Existence and uniqueness of fuzzy control integro-differential equation with perturbed

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Abstract: In the paper, we discuss the existence and uniqueness solution of fuzzy control integro differential equation with perturbed. The method of successive approximations is utilized to establish these results.

1. Introduction

The fuzzy set theory introduced by Zadeh has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations and engineering sciences. Recently, the authors have made important research results in the theory of fuzzy differential equations, integro-differential equations, fuzzy integro-differential equations, ...

The fuzzy integro-differential equation in parametric form is converted to its related crisp form and the variational iteration method is used to approximate the solution of crisp integro-differential equation. Convergence of the obtained solution to exact solution is considered (see [3]). T. Allahviranloo et al considered the existence and uniqueness of solutions of fuzzy Volterra integro-differential equations of the second kind with fuzzy kernel under strongly generalized differentiability. The authors studied fuzzy set control differential equations (FSCDE) and the problem of stability and controllability of FSCDE (see [5]). In [6], the authors consider the existence of solutions of perturbed fuzzy integro differential equations

$$\begin{cases} x'(t) = f(t, x(t)) + \int_{t_0}^t h(t, s, x(s))ds + g\left(t, x(t), \int_{t_0}^t k(t, s, x(s))ds\right) \\ x(t_0) = x_0 \end{cases}$$

In this paper, we consider the existence and uniqueness of fuzzy control integro-differential equation with perturbed of the form

$$\begin{cases} x'(t) = f(t, x(t), u(t)) + \int_{t_0}^t h(t, s, x(s), u(s))ds + g\left(t, x(t), \int_{t_0}^t k(t, s, x(s), u(s))ds, u(t)\right) \\ x(t_0) = x_0 \end{cases}$$

where $u(t)$ is admissible control and some suitable conditions

2. Preliminaries

We recall some notations and concepts presented in detail in recent series works of Prof. V.Lakshmikantham et al ... (see [1,2,]). Let $K_C(R^n)$ denote the collection of all nonempty, compact and convex subsets of R^n . Given A, B in $K_C(R^n)$, the Hausdorff distance between A and B defined as

$$d_H[A, B] = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|_{R^n}, \sup_{b \in B} \inf_{a \in A} \|a - b\|_{R^n}\right\}$$

where $\|\cdot\|_{R^n}$ denotes the Euclidean norm in R^n . It is known that $K_C(R^n), d_H$ is a complete metric space and if the space $K_C(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_C(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

The set $[\omega]^\alpha = \{z \in R^n: \omega(z) \geq \alpha, 0 < \alpha \leq 1\}$ is called the α -level set. For all $0 \leq \alpha \leq \beta \leq 1$ then we have $[\omega]^\beta \subset [\omega]^\alpha \subset \text{left}[\omega^0]$. Set $E^n = \{\omega: R^n \rightarrow [0,1] \text{ such that } \omega(z) \text{ satisfies (i)-(iv) stated below}\}$

- i). ω is normal, that is, there exists an $z_0 \in R^n$ such that $\omega(z_0) = I$;
- ii). ω is fuzzy convex, that is, for $0 \leq \lambda \leq 1$, $\omega(\lambda z_1 + (1 - \lambda)z_2) \geq \min\{\omega(z_1), \omega(z_2)\}$;
- iii). ω is upper semicontinuous;
- iv). $[\omega]^0 = cl\{z \in R^n: \omega(z) > 0\}$ is compact.

The element $\omega \in E^n$ is called a fuzzy number or fuzzy set.

For two fuzzy sets $\omega_1, \omega_2 \in E^n$, we denote $\omega_1 \leq \omega_2$ if and only if $[\omega_1]^\alpha \subset [\omega_2]^\alpha$. Let us denote

$$D_0[\omega_1, \omega_2] = \sup\{d_H[[\omega_1]^\alpha, [\omega_2]^\alpha] : 0 \leq \alpha \leq 1\}$$

the distance between ω_1 and ω_2 in E^n , where $d_H[[\omega_1]^\alpha, [\omega_2]^\alpha]$ is Hausdorff distance between two set $[\omega_1]^\alpha, [\omega_2]^\alpha$ of $K_C(R^n)$. Then (E^n, D_0) is a complete space.

