}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{2} - \frac{5}{6\sqrt{3}}\right)};$$

for $c = 5$,

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{5} + \frac{11i}{15\sqrt{15}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{5} - \frac{11i}{15\sqrt{15}}\right)};$$

for $c = 7$,

$$1 = \sqrt[3]{\frac{1}{2}\left(\frac{1}{7} + \frac{5i}{7\sqrt{7}}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{1}{7} - \frac{5i}{7\sqrt{7}}\right)}.$$

Lemma 3. For $c \in \mathbb{Z}_{>1}$, then

$$1 = \sqrt[3]{\frac{1}{2}\left(-\frac{c+1}{c^2} + \sqrt{\frac{(c+1)^2}{c^4} - \frac{4(c^2+c+1)^3}{27c^6}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{c+1}{c^2} - \sqrt{\frac{(c+1)^2}{c^4} - \frac{4(c^2+c+1)^3}{27c^6}}\right)}.$$

Proof. Simplifying the right-hand side of Theorem 1 and dividing both members for a . \square

Theorem 4. For $n \in \mathbb{Z}_{>1}$, then n has the following representation by the sum of cube roots

$$n = \sum_{k=2}^{n+1} \left[\sqrt[3]{\frac{1}{2}\left(-\frac{k+1}{k^2} + \sqrt{\frac{(k+1)^2}{k^4} - \frac{4(k^2+k+1)^3}{27k^6}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{k+1}{k^2} - \sqrt{\frac{(k+1)^2}{k^4} - \frac{4(k^2+k+1)^3}{27k^6}}\right)} \right].$$

Proof. Using the Lemma 3 and finite induction, this completes the proof. \square

6. Examples

$$1 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)};$$

$$2 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} + \frac{70i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} - \frac{70i}{81\sqrt{3}}\right)};$$

$$3 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} + \frac{70i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} - \frac{70i}{81\sqrt{3}}\right)}$$

$$+ \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} + \frac{9i\sqrt{3}}{32}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} - \frac{9i\sqrt{3}}{32}\right)};$$

$$4 = \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} + \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{3}{4} - \frac{5i}{6\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} + \frac{70i}{81\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{4}{9} - \frac{70i}{81\sqrt{3}}\right)}$$

$$+ \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} + \frac{9i\sqrt{3}}{32}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{5}{16} - \frac{9i\sqrt{3}}{32}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{6}{25} + \frac{308i}{375\sqrt{3}}\right)} + \sqrt[3]{\frac{1}{2}\left(-\frac{6}{25} - \frac{308i}{375\sqrt{3}}\right)}.$$