

On J^* - Class of Π^* - Regular Semigroups

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Abstracts In the paper, we define the equivalence relations on Π^* -regular semigroups, to show J^* -class contains an idempotent with some characterizations.

1 Introduction

According to [1] some good results, in this paper we consider J^* -class of Π^* -regular semigroups and completely Π^* -regular semigroups.(see[2])

Remark The marks we don't illustrate in this paper please see reference ([3], [4],[5]).

Let S be a Π^* -regular semigroup. Define on S the equivalence relations L^*, R^*, H^*, J^* by

$$(a, b)L^*(x, y) \Leftrightarrow S(a, b)^m = S(x, y)^n$$

$$(a, b)R^*(x, y) \Leftrightarrow (a, b)^m S = (x, y)^n S$$

$$H^* \Leftrightarrow L^* \cap R^*$$

$$(a, b)J^*(x, y) \Leftrightarrow S(a, b)^m S = S(x, y)^n S \text{ (see[3])}$$

where m, n are the smallest positive integers such that $(a, b)^m, (x, y)^n$ are regular, i.e. $(a, b)^m, (x, y)^n \in S$. In what follows we will denote by

$$L_{(a,b)}^* \left(R_{(a,b)}^*, H_{(a,b)}^*, J_{(a,b)}^* \right)$$

the L^*, R^*, H^*, J^* -class containing an element (e, e) of S . According to the hypothesis, we had drawn the following conclusions. (see[1])

Lemma 1. Let S be a Π^* -regular semigroup. Then each idempotent (e, e) of S is a right(left, two-sided) identity for regular elements from

$$L_{(e,e)}^* \left(R_{(e,e)}^*, J_{(e,e)}^* \right)$$

Lemma 2. In a Π^* -regular semigroup S every H^* -class contains at most one idempotent.

Lemma 3. Let S be a Π^* -regular semigroup, $(a, b) \in S$ and p the the smallest positive integers such that $(a, b)^p \in S$. Then

$$(a, b)^p \in L_{(a,b)}^* \cap R_{(a,b)}^* = H_{(a,b)}^*.$$

Lemma 4. Let S be a Π^* -regular semigroup. Then

- (1) every J^* -class contains at least one idempotent;
- (2) $G_{(e,e)} \subseteq H_{(e,e)} \subseteq J_{(e,e)}$ for every $e \in E$.

Lemma 5. Let S be a Π^* -regular semigroup. Then (for some $(e, e), (f, f) \in E$)

$$J_{(e,e)}^* = J_{(f,f)}^*, (e, e)(f, f) = (f, f)(e, e) = (f, f) \Rightarrow (e, e) = (f, f).$$

Lemma 6 . Let S be a Π^* -regular semigroup. Then for some $(u, v) \in S, (e, e) \in E$,

$$J_{(e,e)}^* = J_{(e,e)(u,v)(e,e)}^* \implies (e, e)(u, v)(e, e) \in G_{(e,e)}.$$

Theorem 1. Let (e, e) be an idempotent of a Π^* -regular semigroup S . Then $G_{(e,e)} \subseteq H_{(e,e)}^*$, furthermore, if $(u, v) \in H_{(e,e)}^*$ and p is the smallest positive integer such that $(u, v)^p \in S$, then $(u, v)^q \in G_{(e,e)}$ for every $q \geq p$.

Theorem 2. Let S be a Π^* -regular semigroup. Then (for some $(a, b), (x, y) \in S$)

$$J_{(a,b)(x,y)}^* = J_{(x,y)(a,b)}^*.$$

J^* -class plays a very active role in

Π^* -regular semigroups, since the emergence of Idempotents. Here we get some good results.

2 Main Results

Let V be the set of all inverse elements of S ([3]), E is the set of all idempotent of S .

