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# On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions I

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**ABSTRACT.** The main objective of this paper is to develop upper and lower bound for the Andrica conjecture, gaps between primes, using Jacobi elliptic functions.

## 1. INTRODUCTION

In [1, p. 34] Richard K. Guy posted that Dorin Andrica conjectures that, for all natural $n$, we have

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1, \tag{1}$$

consequently, dividing both sides of the equation (1) by $\sqrt{p_n}$, we have

$$\sqrt{\frac{p_{n+1}}{p_n}} - \frac{1}{\sqrt{p_n}} < 1. \tag{2}$$

We will use the following notation for gaps between primes:

$$g(p_n) := p_{n+1} - p_n,$$

that is related to Cramér's conjecture, which states

$$\lim_{n\to\infty} \sup \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1,$$

and the Rosser's theorem [2], which states that $p_n$ is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + \log\log n - 1 < \frac{p_n}{n} < \log n + \log\log n, \tag{3}$$

for $n > 6$.

## 2. THEOREMS

**THEOREM 1**. *Let* $k := \frac{p_n}{p_{n+1}}$ *to be a k modulus and* $n \geq 6$ *, then*

$$\left(\frac{\theta_3 - \theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3 - \theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n},$$

*where* $\theta_2$ *and* $\theta_3$ *are Jacobi theta functions.*

*Proof*. Firstly, we consider the sequence of prime numbers

$$2 < 3 < 5 < 7 < 11 < 13 < 17 < 19 < \cdots p_{n-2} < p_{n-1} < p_n < p_{n+1}. \tag{4}$$

Second, we note that

$$0 < \tfrac{2}{3} < 1, 0 < \tfrac{3}{5} < 1, 0 < \tfrac{5}{7} < 1, 0 < \tfrac{7}{11} < 1, 0 < \tfrac{11}{13} < 1, 0 < \tfrac{13}{17} < 1, 0 < \tfrac{17}{19} < 1, \ldots, \tag{5}$$

$$0<\frac{p_{n-2}}{p_{n-1}}<1,0<\frac{p_{n-1}}{p_n}<1,0<\frac{p_n}{p_{n+1}}<1.$$

Then, we define that

$$k:=k_{n,n+1}=\frac{p_n}{p_{n+1}}\Leftrightarrow p_{n+1}=\frac{p_n}{k}, \tag{6}$$

where $k_{n,n+1}$ is the $k$ modulus.

Substituting (6) in the left-hand side of (1), we find

$$\sqrt{p_{n+1}}-\sqrt{p_n}=\frac{\sqrt{p_n}}{k^{1/2}}-\sqrt{p_n}=\sqrt{p_n}\left(\frac{1-k^{1/2}}{k^{1/2}}\right). \tag{7}$$

In [3, p. 83], we knew that

$$k^{1/2}=\frac{\theta_2}{\theta_3}=\frac{\theta_2(0)}{\theta_3(0)}=\frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \tag{8}$$

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (8) in (7)

$$\sqrt{p_{n+1}}-\sqrt{p_n}=\sqrt{p_n}\left(\frac{1-\frac{\theta_2}{\theta_3}}{\frac{\theta_2}{\theta_3}}\right)=\sqrt{p_n}\left(\frac{\frac{\theta_3-\theta_2}{\theta_3}}{\frac{\theta_2}{\theta_3}}\right)=\sqrt{p_n}\left(\frac{\theta_3-\theta_2}{\theta_2}\right). \tag{9}$$

From (3) and (9), we conclude that

$$\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n+n\log\log n-n}<\sqrt{p_{n+1}}-\sqrt{p_n}<\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n+n\log\log n}.\ \square$$

COROLLARY 1. *Let* $k:=\frac{p_n}{p_{n+1}}$ *to be a k modulus, then Andrica's conjecture is equivalent to*

$$\sqrt{p_n}<\frac{\theta_2}{\theta_3-\theta_2}.$$

*Proof.* Dividing both members of (9) by $\sqrt{p_n}$, we have

$$\sqrt{\frac{p_{n+1}}{p_n}}-\left(\frac{\theta_3-\theta_2}{\theta_2}\right)=1. \tag{10}$$

