On $J^*$-Class of $\Pi^*$-Regular Semigroups

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Abstracts In the paper, we define the equivalence relations on $\Pi^*$-regular semigroups, to show $J^*$-class contains an idempotent with some characterizations.

1 Introduction

According to [1] some good results, in this paper we consider $J^*$-class of $\Pi^*$-regular semigroups and completely $\Pi^*$-regular semigroups.(see[2])

Remark The marks we don’t illustrate in this paper please see reference ([3], [4],[5]).

Let $S$ be a $\Pi^*$-regular semigroup. Define on $S$ the equivalence relations $L^*$, $R^*$, $H^*$, $J^*$ by

$$(a, b) L^* (x, y) \Leftrightarrow S(a, b)^m = S(x, y)^n$$

$$(a, b) R^* (x, y) \Leftrightarrow (a, b)^m S = (x, y)^n S$$

$$H^* \Leftrightarrow L^* \cap R^*$$

$$(a, b) J^* (x, y) \Leftrightarrow S(a, b)^m S = S(x, y)^n S$$

where $m,n$ are the smallest positive integers such that $(a, b)^m, (x, y)^n$ are regular, i.e. $(a, b)^m, (x, y)^n \in S$. In what follows we will denote by

$$L^*_{(a, b)}(R^*_{(a, b)}, H^*_{(a, b)}, J^*_{(a, b)})$$

the $L^*$-, $R^*$-, $H^*$-, $J^*$-class containing an element $(e, e)$ of $S$. According to the hypothesis, we had drawn the following conclusions. (see[1])

Lemma 1. Let $S$ be a $\Pi^*$-regular semigroup. Then each idempotent $(e, e)$ of $S$ is a right(left, two-sided) identity for regular elements from

$$L^*_{(e, e)}\left(R^*_{(e, e)}, J^*_{(e, e)}\right)$$

Lemma 2. In a $\Pi^*$-regular semigroup $S$ every $H^*$-class contains at most one idempotent.

Lemma 3. Let $S$ be a $\Pi^*$-regular semigroup, $(a, b) \in S$ and $p$ the the smallest positive integers such that $(a, b)^p \in S$. Then

$$(a, b)^p \in L^*_{(a, b)} \cap R^*_{(a, b)} = H^*_{(a, b)}.$$ 

Lemma 4. Let $S$ be a $\Pi^*$-regular semigroup. Then

1. every $J^*$-class contains at least one idempotent;

2. $G_{(e, e)} \subseteq H_{(e, e)} \subseteq J_{(e, e)}$ for every $e \in E$.

Lemma 5. Let $S$ be a $\Pi^*$-regular semigroup. Then (for some $(e, e), (f, f) \in E$)

$$J^*_{(e, e)} = J^*_{(f, f)}(e, e)(f, f) = (f, f)(e, e) = (f, f) \Rightarrow (e, e) = (f, f).$$
Lemma 6. Let $S$ be a $\Pi^*$-regular semigroup. Then for some $(u,v) \in S, (e,e) \in E$, 
$$J^*_{(e,e)} = J^*_{(e,e)(u,v)(e,e)} \Rightarrow (e,e)(u,v)(e,e) \in G_{(e,e)}.$$  

Theorem 1. Let $(e,e)$ be an idempotent of a $\Pi^*$-regular semigroup $S$. Then $G_{(e,e)} \subseteq H^*_{(e,e)}$, furthermore, if $(u,v) \in H^*_{(e,e)}$ and $p$ is the smallest positive integer such that $(u,v)^p \in S$, then $(u,v)^q \in G_{(e,e)}$ for every $q \geq p$.

Theorem 2. Let $S$ be a $\Pi^*$-regular semigroup. Then (for some $(a,b),(x,y) \in S$) 
$$J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}.$$  

$J^*$-class plays a very active role in $\Pi^*$-regular semigroups, since the emergence of Idempotents. Here we get some good results.

2 Main Results

Let $V$ be the set of all inverse elements of $S$ ([3]), $E$ is the set of all idempotent of $S$.

