Bulletin of Mathematical Sciences and Applications
ISSN: 2278-9634, Vol. 4, pp 1-5
doi:10.18052/www.scipress.com/BMSA.4.1 Online: 2012-05-01

On J^*- Class of Π^*- Regular Semigroups

Xiaoqiang LUO

Department of mathematics Sichuan University of Arts and Science, Dazhou, Sichuan, 635000,China

Keywords Π^*-regular semigroup, Completely Π^*-regular semigroups, J^*-*class*, Idempotent

Abstracts In the paper, we define the equivalence relations on Π^* -regulur semigroups, to show J^* -class contains an idempotent with some characterizations.

1 Introduction

According to [1] some good results, in this paper we consider J^* *-class* of Π^*-regular semigroups and completely Π^*-regular semigroups.(see[2])

Remark The marks we don't illustrate in this paper please see reference ([3], [4],[5]).

Let S be a Π^*-regular semigroup.Define on S the equivalence relations L^*, R^*, H^*, $J^*$by

$$(a,b)L^*(x,y) \Leftrightarrow S(a,b)^m = S(x,y)^n$$

$$(a,b)R^*(x,y) \Leftrightarrow (a,b)^m S = (x,y)^n S$$

$$H^* \Leftrightarrow L^* \mathrm{I} \ R^*$$

$$(a,b)J^*(x,y) \Leftrightarrow S(a,b)^m S = S(x,y)^n S \text{ (see[3])}$$

where m,n are the smallest positive integers such that $(a,b)^m, (x,y)^n$ are regulare, i.e. $(a,b)^m, (x,y)^n \in S$. In what follows we will denote by

$$L^*_{(a,b)}\left(R^*_{(a,b)}, H^*_{(a,b)}, J^*_{(a,b)}\right)$$

the L^*-, R^*-, H^*-, J^*-class containing an element (e,e) of S. According to the hypothesis,we had drawn the following conclusions. (see[1])

Lemma 1. Let S be a Π^*-regular semigroup.Then each idempotent (e,e) of S is a right(left,two-sided) identity for regular elements from

$$L^*_{(e,e)}\left(R^*_{(e,e)}, J^*_{(e,e)}\right)$$

Lemma 2. In a Π^*-regular semigroup S every H^*-class contains at most one idempotent.

Lemma 3. Let S be a Π^*-regular semigroup, $(a,b) \in S$ and p the the smallest positive integers such that $(a,b)^p \in S$. Then

$$(a,b)^p \in L^*_{(a,b)} \cap R^*_{(a,b)} = H^*_{(a,b)}.$$

Lemma 4. Let S be a Π^*-regular semigroup. Then

(1) every J^*-class contains at least one idempotent;

(2) $G_{(e,e)} \subseteq H_{(e,e)} \subseteq J_{(e,e)}$ for every $e \in E$.

Lemma 5. Let S be a Π^*-regular semigroup. Then (for some $(e,e),(f,f) \in E$)

$$J^*_{(e,e)} = J^*_{(f,f)}, (e,e)(f,f) = (f,f)(e,e) = (f,f) \Rightarrow (e,e) = (f,f).$$

Lemma 6 . Let S be a Π^*-regular semigroup.Then for some $(u,v)\in S,(e,e)\in E$,

$$J^*_{(e,e)} = J^*_{(e,e)(u,v)(e,e)} \Rightarrow (e,e)(u,v)(e,e)\in G_{(e,e)}.$$

Theorem 1. Let (e,e) be an idemptent of a Π^*-regular semigroup S. Then $G_{(e,e)}\subseteq H^*_{(e,e)}$, furthermore ,if $(u,v)\in H^*_{(e,e)}$ and p is the smallest positive integer such that $(u,v)^p\in S$, then $(u,v)^q\in G_{(e,e)}$ for every $q\geq p$.

Theorem 2. Let S be a Π^*-regular semigroup. Then (for some $(a,b),(x,y)\in S$)

$$J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}.$$

J^*-*class* plays a very active role in
Π^*-regular semigroups,since the emergence of Idempotents. Here we get some good results.

2 Main Results

Let V be the set of all inverse elements of S ([3]), E is the set of all idempotent of S .

