On Andrica’s Conjecture, Cramér’s Conjecture, gaps Between Primes and Jacobi Theta Functions I

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Abstract. The main objective of this paper is to develop upper and lower bound for the Andrica conjecture, gaps between primes, using Jacobi elliptic functions.

1. Introduction

In [1, p. 34] Richard K. Guy posted that Dorin Andrica conjectures that, for all natural $n$, we have

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1,$$

consequently, dividing both sides of the equation (1) by $\sqrt{p_n}$, we have

$$\frac{\sqrt{p_{n+1}}}{\sqrt{p_n}} - \frac{1}{\sqrt{p_n}} < 1.$$  

We will use the following notation for gaps between primes:

$$g(p_n) := p_{n+1} - p_n,$$

that is related to Cramér’s conjecture, which states

$$\lim_{n \to \infty} \sup \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1,$$

and the Rosser’s theorem [2], which states that $p_n$ is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + \log \log n - 1 < \frac{p_n}{n} < \log n + \log \log n,$$

for $n > 6$.

2. Theorems

Theorem 1. Let $k := \frac{p_n}{p_{n+1}}$ to be a $k$ modulus and $n \geq 6$, then

$$\left(\frac{\theta_3}{\theta_2}\right)\sqrt{n \log n + n \log \log n} - n < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3}{\theta_2}\right)\sqrt{n \log n + n \log \log n},$$

where $\theta_2$ and $\theta_3$ are Jacobi theta functions.

Proof. Firstly, we consider the sequence of prime numbers

$$2 < 3 < 5 < 7 < 11 < 13 < 17 < 19 < \cdots < p_{n-2} < p_{n-1} < p_n < p_{n+1}.$$  

Second, we note that

$$0 < \frac{2}{3} < 1, 0 < \frac{3}{5} < 1, 0 < \frac{5}{7} < 1, 0 < \frac{7}{11} < 1, 0 < \frac{11}{13} < 1, 0 < \frac{13}{17} < 1, 0 < \frac{17}{19} < 1, \ldots,$$
\[
0 < \frac{p_{n-2}}{p_{n-1}} < 1, 0 < \frac{p_{n-1}}{p_n} < 1, 0 < \frac{p_n}{p_{n+1}} < 1.
\]

Then, we define that
\[
k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \iff p_{n+1} = \frac{p_n}{k},
\]

where \(k_{n,n+1}\) is the \(k\) modulus.

Substituting (6) in the left-hand side of (1), we find
\[
\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{\frac{p_n}{k^{1/2}}} - \sqrt{p_n} = \sqrt{p_n} \left( \frac{1 - k^{1/2}}{k^{1/2}} \right).
\]

In [3, p. 83], we knew that
\[
k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}.
\]

where \(\tau\) is the parameter and \(\theta_2(z|\tau)\) and \(\theta_3(z|\tau)\) are Jacobi theta functions.

We set (8) in (7)
\[
\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{p_n} \left( 1 - \frac{\theta_3}{\theta_2} \right) = \sqrt{p_n} \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) = \sqrt{p_n} \left( \frac{\theta_3 - \theta_2}{\theta_2} \right).
\]

From (3) and (9), we conclude that
\[
\left( \frac{\theta_3 - \theta_2}{\theta_2} \right) \sqrt{n \log n + n \log \log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) \sqrt{n \log n + n \log \log n}.
\]

**COROLLARY 1.** Let \(k := \frac{p_n}{p_{n+1}}\) to be a \(k\) modulus, then Andrica’s conjecture is equivalent to
\[
\sqrt{p_n} < \frac{\theta_2}{\theta_3 - \theta_2}.
\]

**Proof.** Dividing both members of (9) by \(\sqrt{p_n}\), we have
\[
\sqrt{\frac{p_{n+1}}{p_n}} - \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) = 1.
\]

Comparing (2) with (10) and after some algebraic manipulation, we find
\[
\frac{\theta_3 - \theta_2}{\theta_2} < \frac{1}{\sqrt{p_n}}
\]

therefrom,
\[
\sqrt{p_n} < \frac{\theta_2}{\theta_3 - \theta_2}.
\]

**THEOREM 2.** Let \(k := \frac{p_n}{p_{n+1}}\) to be a \(k\) modulus, then
\[
\left( \frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left( \frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n),
\]

where \(g(p_n)\) is a function depending on \(p_n\).
where $\theta_2$ and $\theta_3$ are Jacobi theta functions.

**Proof.** We define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \iff p_{n+1} = \frac{p_n}{k},$$

(12)

where $k_{n,n+1}$ is the $k$ modulus.

We consider that

$$g(p_n) := p_{n+1} - p_n = \frac{p_n}{(k^{1/2})^2} - p_n = p_n \left[ 1 - \left( \frac{1}{(k^{1/2})^2} \right) \right].$$

(13)

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

(14)

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (14) in (13)

$$g(p_n) = p_{n+1} - p_n = p_n \left[ 1 - \left( \frac{\theta_2^2}{\theta_3^2} \right) \right] = p_n \left[ \frac{\theta_2^2 - \theta_3^2}{\theta_3^2} \right].$$

(15)

From (3) and (15), we conclude that

$$\left( \frac{\theta_3^2 - \theta_2^2}{\theta_3^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left( \frac{\theta_3^2 - \theta_2^2}{\theta_3^2} \right) (n \log n + n \log \log n). \quad \Box$$

**Theorem 3.** Let $k := \frac{p_n}{p_{n+1}}$ to be a $k$ modulus, then

$$\left( \frac{\theta_3^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < \theta_{n+1}^2 - \theta_n^2 < \left( \frac{\theta_3^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2,$$

where $\theta_2$ and $\theta_3$ are Jacobi theta functions.

**Proof.** We define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \iff p_{n+1} = \frac{p_n}{k},$$

(16)

where $k_{n,n+1}$ is the $k$ modulus.

We consider that

$$p_{n+1}^2 - p_n^2 = \frac{p_n^2}{(k^{1/2})^4} - p_n^2 = p_n^2 \left[ 1 - \left( \frac{1}{(k^{1/2})^4} \right) \right].$$

(17)

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

(18)

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.
We set (18) in (17)

$$p_{n+1}^2 - p_n^2 = p_n^2 \left( \frac{1 - \theta_2^4}{\theta_3^4} \right) = p_n^2 \left( \frac{\theta_3^4 - \theta_2^4}{\theta_3^4 - \theta_2^4} \right) = p_n^2 \left( \frac{\theta_4^4 - \theta_2^4}{\theta_2^4} \right) = p_n^2 \left( \frac{\theta_4^4}{\theta_2^4} \right).$$

(19)

to see [3, p. 84] which states $\theta_2^4 + \theta_4^4 = \theta_3^4$, the Jacobi identity.

From (3) and (19), we conclude that

$$\left( \frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left( \frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2. \square$$

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REFERENCES