Lemma 7. Let $S$ be a $\Pi^*$-regular semigroup. Then for some $(e,e),(f,f) \in E$ 
$$J^*_{(e,e)} = J^*_{(f,f)} = J^*_{(e,e)(f,f)} = J^*_{(e,e)}.$$  

Proof. Let $S((e,e))S = S((f,f))S$. Then 
$$(e,e) = (a,b)(f,f)(e,e) = (a,b)(f,f)(u,v)(e,e) \in S((a,b),(u,v) \in S)$$

Whence by theorem 2 we have 
$$J^*_{(f,f)} = J^*_{(e,e)} = J^*_{(a,b)(f,f)(u,v)(e,e)} = J^*_{(a,b)(f,f)((f,f)(u,v)(e,e)}$$

And since $(f,f)(u,v)(e,e)(a,b)(f,f) \in E$ by lemma 5 we attain that 
$$(f,f) = (f,f)(u,v)(e,e)(a,b)(f,f)$$

and 
$$(f,f)(u,v)(f,f) \in G_{(f,f)}.$$  

Analogously 
$$(f,f)(e,e)(f,f)^n = ((f,f)(a,b)(f,f)(f,f)(u,v)(f,f))^p \in G_{(f,f)} \subseteq J^*_{(f,f)}$$

for every $n, p \in \mathbb{Z}^+$. Thus 
$$S((f,f))S = S((f,f)(e,e)(f,f))^p \subseteq S((e,e)(f,f))^p \subseteq S.$$  

and it’s opposite inclusion also holds we have 
$$S((f,f))S = S((e,e)(f,f))^p \subseteq S.$$  

Hence 
$$J^*_{(e,e)} = J^*_{(f,f)} = J^*_{(e,e)(f,f)} = J^*_{(e,e)}.$$
Theorem 3. Let $S$ be a $\Pi^*$-regular semigroup. Then (for some $(a,b), (x,y) \in S$)
\[
J^*_S(a,b)(x,y) = J^*_S(e,e)(f,f)(m,n) \in \mathbb{Z}^+
\]

Proof. Let $r$ be the smallest positive integers such that $(a,b)(x,y)^r \in \text{Reg}S$. Then
\[
((a,b)(x,y))^r \in J^*_S(a,b)(x,y)^r \subseteq G_{(h,h)} \subseteq J^*_S(h,h).
\]

Now we prove by induction on $p$ that
\[
(h,h)(a,b)^p(h,h) \in G_{(h,h),(h,h)}(p \geq 0),
\]
for every $P \geq 0$. Suppose $(h,h)(a,b)^p(h,h) \in G_{(h,h),(h,h)}(p \geq 0)$, Then
\[
(h,h)(a,b)^p(h,h)(a,b)(x,y) = (h,h)(a,b)^{p+1}(x,y)(h,h) \in G_{(h,h)}.
\]

Let $(u,v)$ be an inverse element of
\[
(h,h)(a,b)^{p+1}(x,y)(h,h) \text{ in } G_{(h,h)}.
\]

Then
\[
(h,h) = (h,h)(a,b)^{p+1}(x,y)(h,h),
\]
Whence we have that $(h,h)(a,b)^{p+1}$ is regular. Hence by theorem 2
\[
J^*_S(h,h) = J^*_S(h,h)(a,b)^{p+1}(h,h) \cdot J^*_S(h,h)(a,b)^{p+1}(h,h).
\]

Now by lemma 6 $(h,h)(a,b)^{p+1}(h,h) \in G_{(h,h)}$ and since $(h,h)(a,b)^{p+1}(h,h) \in G_{(h,h)}$ the first part is proved. In a similar way we can show the second part.

Since $(a,b)^m \in G_{(e,e)}$ by the first part of (2) we get
\[
(h,h)(e,e)(a,b)^m(h,h) \in G_{(h,h)}.
\]

Let $(u,v)$ be an inverse element of $(h,h)(e,e)(a,b)^m(h,h)$ in $G_{(h,h)}$, Then
\[
(h,h) = (h,h)(e,e)(a,b)^m(h,h) \text{ is regular.}
\]

Thus
\[
J^*_S(h,h) = J^*_S(h,h)(e,e)(h,h) \in G_{(h,h)}.
\]

So
\[
(h,h)(f,f)(h,h) \in G_{(h,h)}.
\]

and analogously
\[
(h,h)(e,e)(f,f)(h,h) \in G_{(h,h)}.
\]

Hence
\[
(h,h)((e,e)(f,f)(h,h)) \in G_{(h,h)}.
\]

Now we show
\[
(h,h)((e,e)(f,f))(h,h) \in G_{(h,h)}.
\]

for every $P \geq 0$.

