

On a Hsu-unified Structure Manifold with a Quarter-symmetric Non-metric Connection

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Keywords: Quarter-symmetric non-metric connection, Hsu-unified structure manifold, Hsu- Kahler manifold, Nijenhuis tensor, Contravariant almost analytic vector field.

Abstract: In the present paper, we have defined a Hsu-unified structure manifold and a Hsu-Kahler manifold and studied some properties of the quarter-symmetric non-metric connection. Certain interesting results on such manifolds have been obtained. We have also studied the properties of the contravariant almost analytic vector field on these manifolds equipped with the quarter-symmetric non-metric connection.

1. Introduction

Quarter-symmetric linear connection was introduced and studied by S. Golab[1] in 1975. Several properties of quarter-symmetric metric and non-metric connections on a differentiable manifold have been studied by Yano and Imai[10], Sular et al. [8], Sengupta and Biswas[7] and many other geometers. In the present paper, we have studied some properties of the quarter-symmetric non-metric connection on a manifold called Hsu-unified structure manifold and a Hsu- Kahler manifold which is a particular case of Hsu-unified structure manifold satisfying a certain condition. It has been shown that the Nijenhuis tensor with respect to quarter-symmetric non-metric connection ∇ and with respect to Riemannian connection D coincide in the Hsu-unified structure manifold but in the Hsu- Kahler manifold Nijenhuis tensor with respect to ∇ vanishes identically i.e. a Hsu- Kahler manifold is integrable. It has also been proved that a contravariant almost analytic vector field V with respect to Riemannian connection D is also contravariant almost analytic with respect to quarter-symmetric non-metric connection ∇ in the Hsu- Kahler manifold but in the Hsu-unified structure manifold, it is possible with a specific condition

2. Preliminaries

If on an even dimensional differentiable manifold $M_n, n = 2m$ of differentiability class C^∞ , there exists a vector valued real linear function F of differentiability class C^∞ , satisfying

$$F^2 X = a^r X \tag{2.1}$$

for arbitrary vector field X . Also there exists a Riemannian metric g , such that

$$g(\bar{X}, \bar{Y}) = a^r g(X, Y) \tag{2.2}$$

where $\bar{X} = FX, 0 \leq r \leq n$ and a is a real or complex number.

Then in view of the equations (2.1) and (2.2), M_n is said to be a Hsu-unified structure manifold.

Let us define a 2-form $'F$ in M_n , given as

$$'F(X, Y)g(\bar{X}, Y) = g(X, \bar{Y}) \tag{2.3}$$

Then it is clear that the 2-form $'F$ satisfies

$$'F(\bar{X}, \bar{Y}) = a^r 'F(X, Y) \tag{2.4}$$

From (2.3), we can also conclude the following properties

$${}'F(\bar{X}, Y) = a^r g(X, Y) \quad (2.5)$$

$${}'F(X, Y) = {}'F(Y, X) \quad (2.6)$$

The equation (2.6) shows that the 2-form $'F$ is symmetric in M_n .

If the Hsu-unified structure manifold M_n satisfies a condition

$$(D_X F)Y = 0 \quad (2.7)$$

Then M_n will be said to be a Hsu-Kähler manifold.

From the equation (2.7), it can be seen that

$$D_X \bar{Y} - \overline{D_X Y} \Leftrightarrow \overline{D_X Y} = a^r (D_X Y) \quad (2.8)$$

3. Quarter-symmetric Non-metric Connection

A linear connection ∇ defined as [9]

$$\nabla_X Y = D_X Y + u(Y)\bar{X} \quad (3.1)$$

or arbitrary vector fields X and Y , is said to be a quarter-symmetric non-metric connection if the torsion tensor S of the connection ∇ and the metric tensor g are given by

$$S(X, Y) = u(Y)\bar{X} - u(X)\bar{Y} \quad (3.2)$$

and

$$(\nabla_X g)(Y, Z) = -u(Y)g(\bar{X}, Z) - u(Z)g(\bar{X}, Y) \quad (3.3)$$

where u is 1-form associated with the vector field U such that

$$u(X) = g(X, U) \quad (3.4)$$

and D is the Riemannian connection.