Lemma 7 . Let S be a Π^* -regular semigroup. Then for some $(e, e), (f, f) \in E$

$$J_{(e,e)}^* = J_{(f,f)}^* = J_{(e,e)(f,f)}^* = J_{(e,e)}^*.$$

Proof. Let $S(e, e)S = S(f, f)S$. Then

$$(e, e) = (a, b)(f, f)(e, e) = (a, b)(f, f)(u, v)(e, e) \quad ((a, b), (u, v) \in S)$$

Whence by theorem 2 we have

$$\begin{aligned} J_{(f,f)}^* &= J_{(e,e)}^* = J_{(a,b)(f,f)(u,v)(e,e)}^* = J_{((a,b)(f,f))((f,f)(u,v)(e,e))}^* \\ &= J_{(f,f)(u,v)(e,e)(a,b)(f,f)}^* \end{aligned}$$

And since $(f, f)(u, v)(e, e)(a, b)(f, f) \in E$ by lemma 5 we attain that E

$$(f, f) = (f, f)(u, v)(e, e)(a, b)(f, f)$$

and

$$(f, f)(u, v)(f, f) \in G_{(f,f)}.$$

Analogously

$$(f, f)(a, b)(f, f) \in G_{(f,f)} \text{ So}$$

$$((f, f)(e, e)(f, f))^n = ((f, f)(a, b)(f, f)(f, f)(u, v)(f, f))^p \in G_{(f,f)} \subseteq J_{(f,f)}^*$$

for every $n, p \in \mathbb{Z}^+$. Thus

$$S(f, f)S = S((f, f)(e, e)(f, f))^p S \subseteq S((e, e)(f, f))^p S.$$

and it's opposite inclusion also holds we have

$$S(f, f)S = S((e, e)(f, f))^p S.$$

Hence

$$J_{(e,e)}^* = J_{(f,f)}^* = J_{(e,e)(f,f)}^* = J_{(e,e)}^*.$$

Theorem 3. Let S be a Π^* -regular semigroup. Then (for some $(a, b), (x, y) \in S$)

$$J_{(a,b)(x,y)}^* = J_{(e,e)(f,f)}^* \cdot (m, n \in \mathbb{Z}^+)$$

Proof. Let r be the the smallest positive integers such that $((a, b)(x, y))^r \in \text{Re } gS$. Then

$$\begin{aligned} ((a, b)(x, y))^r &\in J_{((a,b)(x,y))}^*, ((a, b)(x, y))^r \in G_{(h,h)} \subseteq J_{(h,h)}^*, \\ (h, h)(a, b)(x, y) &= (a, b)(x, y)(h, h) \in G_{(h,h)}, ((h, h) \in E) \end{aligned} \tag{1}$$

Now we prove by induction on p that

$$(h, h)(a, b)^p (h, h) \in G_{(h,h)}, (h, h)(x, y)^p (h, h) \in G_{(h,h)} \tag{2}$$

for every $p \geq 0$. Suppose $(h, h)(a, b)^p (h, h) \in G_{(h,h)} (p \geq 0)$, Then

$$((h, h)(a, b)^p (h, h))(h, h)(a, b)(x, y) = (h, h)(a, b)^{p+1} (x, y)(h, h) \in G_{(h,h)}.$$

Let (u, v) be an inverse element of

$$(h, h)(a, b)^{p+1} (x, y)(h, h) \text{ in } G_{(h,h)},$$

Then

$$(h, h) = (h, h)(a, b)^{p+1} (x, y)(u, v)(h, h),$$

Whence we have that $(h, h)(a, b)^{p+1}$ is regular. Hence by theorem 2

$$J_{(h,h)}^* = J_{(h,h)(a,b)^{p+1}}^* = J_{(h,h)(a,b)^{p+1}(h,h)}^*.$$

Now by lemma 6 $(h, h)(a, b)^{p+1} (h, h) \in G_{(h,h)}$ and since $(h, h)(a, b)^0 (h, h) \in G_{(h,h)}$ the first part is proved. In a similar way we can show the second part.

Since $(a, b)^m \in G_{(e,e)}$ by the first part of (2) we get

$$(h, h)(e, e)(a, b)^m (h, h) \in G_{(h,h)}.$$

Let (u, v) be an inverse element of $(h, h)(e, e)(a, b)^m (h, h)$ in $G_{(h,h)}$, Then

$$(h, h) = (h, h)(e, e)(a, b)^m (u, v)(h, h), (h, h)(e, e) \text{ is regular.}$$

Thus

$$J_{(h,h)}^* = J_{(h,h)(e,e)}^* = J_{(h,h)(e,e)(h,h)}^*$$

So

$$(h, h)(e, e)(h, h) \in G_{(h,h)}$$

and analogously

$$(h, h)(f, f)(h, h) \in G_{(h,h)}.$$

$$\text{Hence } (h, h)(e, e)(h, h), (h, h)(f, f)(h, h) \in G_{(h,h)} \tag{3}$$

$$\text{Now we show } (h, h)((e, e)(f, f))^p (h, h) \in G_{(h,h)} \tag{4}$$

for every $p \geq 0$.

