

# Decomposition in Finite Series for Dedekind sum and Some Hardy sums by Trigonometric Functions

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## 1. INTRODUCTION

The Dedekind sum is defined by

$$s(h, k) := \sum_{\mu(\text{mod } k)} \left( \left( \frac{\mu}{k} \right) \right) \left( \left( \frac{h\mu}{k} \right) \right),$$

being  $h$  an integer and  $k$  a positive integer and

$$\left( (x) \right) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases}$$

where  $[x]$  is the floor function.

By use of finite Fourier series, in [1] and [2], H. Rademacher proved that, for  $\text{gcd}(h, k) = 1$ , then

$$s(h, k) = \frac{1}{4k} \sum_{r=1}^{k-1} \cot\left(\frac{\pi r}{k}\right) \cot\left(\frac{\pi hr}{k}\right). \quad (1)$$

## 2. Preliminares

**Lemma 1.** If  $c = \frac{a}{b}$  and  $b \neq 0$ , then

$$a^2b + ab^2c^2 = a^3 + b^3c^2. \quad (2)$$

*Proof.* Suppose that

$$y = 1 + zy^2,$$

then, we have the parametrization

$$z = \frac{t-1}{t^2}$$

and

$$y = t.$$

Let  $t = \frac{a}{b}$  and  $t^2 = c^2$ , such that  $c = \frac{a}{b}$ . Therefore,

$$\frac{a}{b} = 1 + \frac{\frac{a}{b} - 1}{c^2} \left(\frac{a}{b}\right)^2,$$

$$\frac{a}{b} = 1 + \frac{(a-b)a^2}{b^3c^2},$$

$$\frac{a}{b} = \frac{b^3 c^2 + (a - b)a^2}{b^3 c^2},$$

$$a = \frac{b^3 c^2 + (a - b)a^2}{b^2 c^2},$$

$$ab^2 c^2 = b^3 c^2 + a^3 - a^2 b,$$

$$a^2 b + ab^2 c^2 = a^3 + b^3 c^2. \square$$

**Lemma 2.** For  $x \in \mathbb{R}$ , we have

$$\cot\left(\frac{x}{2}\right) = \frac{1}{2} \left[ \cot^2\left(\frac{x}{2}\right) - 1 \right] \tan(x), \tag{3.a1}$$

$$\cot(x) = \frac{1}{2} \left[ \cot^2(x) - 1 \right] \tan(2x), \tag{3.a2}$$

and

$$\tan\left(\frac{x}{2}\right) = \frac{1}{2} \left[ 1 - \tan^2\left(\frac{x}{2}\right) \right] \tan(x), \tag{3.b1}$$

$$\tan(x) = \frac{1}{2} \left[ 1 - \tan^2(x) \right] \tan(2x), \tag{3.b2}$$

*Proof.* We define that

$$\tan(x) := \frac{\sin(x)}{\cos(x)} \tag{4}$$

and if we set

$$t = \tan\left(\frac{x}{2}\right),$$

then

$$\sin(x) = \frac{2t}{1+t^2}, \tag{5}$$

$$\cos(x) = \frac{1-t^2}{1+t^2}, \tag{6}$$

$$\tan(x) = \frac{2t}{1-t^2}. \tag{7}$$

By Lemma 1, combined with (5), (6) and (7), we have

$$\sin(x)^2 \cos(x) + \sin(x) \cos(x)^2 \tan(x)^2 = \left(\frac{2t}{1+t^2}\right)^3 + \left(\frac{1-t^2}{1+t^2}\right)^3 \left(\frac{2t}{1-t^2}\right)^2,$$

$$\sin(x)^2 \cos(x) + \sin(x)^3 = 8 \left(\frac{t}{1+t^2}\right)^3 + 4 \frac{(1-t^2)t^2}{(1+t^2)^3},$$

$$\sin(x)^2 [\cos(x) + \sin(x)] = \frac{4t^2(-t^2 + 2t + 1)}{(1+t^2)^3},$$

$$\sin(x)^2 [\cos(x) + \sin(x)] = -4t \left(\frac{t}{1+t^2}\right)^3 + 8 \left(\frac{t}{1+t^2}\right)^3 + 4 \frac{1}{1+t^2} \left(\frac{t}{1+t^2}\right)^2,$$

$$\begin{aligned} \sin(x)^2 [\cos(x) + \sin(x)] &= -4t \left(\frac{\sin(x)}{2}\right)^3 + 8 \left(\frac{\sin(x)}{2}\right)^3 + 4 \frac{1}{1+t^2} \left(\frac{\sin(x)}{2}\right)^2, \\ \sin(x)^2 [\cos(x) + \sin(x)] &= -\frac{t}{2} \sin^3(x) + \sin^3(x) + \frac{1}{t} \left(\frac{t}{1+t^2}\right) \sin^2(x), \\ \sin(x)^2 [\cos(x) + \sin(x)] &= -\frac{t}{2} \sin^3(x) + \sin^3(x) + \frac{1}{t} \left(\frac{\sin(x)}{2}\right) \sin^2(x), \\ \sin(x)^2 [\cos(x) + \sin(x)] &= -\frac{t}{2} \sin^3(x) + \sin^3(x) + \frac{1}{2t} \sin^3(x). \end{aligned} \quad (8)$$