Some properties of metric D_0 are as follows.

$$\begin{aligned} D_0[\omega_1 + \omega_3, \omega_2 + \omega_3] &= D_0[\omega_1, \omega_2], \\ D_0[\lambda \omega_1, \lambda \omega_2] &= |\lambda| D_0[\omega_1, \omega_2], \\ D_0[\omega_1, \omega_2] &\leq D_0[\omega_1, \omega_3] + D_0[\omega_3, \omega_2], \end{aligned}$$

for all $\omega_1, \omega_2, \omega_3 \in E^n$ and $\lambda \in R$. Given an interval $I = [T_0, T] \subseteq R_+$

Let us denote $\theta \in E^n$ the zero element of E^n as follows:

$$\theta(z) = \begin{cases} 1 & \text{if } z = \hat{0} \\ 0 & \text{if } z \neq \hat{0} \end{cases}$$

where $\hat{0}$ is the zero element of R^n . Let $u, v \in E^n$. The set $w \in E^n$ satisfying $u = u + w$ is known as the geometric difference of the set u and v and is denoted by the symbol $u - v$. The mapping $F: R_+ \supset I = [T_0, T] \rightarrow E^n$ is said to have a Hukuhara derivative $x'(\tau)$ at a point $\tau \in I$, if

$$\lim_{h \rightarrow 0^+} h^{-1} (x(\tau + h) - x(\tau)) \quad \text{and} \quad \lim_{h \rightarrow 0^+} h^{-1} (x(\tau) - x(\tau - h))$$

exist and equal to $D_H x(\tau)$. Here limits are taken in the metric space (E^n, D_0) . If $x: I \rightarrow E^n$ is continuous, then it is integrable and

$$\int_{t_0}^{t_2} x(s) ds = \int_{t_0}^{t_1} x(s) ds + \int_{t_1}^{t_2} x(s) ds \quad (2.1)$$

If $F, G: I \rightarrow E_n$ are integrable, $\lambda \in R$, then some properties below hold

$$\int_{t_0}^t (x(s) + y(s)) ds = \int_{t_0}^t x(s) ds + \int_{t_0}^t y(s) ds \quad (2.2)$$

$$\int_{t_0}^t \lambda x(s) ds = \lambda \int_{t_0}^t x(s) ds, \lambda \in R, t_0 \leq t \leq T. \quad (2.3)$$

$$D_0 \left[\int_{t_0}^t x(s) ds, \int_{t_0}^t y(s) ds \right] \leq \int_{t_0}^t D_0[x(s), y(s)] ds \quad (2.4)$$

3. Main Results

In the section, we consider the existence and uniqueness of fuzzy control integro-differential equation with perturbed of the form

$$\begin{cases} x'(t) = f(t, x(t), u(t)) + \int_{t_0}^t h(t, s, x(s), u(s))ds + g\left(t, x(t), \int_{t_0}^t k(t, s, x(s), u(s))ds, u(t)\right) \\ x(t_0) = x_0 \end{cases} \quad (3.1)$$

where $f: T \times E^n \times E^P \rightarrow E^n$; $g: T \times E^n \times E^P \times E^P \rightarrow E^n$; $h, k: T \times T \times E^n \times E^P \rightarrow E^n$ are levelwise continuous; the control $u(t) \in E^P$. If $u: T \rightarrow E^P$ is integrable, then it is called an admissible control and the interval $T = \{t: |t - t_0| \leq \sigma \leq s \leq a\}$ with $a > 0$. Let U be a set of all admissible controls. The mapping $(t) \in C^1[T, E^n]$ is said to be a solution of E.q (1.1) on T . The solution of E.q (1.1) is written in the form

$$\begin{cases} x(t) = x_0 + \int_{t_0}^t f(s, x(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau))d\tau ds \\ \quad + \int_{t_0}^t g\left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau))d\tau\right)ds \\ x(t_0) = x_0 \end{cases} \quad (3.2)$$

We denote $Q = T \times B(x_0, b_1) \times B(u_0, b_2)$, $Q_0 = T \times B(x_0, b_1) \times B(x_0, b_1) \times B(u_0, b_2)$ and $Q_1 = T \times T \times B(x_0, b_1) \times B(u_0, b_2)$ where $b_1, b_2 > 0$, $x_0 \in E^n$, $u_0 \in E^P$, $B(x_0, b_1) = \{x(t) \in E^n: D[x, x_0] \leq b_1\}$, $B(u_0, b_2) = \{u(t) \in E^P: D[u, u_0] \leq b_2\}$. Assume that $x, y: T \rightarrow E^n$ satisfy the following hypotheses:

(A1) For every $u(t) \in U$ then admissible control $u(t)$ satisfies: $\int_{t_0}^t D_0[u(s), \theta]ds < B$, with B is fixed real number.

(A2) $f: Q \rightarrow E^n$ is levelwise continuous and for any $(t, x, u), (t, y, v) \in Q$, we have

$$D_0[f(t, x, u), f(t, y, v)] \leq L_f[D_0(x, y) + D_0(u, v)]$$

$$D_0[f(t, x, u), \theta] \leq L_f[D_0(x, \theta) + D_0(u, \theta)]$$

where L_f is a given constant.

