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Decomposition in Finite Series for Dedekind sum and Some Hardy sums by Trigonometric Functions

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1. INTRODUCTION

The Dedekind sum is defined by

$$s(h,k) := \sum_{\mu(\mathrm{mod}k)} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

being h an integer and k a positive integer and

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, \text{if } x \notin \mathbb{Z}, \\ 0, \text{otherwise}, \end{cases}$$

where $\lfloor x \rfloor$ is the floor function.

By use of finite Fourier series, in [1] and [2], H. Rademacher proved that, for $\gcd(h,k) = 1$, then

$$s(h,k) = \frac{1}{4k}\sum_{r=1}^{k-1} \cot\left(\frac{\pi r}{k}\right)\cot\left(\frac{\pi h r}{k}\right). \tag{1}$$

2. Preliminares

Lemma 1. *If* $c = \frac{a}{b}$ *and* $b \neq 0$, *then*

$$a^2b + ab^2c^2 = a^3 + b^3c^2. \tag{2}$$

Proof. Suppose that

$$y = 1 + zy^2,$$

then, we have the parametrization

$$z = \frac{t-1}{t^2}$$

and

$$y = t.$$

Let $t = \frac{a}{b}$ and $t^2 = c^2$, such that $c = \frac{a}{b}$. Therefore,

$$\frac{a}{b} = 1 + \frac{\frac{a}{b} - 1}{c^2}\left(\frac{a}{b}\right)^2,$$

$$\frac{a}{b} = 1 + \frac{(a-b)a^2}{b^3c^2},$$

$$\frac{a}{b} = \frac{b^3c^2 + (a-b)a^2}{b^3c^2},$$

$$a = \frac{b^3c^2 + (a-b)a^2}{b^2c^2},$$

$$ab^2c^2 = b^3c^2 + a^3 - a^2b,$$

$$a^2b + ab^2c^2 = a^3 + b^3c^2. \square$$

Lemma 2. *For $x \in \mathbb{R}$, we have*

$$\cot\left(\frac{x}{2}\right) = \frac{1}{2}\left[\cot^2\left(\frac{x}{2}\right) - 1\right]\tan(x), \tag{3.a1}$$

$$\cot(x) = \frac{1}{2}[\cot^2(x) - 1]\tan(2x), \tag{3.a2}$$

and

$$\tan\left(\frac{x}{2}\right) = \frac{1}{2}\left[1 - \tan^2\left(\frac{x}{2}\right)\right]\tan(x), \tag{3.b1}$$

$$\tan(x) = \frac{1}{2}[1 - \tan^2(x)]\tan(2x), \tag{3.b2}$$

Proof. We define that

$$\tan(\mathrm{x}) := \frac{\sin(\mathrm{x})}{\cos(\mathrm{x})} \tag{4}$$

and if we set

$$t = \tan\left(\frac{x}{2}\right),$$

then

$$\sin(x) = \frac{2t}{1+t^2}, \tag{5}$$

$$\cos(x) = \frac{1-t^2}{1+t^2}, \tag{6}$$

$$\tan(x) = \frac{2t}{1-t^2}. \tag{7}$$

By Lemma 1, combined with (5), (6) and (7), we have

$$\sin(x)^2\cos(x) + \sin(x)\cos(x)^2\tan(x)^2 = \left(\frac{2t}{1+t^2}\right)^3 + \left(\frac{1-t^2}{1+t^2}\right)^3\left(\frac{2t}{1-t^2}\right)^2,$$

$$\sin(x)^2\cos(x) + \sin(x)^3 = 8\left(\frac{t}{1+t^2}\right)^3 + 4\frac{(1-t^2)t^2}{(1+t^2)^3},$$

$$\sin(x)^2[\cos(x) + \sin(x)] = \frac{4t^2(-t^2+2t+1)}{(1+t^2)^3},$$

$$\sin(x)^2[\cos(x) + \sin(x)] = -4t\left(\frac{t}{1+t^2}\right)^3 + 8\left(\frac{t}{1+t^2}\right)^3 + 4\frac{1}{1+t^2}\left(\frac{t}{1+t^2}\right)^2,$$

$$\sin(x)^2\,[\cos(x)+\sin(x)] = -4t\left(\frac{\sin(x)}{2}\right)^3 + 8\left(\frac{\sin(x)}{2}\right)^3 + 4\frac{1}{1+t^2}\left(\frac{\sin(x)}{2}\right)^2,$$

