On Andrica’s Conjecture, Cramér’s Conjecture, gaps Between Primes and Jacobi Theta Functions I

Prof. Dr. Raja Rama Gandhi and Edigles Guedes

1Resource person in Math for Oxford University Press, Professor in Math, BITS-Vizag.
2World order Number Theorist, Pernambuco, Brazil.

ABSTRACT. The main objective of this paper is to develop upper and lower bound for the Andrica conjecture, gaps between primes, using Jacobi elliptic functions.

1. INTRODUCTION

In [1, p. 34] Richard K. Guy posted that Dorin Andrica conjectures that, for all natural \( n \), we have

\[
\sqrt{p_{n+1}} - \sqrt{p_n} < 1,
\]

consequently, dividing both sides of the equation (1) by \( \sqrt{p_n} \), we have

\[
\sqrt{\frac{p_{n+1}}{p_n}} - \frac{1}{\sqrt{p_n}} < 1.
\]

We will use the following notation for gaps between primes:

\[ g(p_n) := p_{n+1} - p_n, \]

that is related to Cramér’s conjecture, which states

\[
\lim_{n \to \infty} \sup \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1,
\]

and the Rosser’s theorem [2], which states that \( p_n \) is larger than \( n \log n \). This can be improved by the following pair of bounds:

\[
\log n + \log \log n - 1 < \frac{p_n}{n} < \log n + \log \log n,
\]

for \( n > 6 \).

2. THEOREMS

THEOREM 1. Let \( k := \frac{p_n}{p_{n+1}} \) to be a \( k \) modulus and \( n \geq 6 \), then

\[
\left(\frac{\theta_3 - \theta_2}{\theta_2}\right)\sqrt{n \log n + n \log \log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3 - \theta_2}{\theta_2}\right)\sqrt{n \log n + n \log \log n},
\]

where \( \theta_2 \) and \( \theta_3 \) are Jacobi theta functions.

Proof. Firstly, we consider the sequence of prime numbers

\[
2 < 3 < 5 < 7 < 11 < 13 < 17 < 19 < \cdots < p_{n-2} < p_{n-1} < p_n < p_{n+1}.
\]

Second, we note that

\[
0 < \frac{2}{3} < 1, \quad \frac{3}{5} < 1, \quad \frac{5}{7} < 1, \quad \frac{7}{11} < 1, \quad \frac{11}{13} < 1, \quad \frac{13}{17} < 1, \quad 1, \quad \ldots,
\]
Then, we define that

\[ k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \iff p_{n+1} = \frac{p_n}{k}, \tag{6} \]

where \( k_{n,n+1} \) is the \( k \) modulus.

Substituting (6) in the left-hand side of (1), we find

\[ \sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{\frac{p_n}{k^{1/2}}} - \sqrt{p_n} = \sqrt{p_n} \left( \frac{1-k^{1/2}}{k^{1/2}} \right). \tag{7} \]

In [3, p. 83], we knew that

\[ k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)}, \tag{8} \]

where \( \tau \) is the parameter and \( \theta_2(z|\tau) \) and \( \theta_3(z|\tau) \) are Jacobi theta functions.

We set (8) in (7)

\[ \sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{p_n} \left( 1 - \theta_2 \theta_3 \right) = \sqrt{p_n} \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) = \sqrt{p_n} \left( \frac{\theta_3 - \theta_2}{\theta_2} \right). \tag{9} \]

From (3) and (9), we conclude that

\[ \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) \sqrt{n \log n + n \log \log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) \sqrt{n \log n + n \log \log n}. \]

**COROLLARY 1.** Let \( k := \frac{p_n}{p_{n+1}} \) to be a \( k \) modulus, then Andrica’s conjecture is equivalent to

\[ \sqrt{p_n} < \frac{\theta_2}{\theta_3 - \theta_2}. \]

**Proof.** Dividing both members of (9) by \( \sqrt{p_n} \), we have

\[ \sqrt{\frac{p_{n+1}}{p_n}} - \left( \frac{\theta_3 - \theta_2}{\theta_2} \right) = 1. \tag{10} \]

Comparing (2) with (10) and after some algebraic manipulation, we find

\[ \frac{\theta_3 - \theta_2}{\theta_2} < \frac{1}{\sqrt{p_n}} \]

therefrom,

\[ \sqrt{p_n} \leq \frac{\theta_2}{\theta_3 - \theta_2}. \]

**THEOREM 2.** Let \( k := \frac{p_n}{p_{n+1}} \) to be a \( k \) modulus, then

\[ \left( \frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left( \frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n), \]
where $\theta_2$ and $\theta_3$ are Jacobi theta functions.

**Proof.** We define that

$$k := k_{n+1} = \frac{p_n}{p_{n+1}} \iff p_{n+1} = \frac{p_n}{k},$$

where $k_{n+1}$ is the $k$ modulus.

We consider that

$$g(p_n) := p_{n+1} - p_n = \frac{p_n}{(k^{1/2})^2} - p_n = p_n \left[ \frac{1-\left(k^{1/2}\right)^2}{(k^{1/2})^2} \right].$$

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (14) in (13)

$$g(p_n) = p_{n+1} - p_n = p_n \left( \frac{1-\theta_2^2/\theta_3^2}{\theta_2^2/\theta_3^2} \right) = p_n \left( \frac{\theta_2^2/\theta_3^2}{\theta_2^2} \right) = p_n \left( \frac{\theta_2^2/\theta_3^2}{\theta_2^2} \right).$$

From (3) and (15), we conclude that

$$\left( \frac{\theta_2^2/\theta_3^2}{\theta_2^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left( \frac{\theta_3^2/\theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n). \blacksquare$$

**THEOREM 3.** Let $k := \frac{p_n}{p_{n+1}}$ to be a $k$ modulus, then

$$\left( \frac{\theta_2^4/\theta_3^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left( \frac{\theta_3^4/\theta_2^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2,$$

where $\theta_2$ and $\theta_3$ are Jacobi theta functions.

**Proof.** We define that

$$k := k_{n+1} = \frac{p_n}{p_{n+1}} \iff p_{n+1} = \frac{p_n}{k},$$

where $k_{n+1}$ is the $k$ modulus.

We consider that

$$p_{n+1}^2 - p_n^2 = \frac{p_n^2}{(k^{1/2})^2} - p_n^2 = p_n^2 \left[ \frac{1-\left(k^{1/2}\right)^4}{(k^{1/2})^4} \right].$$

In [2, p. 83], we knew that

$$k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

where $\tau$ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.
We set (18) in (17)

\[ p_{n+1}^2 - p_n^2 = p_n^2 \left( \frac{1 - \theta_4^4}{\theta_3^4} \right) = p_n^2 \left( \frac{\theta_4^4 - \theta_3^4}{\theta_3^4} \right) = p_n^2 \left( \frac{\theta_4^4 - \theta_2^4}{\theta_2^4} \right) = p_n^2 \left( \frac{\theta_4^4}{\theta_2^4} \right), \]  

(19)

to see [3, p. 84] which states \( \theta_2^4 + \theta_4^4 = \theta_3^4 \), the Jacobi identity.

From (3) and (19), we conclude that

\[ \left( \frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left( \frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2. \] \( \square \)

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REFERENCES