Lemma 7 . Let S be a Π^*-regular semigroup.Then for some $(e,e),(f,f)\in E$)

$$J^*_{(e,e)} = J^*_{(f,f)} = J^*_{(e,e)(f,f)} = J^*_{(e,e)}.$$

Proof. Let $S(e,e)S=S(f,f)S$. Then

$$(e,e)=(a,b)(f,f)(e,e)=(a,b)(f,f)(u,v)(e,e) \quad ((a,b),(u,v)\in S)$$

Whence by theorem 2 we have

$$J^*_{(f,f)} = J^*_{(e,e)} = J^*_{(a,b)(f,f)(u,v)(e,e)} = J^*_{((a,b)(f,f))((f,f)(u,v)(e,e))}$$

$$= J^*_{(f,f)(u,v)(e,e)(a,b)(f,f)}$$

And since $(f,f)(u,v)(e,e)(a,b)(f,f)\in E$ by lemma 5 we attain that E

$$(f,f)=(f,f)(u,v)(e,e)(a,b)(f,f)$$

and

$$(f,f)(u,v)(f,f)\in G_{(f,f)}$$

Analogously

$$(f,f)(a,b)(f,f)\in G_{(f,f)}$$

So

$$((f,f)(e,e)(f,f))^n = ((f,f)(a,b)(f,f)(f,f)(u,v)(f,f))^p \in G_{(f,f)} \subseteq J^*_{(f,f)}$$

for every $n,p\in Z^+$. Thus

$$S(f,f)S=S((f,f)(e,e)(f,f))^p S\subseteq S((e,e)(f,f))^p S.$$

and it's opposite inclusion also holds we have

$$S(f,f)S=S((e,e)(f,f))^p S.$$

Hence

$$J^*_{(e,e)} = J^*_{(f,f)} = J^*_{(e,e)(f,f)} = J^*_{(e,e)}.$$

Theorem 3. Let S be a Π^*-regular semigroup. Then (for some $(a,b),(x,y)\in S$)

$$J^*_{(a,b)(x,y)} = J^*_{(e,e)(f,f)} \quad .(m,n\in Z^+)$$

Proof. Let r be the the smallest positive integers such that $((a,b)(x,y))^r \in \operatorname{Reg} S$. Then

$$((a,b)(x,y))^r \in J^*_{((a,b)(x,y))},\ ((a,b)(x,y))^r \in G_{(h,h)} \subseteq J^*_{(h,h)},$$

$$(h,h)(a,b)(x,y) = (a,b)(x,y)(h,h) \in G_{(h,h)},((h,h)\in E) \tag{1}$$

Now we prove by induction on p that

$$(h,h)(a,b)^p(h,h)\in G_{(h,h)},(h,h)(x,y)^p(h,h)\in G_{(h,h)} \tag{2}$$

for every $p\geq 0$. Suppose $(h,h)(a,b)^p(h,h)\in G_{(h,h)}(p\geq 0)$, Then

$$((h,h)(a,b)^p(h,h))(h,h)(a,b)(x,y) = (h,h)(a,b)^{p+1}(x,y)(h,h)\in G_{(h,h)}.$$

Let (u,v) be an inverse element of

$$(h,h)(a,b)^{p+1}(x,y)(h,h) \text{ in } G_{(h,h)},$$

Then

$$(h,h) = (h,h)(a,b)^{p+1}(x,y)(u,v)(h,h),$$

Whence we have that $(h,h)(a,b)^{p+1}$ is regulare.Hence by theorem 2

$$J^*_{(h,h)} = J^*_{(h,h)(a,b)^{p+1}} = J^*_{(h,h)(a,b)^{p+1}(h,h)}.$$

Now by lemma 6 $(h,h)(a,b)^{p+1}(h,h)\in G_{(h,h)}$ and since $(h,h)(a,b)^0(h,h)\in G_{(h,h)}$ the first part is proved.In a similar way we can show the second part .

Since $(a,b)^m\in G_{(e,e)}$ by the first part of (2) we get

$$(h,h)(e,e)(a,b)^m(h,h)\in G_{(h,h)}.$$

Let (u,v) be an inverse element of $(h,h)(e,e)(a,b)^m(h,h)$ in $G_{(h,h)}$, Then $(h,h) = (h,h)(e,e)(a,b)^m(u,v)(h,h)$,$(h,h)(e,e)$ is regular.

Thus

$$J^*_{(h,h)} = J^*_{(h,h)(e,e)} = J^*_{(h,h)(e,e)(h,h)}$$

So

$$(h,h)(e,e)(h,h)\in G_{(h,h)}$$

and analogously

$$(h,h)(f,f)(h,h)\in G_{(h,h)}.$$

Hence $(h,h)(e,e)(h,h),(h,h)(f,f)(h,h)\in G_{(h,h)}$ (3)

Now we show $(h,h)((e,e)(f,f))^p(h,h)\in G_{(h,h)}$ (4)

for every $p\geq 0$.

