

An Elementary Identity with Cotangent Hyperbolic Function

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LEMMA 1. For $a \in \mathbb{R} \setminus \{0\}$, then

$$\frac{1}{a} = \sum_{k=1}^{\infty} \frac{1}{(a+k)(a+k-1)}. \quad (1)$$

Proof. We knew that

$$\begin{aligned} \frac{a}{b} &= \prod_{k=1}^{\infty} \frac{(b+k)(a+k-1)}{(a+k)(b+k-1)}, \\ \ln a - \ln b &= \sum_{k=1}^{\infty} \ln \left[\frac{(b+k)(a+k-1)}{(a+k)(b+k-1)} \right], \\ \frac{\partial}{\partial a} (\ln a - \ln b) &= \sum_{k=1}^{\infty} \frac{\partial}{\partial a} \ln \left[\frac{(b+k)(a+k-1)}{(a+k)(b+k-1)} \right], \\ \frac{1}{a} &= \sum_{k=1}^{\infty} \frac{1}{(a+k)(a+k-1)}. \quad \square \end{aligned}$$

THEOREM 1. For $x \in \mathbb{R}$, then

$$\begin{aligned} \frac{1}{\pi x} \left[\sum_{k=1}^{\infty} \left(\frac{1}{x^2+k} - \frac{1}{x^2+k-1} + \frac{1}{x+k-1} - \frac{1}{x+k} \right) \right] &= \\ = \frac{\cot(\pi x)}{x} + \sum_{k=1}^{\infty} \left[\frac{\cot(\pi\sqrt{x^2+k})}{\sqrt{x^2+k}} - \frac{\cot(\pi\sqrt{x^2+k-1})}{\sqrt{x^2+k-1}} \right]. \end{aligned} \quad (2)$$

Proof. We knew that

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2-n^2}. \quad (3)$$

Substituting (1) in (3), we encounter

$$\begin{aligned} \pi \cot(\pi x) &= \sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k-1)} + \sum_{n=1}^{\infty} 2x \sum_{k=1}^{\infty} \frac{1}{(x^2-n^2+k)(x^2-n^2+k-1)} \\ &= \sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k-1)} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{2x}{(x^2-n^2+k)(x^2-n^2+k-1)} \\ &= \sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k-1)} - \sum_{k=1}^{\infty} \frac{x}{(x^2+k)(x^2+k-1)} + \pi x \sum_{k=1}^{\infty} \frac{\cot(\pi\sqrt{x^2+k-1})}{\sqrt{x^2+k-1}} \\ &\quad - \pi x \sum_{k=1}^{\infty} \frac{\cot(\pi\sqrt{x^2+k})}{\sqrt{x^2+k}}, \end{aligned}$$

ergo,

$$\frac{1}{\pi x} \left[\sum_{k=1}^{\infty} \left(\frac{1}{x^2+k} - \frac{1}{x^2+k-1} + \frac{1}{x+k-1} - \frac{1}{x+k} \right) \right] = \frac{\cot(\pi x)}{x} + \sum_{k=1}^{\infty} \left[\frac{\cot(\pi\sqrt{x^2+k})}{\sqrt{x^2+k}} - \frac{\cot(\pi\sqrt{x^2+k-1})}{\sqrt{x^2+k-1}} \right]. \square$$

Note: But the equation $\sum_{k=1}^{\infty} \left[\frac{\cot(\pi\sqrt{x^2+k})}{\sqrt{x^2+k}} - \frac{\cot(\pi\sqrt{x^2+k-1})}{\sqrt{x^2+k-1}} \right]$ of right-hand side not converge. The Sir and I can modify the theorem to:

$$\frac{\cot(\pi x)}{x} = \frac{1}{\pi x} \left[\sum_{k=1}^{\infty} \left(\frac{1}{x^2+k} - \frac{1}{x^2+k-1} + \frac{1}{x+k-1} - \frac{1}{x+k} \right) \right] - \pi x (\theta_{\cot(\pi x)})?$$

for θ in function of $\cot(\pi x)$?