If we put (3.1) as

$$\nabla_X Y = D_X Y + H(X, Y) \quad (3.5)$$

Then

$$H(X, Y) = u(Y)\bar{X} \quad (3.6)$$

Let us define

$${}'H(X, Y, Z) = g(H(X, Y), Z) \quad (3.7)$$

By virtue of the equation (3.6), (3.7) becomes

$${}'H(X, Y, Z) = u(Y)g(\bar{X}, Z)$$

4. Hsu-unified structure manifold equipped with the Quarter-symmetric non-metric connection

In this section, we have the following theorems

Theorem 4.1: For a Hsu-unified structure manifold M_n equipped with a quarter-symmetric non-metric connection ∇ , the following results hold good

$$\begin{aligned} (i) \quad & H(\bar{X}, Y) = a^r H(X, \bar{Y}) \quad \text{iff} \quad u(Y)X = u(\bar{Y})\bar{X} \\ (ii) \quad & H(\bar{X}, Y) = H(\bar{Y}, X) \quad \text{iff} \quad u(Y)X = u(X)Y \\ (iii) \quad & H(\bar{X}, \bar{Y}) = a^r H(X, Y) \quad \text{iff} \quad u(\bar{Y})X = u(Y)\bar{X} \end{aligned} \quad (4.1)$$

Proof: From the equations (3.6) and (2.1), we have

$$\begin{aligned} (i) \quad & H(\bar{X}, Y) = a^r u(Y)X \\ (ii) \quad & H(X, \bar{Y}) = u(\bar{Y})\bar{X} \end{aligned} \tag{4.2}$$

Clearly (4.1)) (i) follows from (4.2)) (i) and (4.2)) (ii) .

Interchanging X and Y in (3.6), we get

$$H(Y, X) = u(X)\bar{Y} \tag{4.3}$$

Barring Y in (4.3) and using (2.1), we have

$$H(\bar{Y}, X) = a^r u(X)Y \tag{4.4}$$

Now the result of (4.1)) (ii) follows from (4.2)) (i) and (4.4).

Again barring X and Y in (3.6) and using (2.1), we obtain

$$H(\bar{X}, \bar{Y}) = a^r u(\bar{Y})X \tag{4.5}$$

By virtue of the equations (3.6) and (4.5), the result of (4.1)) (iii) follows.

Theorem 4.2: In a Hsu-unified structure manifold M_n with a quarter-symmetric non-metric connection ∇ , we have

$$\begin{aligned} (i) \quad & 'H(\bar{X}, \bar{Y}, Z) = 'F(\bar{X}, Z)u(\bar{Y}) = 'H(X, \bar{Y}, \bar{Z}) \\ (ii) \quad & 'H(\bar{X}, \bar{Y}, \bar{Z}) = 'F(\bar{X}, \bar{Z})u(\bar{Y}) = 'H(\bar{Z}, \bar{Y}, \bar{X}) \end{aligned} \tag{4.6}$$

Proof: Barring X and Y in (3.8) and using the equations (2.3) and (2.5), we get

$$'H(\bar{X}, \bar{Y}, Z) = 'F(\bar{X}, Z)u(\bar{Y}) \tag{4.7}$$

Again barring Y and Z in (3.8) and using the equations (2.3) and (2.5), we obtain

$$'H(X, \bar{Y}, \bar{Z}) = 'F(\bar{X}, \bar{Z})u(\bar{Y}) \tag{4.8}$$

Hence the equations (4.7) and (4.8) prove the result of (4.6)) (i) .

Barring Z in (4.7), we have

$$'H(\bar{X}, \bar{Y}, \bar{Z}) = 'F(\bar{X}, \bar{Z})u(\bar{Y}) \tag{4.8}$$

Interchanging \bar{X} and \bar{Z} in (4.8), we get

$$'H(\bar{Z}, \bar{Y}, \bar{X}) = 'F(\bar{Z}, \bar{X})u(\bar{Y}) \tag{4.9}$$

In view of the equations (4.8), (4.9) and (2.6), the result of (4.6)) (ii) follows easily

Theorem 4.3: A Hsu-unified structure manifold M_n with a quarter-symmetric non-metric Connection ∇ satisfies the following relation

$$(\nabla_{\bar{X}}F)\bar{Y} = -a^r(\nabla_XF)Y \text{ iff } (D_{\bar{X}}F)\bar{Y} = -a^r(D_XF)Y \quad (4.10)$$