From $(h,h)(f,f)(h,h)\in G_{(h,h)}$ we attain

$$(h,h) = (h,h)(e,e)(u,v)(f,f)(h,h) = (h,h)(f,f)(h,h)(e,e)(s,t),$$

Here $(f,f)(h,h)(e,e)$ and $(e,e)(f,f)(h,h)$ are regular

And $J^*_{(h,h)} = J^*_{(e,e)(f,f)(h,h)} = J^*_{(f,f)(h,h)(e,e)} = J^*_{(h,h)(e,e)(f,f)(h,h)}$

So $(h,h)(e,e)(f,f)(h,h) \in G_{(h,h)}$, i.e.

The condition $(h,h)((e,e)(f,f))^p(h,h) \in G_{(h,h)}$ is right for $p=1$.

According to the above assumptions $p=2$ we have

$$S(h)S \subseteq S((e,e)(f,f))^p(h,h)(e,e)(f,f)S$$

and the other hand

$$S((e,e)(f,f))^p(h,h)(e,e)(f,f)S \subseteq S(h,h)S$$

So $S(h,h)S=S((e,e)(f,f))^p(h,h)(e,e)(f,f)S$.

Thus $J^*_{(h,h)} = J^*_{((e,e)(f,f))^p(h,h)(e,e)(f,f)} = J^*_{(h,h)((e,e)(f,f))^{p+1}(h,h)}$

Whence $(h,h)((e,e)(f,f))^{p+1}(h,h) \in G_{(h,h)}$,

so by induction that $(h,h)((e,e)(f,f))^p(h,h) \in G_{(h,h)}$ holds.

Let q be the the smallest positive integers such that $((e,e)(f,f))^q \in \operatorname{Reg}S$.
Then

$$((e,e)(f,f))^q \in J^*_{(e,e)(f,f)}, ((e,e)(f,f))^q \in G_{(k,k)} \subseteq J^*_{(k,k)}((k,k)\in E). \tag{5}$$

Now we prove

$$(k,k)((a,b)(x,y))^p \in G_{(k,k)} \tag{6}$$

for every $p \geq 0$.If p=1. By induction that

$$((e,e)(f,f))^{2q} \in G_{(k,k)} \subseteq J^*_{(k,k)},$$

so $((e,e)(f,f))^{2q} = ((e,e)(f,f))^{2q}(u,v)((e,e)(f,f))^{2q}((u,v)\in G_{(k,k)})$.

Therefor we have

$$J^*_{(k,k)} = J^*_{(a,b)((e,e)(f,f))^{2q}(x,y)} = J^*_{(x,y)(a,b)((e,e)(f,f))^{2q}} = J^*_{(k,k)((e,e)(f,f))^q(x,y)(a,b)((e,e)(f,f))^q}.$$

whence

$$((e,e)(f,f))^q(a,b)(x,y)((e,e)(f,f))^q =$$

$$(k,k)((e,e)(f,f))^q(x,y)(a,b)((e,e)(f,f))^q(k,k) \in G_{(k,k)},$$

by (1),(4),(5),(6) we obtain

$$S((a,b)(x,y))^rS = S(h,h)S = S(h,h)((e,e)(f,f))^w(h,h)S \subseteq S((e,e)(f,f))^qS;$$

$$S((e,e)(f,f))^qS = S(k,k)S = S(k,k)((x,y)(a,b))^r(k,k)S \subseteq S((x,y)(a,b))^rS.$$

and we consider $J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}$, then

$$S((a,b)(x,y))^rS = S((e,e)(f,f))^qS.$$

Hence

$$J^*_{(a,b)(x,y)} = J^*_{(e,e)(f,f)}.$$

References

[1] LUO Xiaoqiang. The Subclasses of Characterization on Π^*-Regular Semigroups, Progress in Applied Mathmatics(Canadian),5(2),2013,1-5.
[2] Xiaoqiang LUO. Π^*-Regular Semigroups.Bulletin of Mathematical Science &Applications (India), 1(1), (2012), 63-70.
[3] Mario Petrich and Norman R .Reilly..Completely Regular Semigroups.New York, John Wiley & Sons ,Inc. Press,1999.
[4] Howie J M .().An Introduction to Semigroup Theory.New York ,Academic Press,1976.
[5] Xiaoqiang LUO. Another Property of Congruence-free Regular Semigroups.South Asian Journal of Mathematics, 2(3), 2012,238-241.