Comparing (2) with (10) and after some algebraic manipulation, we find

$$\frac{\theta_3-\theta_2}{\theta_2}<\frac{1}{\sqrt{p_n}}, \tag{11}$$

therefrom,

$$\sqrt{p_n}<\frac{\theta_2}{\theta_3-\theta_2}.\ \square$$

**THEOREM 2.** *Let* $k:=\frac{p_n}{p_{n+1}}$ *to be a k modulus, then*

$$\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n-n)<g(p_n)<\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n),$$

*where $\theta_2$ and $\theta_3$ are Jacobi theta functions.*

*Proof.* We define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \Leftrightarrow p_{n+1} = \frac{p_n}{k}, \tag{12}$$

where $k_{n,n+1}$ is the $k$ modulus.

We consider that

$$g(p_n) := p_{n+1} - p_n = \frac{p_n}{\left(k^{1/2}\right)^2} - p_n = p_n\left[\frac{1-\left(k^{1/2}\right)^2}{\left(k^{1/2}\right)^2}\right]. \tag{13}$$

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \tag{14}$$

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (14) in (13)

$$g(p_n) = p_{n+1} - p_n = p_n\left(\frac{1-\frac{\theta_2^2}{\theta_3^2}}{\frac{\theta_2^2}{\theta_3^2}}\right) = p_n\left(\frac{\frac{\theta_3^2-\theta_2^2}{\theta_3^2}}{\frac{\theta_2}{\theta_3^2}}\right) = p_n\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right). \tag{15}$$

From (3) and (15), we conclude that

$$\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n - n) < g(p_n) < \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2}\right)(n\log n + n\log\log n). \square$$

**THEOREM 3.** *Let $k := \frac{p_n}{p_{n+1}}$ to be a k modulus, then*

$$\left(\frac{\theta_4^4}{\theta_2^4}\right)(n\log n + n\log\log n - n)^2 < p_{n+1}^2 - p_n^2 < \left(\frac{\theta_4^4}{\theta_2^4}\right)(n\log n + n\log\log n)^2,$$

*where $\theta_2$ and $\theta_3$ are Jacobi theta functions.*

*Proof.* We define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \Leftrightarrow p_{n+1} = \frac{p_n}{k}, \tag{16}$$

where $k_{n,n+1}$ is the $k$ modulus.

We consider that

$$p_{n+1}^2 - p_n^2 = \frac{p_n^2}{\left(k^{1/2}\right)^4} - p_n^2 = p_n^2\left[\frac{1-\left(k^{1/2}\right)^4}{\left(k^{1/2}\right)^4}\right]. \tag{17}$$

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \tag{18}$$

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (18) in (17)

$$p_{n+1}^2 - p_n^2 = p_n^2\left(\frac{1-\frac{\theta_2^4}{\theta_3^4}}{\frac{\theta_2^4}{\theta_3^4}}\right) = p_n^2\left(\frac{\frac{\theta_3^4-\theta_2^4}{\theta_3^4}}{\frac{\theta_2^4}{\theta_3^4}}\right) = p_n^2\left(\frac{\theta_3^4-\theta_2^4}{\theta_2^4}\right) = p_n^2\left(\frac{\theta_4^4}{\theta_2^4}\right), \tag{19}$$

to see [3, p. 84] which states $\theta_2^4 + \theta_4^4 = \theta_3^4$, the Jacobi identity.

From (3) and (19), we conclude that

$$\left(\frac{\theta_4^4}{\theta_2^4}\right)(n\log n + n\log\log n - n)^2 < p_{n+1}^2 - p_n^2 < \left(\frac{\theta_4^4}{\theta_2^4}\right)(n\log n + n\log\log n)^2. \square$$

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