Proof: Replacing Y by \bar{Y} in (3.1), we have

$$\nabla_X\bar{Y} = D_X\bar{Y} + u(\bar{Y})\bar{X} \quad (4.11)$$

The above equation can also be modified as

$$(\nabla_XF)Y = D_X\bar{Y} + u(\bar{Y})\bar{X} - \nabla_X\bar{Y} \quad (4.12)$$

Operation of F on both sides of the equation (3.1) yields

$$\overline{\nabla_XY} = \overline{D_XY} + a^r u(Y)X \quad (4.13)$$

From the equations (4.12) and (4.13), we get

$$(\nabla_XF)Y = (D_XF)Y + u(\bar{Y})\bar{X} - a^r u(Y)X \quad (4.14)$$

Barring X and Y in (4.14), we have

$$(\nabla_{\bar{X}}F)\bar{Y} = (D_{\bar{X}}F)\bar{Y} + a^{2r}u(Y)X - a^r u(\bar{Y})\bar{X} \quad (4.15)$$

Subtraction of the equations (4.14) and (4.15) gives

$$(\nabla_{\bar{X}}F)\bar{Y} - (\nabla_XF)Y = (D_{\bar{X}}F)\bar{Y} - (D_XF)Y + (a^r + 1)[a^r u(Y)X - u(\bar{Y})\bar{X}] \quad (4.16)$$

Now in view of the equations (4.14) and (4.16), we obtain

$$(\nabla_{\bar{X}}F)\bar{Y} + a^r(\nabla_XF)Y = (D_{\bar{X}}F)\bar{Y} + a^r(D_XF)Y \quad (4.17)$$

Clearly, the equation (4.17) proves the statement of the theorem.

Theorem 4.4: If a Hsu-unified structure manifold M_n admits a quarter-symmetric non-metric Connection ∇ , then the Nijenhuis tensors of the Riemannian connection D and ∇ coincide.

Proof: From the equation (4.14), we have

$$(D_XF)Y = (\nabla_XF)Y - u(\bar{Y})\bar{X} + a^r u(Y)X \quad (4.18)$$

Replacing X by \bar{X} in (4.18), we get

$$(D_{\bar{X}}F)Y = (\nabla_{\bar{X}}F)Y - a^r u(\bar{Y})X + a^r u(Y)\bar{X} \quad (4.19)$$

Interchanging X and Y in (4.19), we find

$$(D_{\bar{Y}}F)X = (\nabla_{\bar{Y}}F)X - a^r u(\bar{X})Y + a^r u(X)\bar{Y} \quad (4.20)$$

Operating both sides of the equation (4.18) by F , we obtain

$$\overline{(D_XF)Y} = \overline{(\nabla_XF)Y} - a^r u(\bar{Y})X + a^r u(Y)\bar{X} \quad (4.21)$$

Again interchanging X and Y in the above equation, we have

$$\overline{(D_YF)X} = \overline{(\nabla_YF)X} - a^r u(\bar{X})Y + a^r u(X)\bar{Y} \quad (4.22)$$

The Nijenhuis tensor with respect to F is a vector valued bilinear function, defined as [2], [3]

$$N(X, Y) = [\bar{X}, \bar{Y}] - [\bar{X}, Y] - [X, \bar{Y}] + [X, Y]$$

In view of the equation (2.1), the above expression can be written in the form

$$N(X, Y) = [\overline{X}, \overline{Y}] - [\overline{X}, Y] - [X, \overline{Y}] + a^r[X, Y]$$

The Nijenhuis tensor with respect to the Riemannian connection D is given as

$$N(X, Y) = (D_{\overline{X}}F)Y - (D_{\overline{Y}}F)X - \overline{(D_X F)Y} + \overline{(D_Y F)X} \tag{4.23}$$

By virtue of the equations (4.19), (4.20), (4.21) and (4.22), (4.23) becomes

$$\begin{aligned} N(X, Y) &= (\nabla_{\overline{X}}F)Y - (\nabla_{\overline{Y}}F)X - \overline{(\nabla_X F)Y} + \overline{(\nabla_Y F)X} \\ &= N^*(X, Y) \end{aligned}$$

Where $N^*(X, Y)$ denotes the Nijenhuis tensor with respect to the quarter-symmetric non-metric connection ∇ . Hence we have the statement of the theorem.