$$\sin(x)^2\,[\cos(x)+\sin(x)] = -\frac{t}{2}\sin^3(x) + \sin^3(x) + \frac{1}{t}\left(\frac{t}{1+t^2}\right)\sin^2(x),$$

$$\sin(x)^2\,[\cos(x)+\sin(x)] = -\frac{t}{2}\sin^3(x) + \sin^3(x) + \frac{1}{t}\left(\frac{\sin(x)}{2}\right)\sin^2(x),$$

$$\sin(x)^2\,[\cos(x)+\sin(x)] = -\tfrac{t}{2}\sin^3(x) + \sin^3(x) + \tfrac{1}{2t}\sin^3(x). \tag{8}$$

Eliminating $\sin^2(x)$ in both members of (8) and manipulating algebraically, we encounter

$$2\cos(x)\,t + 2\sin(x)\,t = -\sin(x)\,t^2 + 2\sin(x)\,t + \sin(x),$$

$$-\sin(x)\,t^2 - 2\cos(x)\,t + \sin(x) = 0,$$

$$\sin(x)\,t^2 + 2\cos(x)\,t - \sin(x) = 0. \tag{9}$$

We can easily verify that the above equation has the following solutions

$$t = -\cot\left(\tfrac{x}{2}\right),\, t = \tan\left(\tfrac{x}{2}\right). \tag{10}$$

From (9) and (10), it follows that

$$\cot\left(\frac{x}{2}\right) = \frac{\cot^2\left(\frac{x}{2}\right) - 1}{2\cos(x)}\sin(x) = \frac{1}{2}\left[\cot^2\left(\frac{x}{2}\right) - 1\right]\tan(x),$$

and

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{2\cos(x)}\sin(x) = \frac{1}{2}\left[1 - \tan^2\left(\frac{x}{2}\right)\right]\tan(x).$$

To complete the proof let $x \to 2x$ in the two previous equations. □

3. Dedekind sum

Theorem 1. *Let* $\gcd(h,k) = 1$, *if h is even and k is odd, then*

$$\begin{aligned} s(h,k) = \frac{1}{16k}\sum_{r=1}^{k-1}\Bigg[&\tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi r}{k}\right)\cot^2\left(\frac{\pi hr}{k}\right) \\ &- \tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi r}{k}\right) \\ &+ \tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\Bigg]. \end{aligned}$$

Proof. We substitute (3.a2) in (1), as follows

$$s(h,k) = \frac{1}{16k}\sum_{r=1}^{k-1}\left[\tan\left(\frac{2\pi r}{k}\right)\cot^2\left(\frac{\pi r}{k}\right) - \tan\left(\frac{2\pi r}{k}\right)\right]\left[\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi hr}{k}\right)\right]$$

$$= \frac{1}{16k}\sum_{r=1}^{k-1}\left[\tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi r}{k}\right)\cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi hr}{k}\right)\right.$$
$$\left. - \tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\cot^2\left(\frac{\pi r}{k}\right) + \tan\left(\frac{2\pi r}{k}\right)\tan\left(\frac{2\pi hr}{k}\right)\right]. \square$$

4. INSIGHTS WITHOUT PROOFS

Let $\gcd(h,k) = 1$, *if* h *is even and* k *is odd, then*

$$s(h,k) = \frac{1}{16k}\sum_{r=1}^{k-1}\left[\tan\left(\frac{\pi r}{k}\right)\cot\left(\frac{\pi hr}{k}\right)\cot^2\left(\frac{\pi r}{k}\right) - \tan\left(\frac{2\pi r}{k}\right)\cot\left(\frac{\pi hr}{k}\right)\right];$$

$$s(h,k) = \frac{1}{16k}\sum_{r=1}^{k-1}\left[\tan\left(\frac{\pi hr}{k}\right)\cot\left(\frac{\pi r}{k}\right)\cot^2\left(\frac{\pi hr}{k}\right) - \tan\left(\frac{2\pi hr}{k}\right)\cot\left(\frac{\pi r}{k}\right)\right].$$

5. Hardy sums

I not solve the Hardy sums, because I not have the *Collected Papers of G. H. Hardy,* Vol IV, in *On Certain Series of Discontinuous Functions Connected with the Modular Functions*. Then, I leave these problems for Sir.

REFERENCES

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