On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions I

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ABSTRACT. The main objective of this paper is to develop upper and lower bound for the Andrica conjecture, gaps between primes, using Jacobi elliptic functions.

1. INTRODUCTION

In [1, p. 34] Richard K. Guy posted that Dorin Andrica conjectures that, for all natural n , we have

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1, \quad (1)$$

consequently, dividing both sides of the equation (1) by $\sqrt{p_n}$, we have

$$\sqrt{\frac{p_{n+1}}{p_n}} - \frac{1}{\sqrt{p_n}} < 1. \quad (2)$$

We will use the following notation for gaps between primes:

$$g(p_n) := p_{n+1} - p_n,$$

that is related to Cramér's conjecture, which states

$$\limsup_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1,$$

and the Rosser's theorem [2], which states that p_n is larger than $n \log n$. This can be improved by the following pair of bounds:

$$\log n + \log \log n - 1 < \frac{p_n}{n} < \log n + \log \log n, \quad (3)$$

for $n > 6$.

2. THEOREMS

THEOREM 1. Let $k := \frac{p_n}{p_{n+1}}$ to be a k modulus and $n \geq 6$, then

$$\left(\frac{\theta_3 - \theta_2}{\theta_2}\right) \sqrt{n \log n + n \log \log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3 - \theta_2}{\theta_2}\right) \sqrt{n \log n + n \log \log n},$$

where θ_2 and θ_3 are Jacobi theta functions.

Proof. Firstly, we consider the sequence of prime numbers

$$2 < 3 < 5 < 7 < 11 < 13 < 17 < 19 < \dots p_{n-2} < p_{n-1} < p_n < p_{n+1}. \quad (4)$$

Second, we note that

$$0 < \frac{2}{3} < 1, 0 < \frac{3}{5} < 1, 0 < \frac{5}{7} < 1, 0 < \frac{7}{11} < 1, 0 < \frac{11}{13} < 1, 0 < \frac{13}{17} < 1, 0 < \frac{17}{19} < 1, \dots, \quad (5)$$

$$0 < \frac{p_{n-2}}{p_{n-1}} < 1, 0 < \frac{p_{n-1}}{p_n} < 1, 0 < \frac{p_n}{p_{n+1}} < 1.$$

Then, we define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \Leftrightarrow p_{n+1} = \frac{p_n}{k}, \quad (6)$$

where $k_{n,n+1}$ is the k modulus.

Substituting (6) in the left-hand side of (1), we find

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \frac{\sqrt{p_n}}{k^{1/2}} - \sqrt{p_n} = \sqrt{p_n} \left(\frac{1-k^{1/2}}{k^{1/2}} \right). \quad (7)$$

In [3, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \quad (8)$$

where τ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (8) in (7)

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{p_n} \left(\frac{1 - \frac{\theta_2}{\theta_3}}{\frac{\theta_2}{\theta_3}} \right) = \sqrt{p_n} \left(\frac{\frac{\theta_3 - \theta_2}{\theta_3}}{\frac{\theta_2}{\theta_3}} \right) = \sqrt{p_n} \left(\frac{\theta_3 - \theta_2}{\theta_2} \right). \quad (9)$$

From (3) and (9), we conclude that

$$\left(\frac{\theta_3 - \theta_2}{\theta_2} \right) \sqrt{n \log n + n \log \log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3 - \theta_2}{\theta_2} \right) \sqrt{n \log n + n \log \log n}. \square$$

COROLLARY 1. Let $k := \frac{p_n}{p_{n+1}}$ to be a k modulus, then Andrica's conjecture is equivalent to

$$\sqrt{p_n} < \frac{\theta_2}{\theta_3 - \theta_2}.$$

Proof. Dividing both members of (9) by $\sqrt{p_n}$, we have

$$\sqrt{\frac{p_{n+1}}{p_n}} - \left(\frac{\theta_3 - \theta_2}{\theta_2} \right) = 1. \quad (10)$$

Comparing (2) with (10) and after some algebraic manipulation, we find

$$\frac{\theta_3 - \theta_2}{\theta_2} < \frac{1}{\sqrt{p_n}}, \quad (11)$$

therefrom,

$$\sqrt{p_n} < \frac{\theta_2}{\theta_3 - \theta_2}. \square$$

THEOREM 2. Let $k := \frac{p_n}{p_{n+1}}$ to be a k modulus, then

$$\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n),$$

where θ_2 and θ_3 are Jacobi theta functions.

Proof. We define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \Leftrightarrow p_{n+1} = \frac{p_n}{k}, \quad (12)$$

where $k_{n,n+1}$ is the k modulus.

We consider that

$$g(p_n) := p_{n+1} - p_n = \frac{p_n}{(k^{1/2})^2} - p_n = p_n \left[\frac{1 - (k^{1/2})^2}{(k^{1/2})^2} \right]. \quad (13)$$

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \quad (14)$$

where τ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (14) in (13)

$$g(p_n) = p_{n+1} - p_n = p_n \left(\frac{1 - \frac{\theta_2^2}{\theta_3^2}}{\frac{\theta_2^2}{\theta_3^2}} \right) = p_n \left(\frac{\frac{\theta_3^2 - \theta_2^2}{\theta_3^2}}{\frac{\theta_2^2}{\theta_3^2}} \right) = p_n \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right). \quad (15)$$

From (3) and (15), we conclude that

$$\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n). \square$$

THEOREM 3. Let $k := \frac{p_n}{p_{n+1}}$ to be a k modulus, then

$$\left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2,$$

where θ_2 and θ_3 are Jacobi theta functions.

Proof. We define that

$$k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \Leftrightarrow p_{n+1} = \frac{p_n}{k}, \quad (16)$$

where $k_{n,n+1}$ is the k modulus.

We consider that

$$p_{n+1}^2 - p_n^2 = \frac{p_n^2}{(k^{1/2})^4} - p_n^2 = p_n^2 \left[\frac{1 - (k^{1/2})^4}{(k^{1/2})^4} \right]. \quad (17)$$

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \quad (18)$$

where τ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (18) in (17)

$$p_{n+1}^2 - p_n^2 = p_n^2 \left(\frac{1 - \frac{\theta_2^4}{\theta_3^4}}{\frac{\theta_2^4}{\theta_3^4}} \right) = p_n^2 \left(\frac{\frac{\theta_3^4 - \theta_2^4}{\theta_3^4}}{\frac{\theta_2^4}{\theta_3^4}} \right) = p_n^2 \left(\frac{\theta_3^4 - \theta_2^4}{\theta_2^4} \right) = p_n^2 \left(\frac{\theta_4^4}{\theta_2^4} \right), \quad (19)$$

to see [3, p. 84] which states $\theta_2^4 + \theta_4^4 = \theta_3^4$, the Jacobi identity.

From (3) and (19), we conclude that

$$\left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2. \square$$

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