5. Hsu-Kahler manifold with the Quarter-symmetric non-metric connection

As discussed earlier that if the Hsu-unified structure manifold M_n satisfies the condition (2.7), then M_n is called a Hsu-Kahler manifold. In this section, we have some following theorems

Theorem 5.1: If M_n be a Hsu-Kahler manifold admitting a quarter-symmetric non-metric connection ∇ , then in M_n

$$\begin{aligned} (i) \quad &(\nabla_{\overline{X}}F)\overline{Y} = -a^r(\nabla_X F)Y \\ (ii) \quad &(\nabla_X F)Y = 0 \quad \text{iff} \quad u(\overline{Y})\overline{X} = a^r u(Y)X \end{aligned} \tag{5.1}$$

Proof: In a Hsu-Kahler manifold $n M$, the equation (4.11) can be written as

$$(\nabla_X F)Y = \overline{D_X Y} + u(\overline{Y})\overline{X} - \overline{\nabla_X Y} \tag{5.2}$$

From the equations (5.2) and (4.13), it is clear that

$$(\nabla_X F)Y = u(\overline{Y})\overline{X} - a^r u(Y)X \tag{5.3}$$

Barring X and Y in (5.3) and using (2.1), we get

$$(\nabla_{\overline{X}}F)\overline{Y} = -a^r [u(\overline{Y})\overline{X} - a^r u(Y)X] \tag{5.4}$$

By virtue of the equations (5.3) and (5.4), the required result of (5.1) (i) follows.

Also in view of the equation (5.3), (5.1) (ii) is obvious.

Theorem 5.2: A Hsu-Kahler manifold M_n with a quarter-symmetric non-metric connection ∇ satisfies the following relation

$$d^r F(X, Y, Z) = -2[u(X)g(\overline{Y}, \overline{Z}) + u(Y)g(\overline{Z}, \overline{X}) + u(Z)g(\overline{X}, \overline{Y})] \tag{5.5}$$

Proof: By definition, we know that

$$d^r F(X, Y, Z) = (\nabla_X {}^r F)(Y, Z) + (\nabla_Y {}^r F)(Z, X) + (\nabla_Z {}^r F)(X, Y) \tag{5.6}$$

From the equation (2.3), we have

$${}^r F(Y, Z) = g(\overline{Y}, \overline{Z}) \tag{5.7}$$

Differentiating the equation (5.7) covariantly with respect to X and using the expressions of the equations (2.2), (2.3), (3.1) and (3.3), we obtain

$$(\nabla_X {}^r F)(Y, Z) = -[u(Z)g(\overline{X}, \overline{Y}) + u(Y)g(\overline{Z}, \overline{X})] \tag{5.8}$$

Proceeding in the same way, we can also get

$$(\nabla_Y {}^r F)(Z, X) = -[u(X)g(\overline{Y}, \overline{Z}) + u(Z)g(\overline{X}, \overline{Y})] \tag{5.9}$$

$$(\nabla_Z {}^r F)(X, Y) = -[u(Y)g(\overline{Z}, \overline{X}) + u(X)g(\overline{Y}, \overline{Z})] \tag{5.10}$$

In consequence with the equations (5.6), (5.8), (5.9) and (5.10), the required result follows.

Theorem 5.3: The Nijenhuis tensor with respect to the quarter-symmetric non-metric connection ∇ in a Hsu-Kähler manifold M_n vanishes i.e. the manifold M_n is integrable.

Proof: The Nijenhuis tensor with respect to the quarter-symmetric non-metric connection ∇ is defined as

$$N^*(X, Y) = (\nabla_{\bar{X}}F)Y - (\nabla_{\bar{Y}}F)X - \overline{(\nabla_X F)Y} + \overline{(\nabla_Y F)X} \quad (5.11)$$

Replacing X by \bar{X} in (5.3), we get

$$(\nabla_{\bar{X}}F)Y = a^r[u(\bar{Y})X - u(Y)\bar{X}] \quad (5.12)$$

Interchanging X and Y in (5.12), we obtain

$$(\nabla_{\bar{Y}}F)X = a^r[u(\bar{X})Y - u(X)\bar{Y}] \quad (5.13)$$

Operation of F on both sides of the equation (5.3) yields

$$\overline{(\nabla_X F)Y} = a^r[u(\bar{Y})X - u(Y)\bar{X}] \quad (5.14)$$

Interchanging X and Y in (5.14), we have

$$\overline{(\nabla_Y F)X} = a^r[u(\bar{X})Y - u(X)\bar{Y}] \quad (5.15)$$

In view of the equations (5.12), (5.13), (5.14) and (5.15), the equation (5.11) takes the form

$$N^*(X, Y) = 0$$

which proves the theorem.