From $(h, h)(f, f)(h, h) \in G_{(h,h)}$ we attain

$$(h, h) = (h, h)(e, e)(u, v)(f, f)(h, h) = (h, h)(f, f)(h, h)(e, e)(s, t),$$

Here $(f, f)(h, h)(e, e)$ and $(e, e)(f, f)(h, h)$ are regular

And $J_{(h,h)}^* = J_{(e,e)(f,f)(h,h)}^* = J_{(f,f)(h,h)(e,e)}^* = J_{(h,h)(e,e)(f,f)(h,h)}^*$

So $(h, h)(e, e)(f, f)(h, h) \in G_{(h,h)}$, i.e.

The condition $(h, h)((e, e)(f, f))^p (h, h) \in G_{(h,h)}$ is right for $p=1$.

According to the above assumptions $p=2$ we have

$$S(h)S \subseteq S((e, e)(f, f))^p (h, h)(e, e)(f, f)S$$

and the other hand

$$S((e, e)(f, f))^p (h, h)(e, e)(f, f)S \subseteq S(h, h)S$$

So $S(h, h)S = S((e, e)(f, f))^p (h, h)(e, e)(f, f)S$.

Thus $J_{(h,h)}^* = J_{((e,e)(f,f))^p (h,h)(e,e)(f,f)}^* = J_{(h,h)((e,e)(f,f))^{p+1}(h,h)}^*$

Whence $(h, h)((e, e)(f, f))^{p+1} (h, h) \in G_{(h,h)}$,

so by induction that $(h, h)((e, e)(f, f))^p (h, h) \in G_{(h,h)}$ holds.

Let q be the the smallest positive integers such that $((e, e)(f, f))^q \in \text{Reg}S$.

Then

$$((e, e)(f, f))^q \in J_{(e,e)(f,f)}^*, ((e, e)(f, f))^q \in G_{(k,k)} \subseteq J_{(k,k)}^* ((k, k) \in E). \quad (5)$$

Now we prove

$$(k, k)((a, b)(x, y))^p \in G_{(k,k)} \quad (6)$$

for every $p \geq 0$. If $p=1$. By induction that

$$((e, e)(f, f))^{2q} \in G_{(k,k)} \subseteq J_{(k,k)}^*,$$

so $((e, e)(f, f))^{2q} = ((e, e)(f, f))^{2q}(u, v)((e, e)(f, f))^{2q}((u, v) \in G_{(k,k)})$.

Therefor we have

$$J_{(k,k)}^* = J_{(a,b)((e,e)(f,f))^{2q}(x,y)}^* = J_{(x,y)(a,b)((e,e)(f,f))^{2q}}^* = J_{(k,k)((e,e)(f,f))^q(x,y)(a,b)((e,e)(f,f))^q}^*.$$

whence

$$\begin{aligned} & ((e, e)(f, f))^q (a, b)(x, y)((e, e)(f, f))^q = \\ & (k, k)((e, e)(f, f))^q (x, y)(a, b)((e, e)(f, f))^q (k, k) \in G_{(k,k)} \end{aligned}$$

by (1),(4),(5),(6) we obtain

$$S((a, b)(x, y))^r S = S(h, h)S = S(h, h)((e, e)(f, f))^w (h, h)S \subseteq S((e, e)(f, f))^q S;$$

$$S((e, e)(f, f))^q S = S(k, k)S = S(k, k)((x, y)(a, b))^r (k, k)S \subseteq S((x, y)(a, b))^r S.$$

and we consider $J_{(a,b)(x,y)}^* = J_{(x,y)(a,b)}^*$, then

$$S((a, b)(x, y))^r S = S((e, e)(f, f))^q S.$$

Hence

$$J_{(a,b)(x,y)}^* = J_{(e,e)(f,f)}^* .$$

References

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