From
\[
(h,h) = (h,h)(e,e)(u,v)(f,f)(h,h) = (h,h)(f,f)(h,h)(e,e)(s,t),
\]
we attain
\[
(h,h) = (h,h)(e,e)(u,v)(f,f)(h,h) = (h,h)(f,f)(h,h)(e,e)(s,t),
\]
Here \((f, f) (h, h)(e, e)\) and \((e, e)(f, f)(h, h)\) are regular
\[ J^*_{(h, h)} = J^*_{(e, e)\langle f, f \rangle (h, h)} = J^*_{(f, f)(h, h) (e, e)} = J^*_{(h, h)(e, e)(f, f)(h, h)} \]
And
\[(h, h)(e, e)(f, f)(h, h) \in G_{(h, h)}, \text{ i.e.} \]
The condition \((h, h)((e, e)(f, f))^P(h, h) \in G_{(h, h)}\) is right for \(p = 1\).
According to the above assumptions \(p = 2\) we have
\[ S(h) S \subseteq S((e, e)(f, f))^P(h, h)(e, e)(f, f) S \]
and the other hand
\[ S((e, e)(f, f))^P(h, h)(e, e)(f, f) S \subseteq S(h, h) S \]
So
\[ S(h, h) S = S((e, e)(f, f))^P(h, h)(e, e)(f, f) S. \]
Thus
\[ (h, h)((e, e)(f, f))^P(h, h)(e, e)(f, f) \in G_{(h, h)}, \]
Whence
\[ (h, h)((e, e)(f, f))^{p+1}(h, h) \in G_{(h, h)}, \]
so by induction that \((h, h)((e, e)(f, f))^p(h, h) \in G_{(h, h)}\) holds.
Let \(q\) be the the smallest positive integers such that \(((e, e)(f, f))^q \in \text{Reg}\ S\).
Then
\[ ((e, e)(f, f))^q \in J^*_{(e, e)(f, f)} ((e, e)(f, f))^q \in G_{(k, k)} \subseteq J^*_{(k, k)} ((k, k) \in E). \] (5)
Now we prove
\[ (k, k)((a, b)(x, y))^p \in G_{(k, k)} \] (6)
for every \(p \geq 0\). If \(p = 1\). By induction that
\[ ((e, e)(f, f))^q \in G_{(k, k)} \subseteq J^*_{(k, k)}, \]
so
\[ ((e, e)(f, f))^q = ((e, e)(f, f))^q ((u, v)) \in G_{(k, k)}. \]
Therefor we have
\[ J^*_{(k, k)} = J^*_{(a, b)(x, y)((e, e)(f, f))^q} = J^*_{(x, y)(a, b)((e, e)(f, f))^q} = J^*_{((x, y)(a, b)((e, e)(f, f))^q}, \]
whence
\[ ((e, e)(f, f))^q (a, b)(x, y)((e, e)(f, f))^q = (k, k)((e, e)(f, f))^q (x, y)(a, b)((e, e)(f, f))^q (k, k) \in G_{(k, k)}, \]
by (1), (4), (5), (6) we obtain
\[ S((a, b)(x, y))^q S = S(h, h) S = S((e, e)(f, f))^q (h, h) S \subseteq S((e, e)(f, f))^q S; \]
\[ S((e, e)(f, f))^q S = S((k, k) S = S((k, k)((x, y)(a, b))^q (k, k) S \subseteq S((x, y)(a, b))^q S. \]
and we consider \[ J^*_{(a, b)(x, y)} = J^*_{(x, y)(a, b)} \], then
\[ S((a, b)(x, y))^q S = S((e, e)(f, f))^q S. \]
Hence

\[ J^*_\ast(ax,by) = J^*_\ast(ex,e(f \cdot f)) \]

References

[1] LUO Xiaoqiang. The Subclasses of Characterization on \( \Pi^\ast \)-Regular Semigroups, Progress in Applied Mathematics(Canadian), 5(2), 2013, 1-5.