6. Contravariant almost analytic vector field

A vector field V is said to be contravariant almost analytic if the Lie-derivative of the tensor field F with respect to V vanishes identically i.e. [2], [3]

$$(L_V F)X = 0 \quad \text{for all } X \quad (6.1)$$

The equation (6.1) can also be written as

$$[V, \bar{X}] = \overline{[V, X]} \quad (6.2)$$

In a Hsu-unified structure manifold, the equation (6.2) takes the form

$$(D_V F)X - D_{\bar{X}}V + \overline{D_X V} = 0 \quad (6.3)$$

In a Hsu-Kähler manifold, the equation (6.2) becomes

$$D_{\bar{X}}V - \overline{D_X V} = 0 \Leftrightarrow \overline{D_X V} = a^r(D_X V) \quad (6.4)$$

Now we have the following theorems

Theorem 6.1: On a Hsu-unified structure manifold, a contravariant almost analytic vector field V with respect to the Riemannian connection D is also contravariant almost analytic with respect to the quarter-symmetric non-metric connection ∇ if

$$u(\bar{X})\bar{V} = a^r u(X)V$$

Proof: For any vector field V , the equation (3.1) takes the form

$$\nabla_X V = D_X V + u(V)\bar{X} \quad (6.5)$$

Replacing X by \bar{X} in (6.5), we have

$$\nabla_{\bar{X}}V = D_{\bar{X}}V + a^r u(V)X \quad (6.6)$$

Operating both sides of the equation (6.5) by F , we get

$$\overline{\nabla}_X V = \overline{D}_X V + a^r u(V)X \tag{6.7}$$

Interchanging X and V in (6.5), we obtain

$$\nabla_V X = D_V X + u(X)\overline{V} \tag{6.8}$$

From (6.8), it is clear that

$$\begin{aligned} (i) \nabla_V \overline{X} &= D_V \overline{X} + u(\overline{X})\overline{V} \\ (ii) \overline{\nabla}_V X &= \overline{D}_V X + a^r u(X)V \end{aligned} \tag{6.9}$$

As we know that

$$(\nabla_V F)X = \nabla_V \overline{X} - \overline{\nabla}_V X$$

In consequence with the equation (6.9), the last expression becomes

$$(\nabla_V F)X = D_V \overline{X} - \overline{D}_V X + u(\overline{X})\overline{V} - a^r u(X)V \tag{6.10}$$

Now by virtue of the equations (6.6), (6.7) and (6.10), we find that

$$(\nabla_V F)X - \nabla_{\overline{X}} V + \overline{\nabla}_X V = (D_V F)X - D_{\overline{X}} V + \overline{D}_X V + u(\overline{X})\overline{V} - a^r u(X)V \tag{6.11}$$

Since V is contravariant almost analytic vector field with respect to the Riemannian connection D , so using (6.3), the equation (6.11) is transformed into

$$(\nabla_V F)X - \nabla_{\overline{X}} V + \overline{\nabla}_X V = u(\overline{X})\overline{V} - a^r u(X)V \tag{6.12}$$

If the vector field V is also contravariant almost analytic with respect to the quarter-symmetric non-metric connection ∇, then it will satisfy

$$(\nabla_V F)X - \nabla_{\overline{X}} V + \overline{\nabla}_X V = 0$$

Hence the equation (6.12) proves the statement of the theorem.

Theorem 6.2: On a Hsu-Kahler manifold, a contravariant almost analytic vector field V with respect to the Riemannian connectionD is also contravariant almost analytic with respect to the quarter-symmetric non-metric connection∇.

Proof: Subtraction of the equation (6.7) from (6.6) gives

$$\nabla_{\overline{X}} V - \overline{\nabla}_X V = D_{\overline{X}} V - \overline{D}_X V \tag{6.13}$$

Since V is contravariant almost analytic vector field with respect to the Riemannian connection D in the Hsu-Kahler manifold, so with the help of the equation (6.4), (6.13) yields

$$\nabla_{\overline{X}} V - \overline{\nabla}_X V = 0 \tag{6.14}$$

which implies that V is also contravariant almost analytic vector field with respect to the quarter-symmetric non-metric connection ∇.

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