Eliminating  $\sin^2(x)$  in both members of (8) and manipulating algebraically, we encounter

$$\begin{aligned} 2 \cos(x) t + 2 \sin(x) t &= -\sin(x) t^2 + 2 \sin(x) t + \sin(x), \\ -\sin(x) t^2 - 2 \cos(x) t + \sin(x) &= 0, \\ \sin(x) t^2 + 2 \cos(x) t - \sin(x) &= 0. \end{aligned} \quad (9)$$

We can easily verify that the above equation has the following solutions

$$t = -\cot\left(\frac{x}{2}\right), t = \tan\left(\frac{x}{2}\right). \quad (10)$$

From (9) and (10), it follows that

$$\cot\left(\frac{x}{2}\right) = \frac{\cot^2\left(\frac{x}{2}\right) - 1}{2 \cos(x)} \sin(x) = \frac{1}{2} \left[ \cot^2\left(\frac{x}{2}\right) - 1 \right] \tan(x),$$

and

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{2 \cos(x)} \sin(x) = \frac{1}{2} \left[ 1 - \tan^2\left(\frac{x}{2}\right) \right] \tan(x).$$

To complete the proof let  $x \rightarrow 2x$  in the two previous equations.  $\square$

### 3. Dedekind sum

**Theorem 1.** Let  $\gcd(h, k) = 1$ , if  $h$  is even and  $k$  is odd, then

$$\begin{aligned} s(h, k) &= \frac{1}{16k} \sum_{r=1}^{k-1} \left[ \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi r}{k}\right) \cot^2\left(\frac{\pi hr}{k}\right) \right. \\ &\quad - \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi r}{k}\right) \\ &\quad \left. + \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \right]. \end{aligned}$$

*Proof.* We substitute (3.a2) in (1), as follows

$$s(h, k) = \frac{1}{16k} \sum_{r=1}^{k-1} \left[ \tan\left(\frac{2\pi r}{k}\right) \cot^2\left(\frac{\pi r}{k}\right) - \tan\left(\frac{2\pi r}{k}\right) \right] \left[ \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi hr}{k}\right) \right]$$

$$= \frac{1}{16k} \sum_{r=1}^{k-1} \left[ \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi r}{k}\right) \cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi hr}{k}\right) \right. \\ \left. - \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \cot^2\left(\frac{\pi r}{k}\right) + \tan\left(\frac{2\pi r}{k}\right) \tan\left(\frac{2\pi hr}{k}\right) \right]. \square$$

#### 4. INSIGHTS WITHOUT PROOFS

Let  $\gcd(h, k) = 1$ , if  $h$  is even and  $k$  is odd, then

$$s(h, k) = \frac{1}{16k} \sum_{r=1}^{k-1} \left[ \tan\left(\frac{\pi r}{k}\right) \cot\left(\frac{\pi hr}{k}\right) \cot^2\left(\frac{\pi r}{k}\right) - \tan\left(\frac{2\pi r}{k}\right) \cot\left(\frac{\pi hr}{k}\right) \right];$$

$$s(h, k) = \frac{1}{16k} \sum_{r=1}^{k-1} \left[ \tan\left(\frac{\pi hr}{k}\right) \cot\left(\frac{\pi r}{k}\right) \cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi hr}{k}\right) \cot\left(\frac{\pi r}{k}\right) \right].$$

#### 5. Hardy sums

I not solve the Hardy sums, because I not have the *Collected Papers of G. H. Hardy*, Vol IV, in *On Certain Series of Discontinuous Functions Connected with the Modular Functions*. Then, I leave these problems for Sir.

#### REFERENCES

- [1] Rademacher, *Egy Reciprocitásképletről a Modulfüggevények Elméletéből*, Mat. Fiz. Lapok 40 (1933), pp. 24-34.
- [2] Rademacher, H., *Some remarks on certain generalized Dedekind sums*, Acta Arith. 9 (1964), pp. 97-105.