On Andrica’s Conjecture, Cramér’s Conjecture, gaps Between Primes and Jacobi Theta Functions II: A Simple Proof of Asymptotic for Andrica’s Conjecture

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1. PRELIMINARES

In [1, p. 185] states that the prime number theorem yields:

\[ p_n \sim n \log n, \]

that is,

\[ \lim_{n \to \infty} \frac{p_n}{n \log n} = 1. \]

LEMMA 1. The Andrica’s conjecture is equivalent to

\[ p_{n+1} < 1 + 2\sqrt{p_n} - p_n. \]  

Proof. Step 1. In [2, p. ___], we conclude that Andrica’s conjecture is equivalent to

\[ \sqrt{p_n} < \sqrt{\theta_3 - \theta_2} \Rightarrow \frac{1}{\sqrt{\theta_2}} < \sqrt{\frac{p_n}{\theta_3}} \]

where \( k := \frac{p_n}{p_{n+1}} \) is a k modulus. In [3, p. 83], we encounter

\[ p_n \sim k^{1/2}. \]

Substituting (5) in (4) and considering \( k = \frac{p_n}{p_{n+1}} \), we have

\[ \sqrt{p_n} < \frac{\theta_2}{\theta_3 - \theta_2} \Leftrightarrow \frac{\sqrt{p_{n+1}}}{\sqrt{p_n}} < 1 + \frac{1}{\sqrt{p_n}} \]

ergo, squaring both sides of the equation (6),

\[ \frac{p_{n+1}}{p_n} < 1 + \frac{2}{\sqrt{p_n}} + \frac{1}{p_n} \Leftrightarrow p_{n+1} < 1 + 2\sqrt{p_n} + p_n. \]

2. THEOREM

THEOREM 1 (Asymptotic for Andrica’s Conjecture). Let \( n \in \mathbb{N} \) and \( n \) sufficiently large, then

\[ \sqrt{p_{n+1}} - \sqrt{p_n} < 1. \]

Proof. Henceforth, we will use the reductio ad absurdum to prove the Lemma 1. We assume that

\[ p_{n+1} \geq 1 + 2\sqrt{p_n} + p_n. \]
For \( n \) sufficiently large, we set (1) in (7), as follows

\[
(n + 1) \log(n + 1) \geq 1 + 2\sqrt{n \log n} + n \log n. \tag{8}
\]

Dividing (8) by \( \log(n + 1) \), we encounter

\[
n + 1 \geq \frac{1}{\log(n+1)} + 2\sqrt{\frac{n \log n}{\log(n+1)}} + n \frac{\log n}{\log(n+1)}. \tag{9}
\]

On the other hand, it is easy to see that, as \( n \) is sufficiently large, then

\[
\frac{\log n}{\log(n+1)} \to 1, \quad \frac{1}{\log(n+1)} \to 0, \quad \frac{\sqrt{n \log n}}{\log(n+1)} \to \infty. \tag{10}
\]

namely,

\[
\lim_{n \to \infty} \frac{\log n}{\log(n+1)} = 1, \quad \lim_{n \to \infty} \frac{1}{\log(n+1)} = 1, \quad \lim_{n \to \infty} \frac{\sqrt{n \log n}}{\log(n+1)} = \infty.
\]

Substituting (10) in (9), we find

\[
n + 1 \geq 0 + 2(\infty) + n \cdot 1 \iff 1 > 2(\infty), \tag{9}
\]

which is false. Therefore, for \( n \) sufficiently large, \( p_{n+1} < 1 + 2\sqrt{p_n} + p_n \). In face of Lemma 1, the asymptotic for Andrica's conjecture is proved. \( \square \)

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REFERENCES


[2] Prof. Dr. Raja Rama Gandhi and Guedes, Edigles, On Andrica’s Conjecture, gaps Between Primes and Jacobi Theta Functions.