(A3) $g: Q_0 \rightarrow E^n$ is levelwise continuous and for any $(t, x_1, y_1, u), (t, x_2, y_2, v) \in Q$, we have

$$D_0[g(t, x, y, u), \theta] \leq L_g(D_0[x, \theta] + D_0[y, \theta] + D_0[u, \theta])$$

$$D_0[g(t, x_1, y_1, u), g(t, x_2, y_2, v)] \leq L_g(D_0[x_1, y_1] + D_0[x_2, y_2] + D_0[u, v])$$

where L_g is a given constant.

(A4) $h, k: Q_1 \rightarrow E^n$ is levelwise continuous and for any $(t, s, x, u), (t, s, x, v) \in Q_1$ we have

$$D_0[h(t, s, x, u), \theta] \leq L_h(D_0[x, \theta] + D_0[u, \theta])$$

$$D_0[k(t, s, x, u), \theta] \leq L_k(D_0[x, \theta] + D_0[u, \theta])$$

where L_h, L_k is a given constant.

Theorem . If the conditions (A0)-(A3) holds then there exists a unique solution $x(t)$ of (3.1) defined $|t - t_0| \leq \sigma$. Moreover, there exists $x(t): T \rightarrow E^n$ such that $D(x_n(t), x(t)) \rightarrow 0$ on $|t - t_0| \leq \delta$ as $n \rightarrow \infty$

Proof. Let us define a sequence $x_n(t)$ in E^n , $n=0, 1, 2, \dots$ of successive approximations as follows

$$\begin{cases} x_n(t) = x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau))d\tau ds \\ \quad + \int_{t_0}^t g\left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s)\right)ds \\ x(t_0) = x_0 \end{cases} \quad (3.3)$$

For $n=1$, $t \in T$ we have

$$\begin{aligned} x_1(t) = x_0 + \int_{t_0}^t f(s, x_0(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_0(\tau), u(\tau))d\tau ds \\ + \int_{t_0}^t g\left(s, x_0(s), \int_{t_0}^s k(s, \tau, x_0(\tau), u(\tau))d\tau, u(s)\right)ds \end{aligned}$$

which proves that $x(t)$ is levelwise continuous on $|t - t_0| \leq a$ and hence $|t - t_0| \leq \sigma$. Moreover, we observe that

$$\begin{aligned} \int_{t_0}^t D_0[f(s, x_0(s), u(s)), \theta]ds &\leq \int_{t_0}^t [D_0[x_0, \theta] + D_0[u(s), \theta]]ds \\ &\leq |t - t_0| L_f (D_0[x_0, \theta] + B) \end{aligned}$$

and

$$\begin{aligned} \int_{t_0}^t \int_{t_0}^s D_0[h(s, \tau, x_0(\tau), u(\tau)), \theta]d\tau ds &\leq \int_{t_0}^t \int_{t_0}^s (D_0[x_0, \theta] + D_0[u(\tau), \theta])d\tau ds \\ &\leq \frac{(t - t_0)^2}{2!} L_h (D_0[x_0, \theta] + B) \end{aligned}$$

and

$$\begin{aligned} \int_{t_0}^t D_0\left[g\left(s, x_0(s), \int_{t_0}^s k(s, \tau, x_0(\tau), u(\tau))d\tau, u(s)\right), \theta\right]ds \\ \leq \int_{t_0}^t L_g \left(D_0[x_0, \theta] + D_0[u(s), \theta] + \int_{t_0}^s D_0[k(s, \tau, x_0(\tau), u(\tau)), \theta]d\tau\right)ds \\ \leq |t - t_0| L_g (D_0[x_0, \theta] + B) + \frac{(t - t_0)^2}{2!} L_g L_k (D_0[x_0, \theta] + B) \end{aligned}$$

From these results, we have the following evaluation

$$\begin{aligned} D_0[x_1(t), x_0] &= D_0\left[x_0 + \int_{t_0}^t f(s, x_0(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_0(\tau), u(\tau))d\tau ds \right. \\ &\quad \left. + \int_{t_0}^t g\left(s, x_0(s), \int_{t_0}^s k(s, \tau, x_0(\tau), u(\tau))d\tau, u(s)\right)ds, x_0\right] \\ &\leq \int_{t_0}^t D_0[f(s, x_0(s), u(s)), \theta]ds + \int_{t_0}^t \int_{t_0}^s D_0[h(s, \tau, x_0(\tau), u(\tau)), \theta]d\tau ds \\ &\quad + \int_{t_0}^t D_0\left[g\left(s, x_0(s), \int_{t_0}^s k(s, \tau, x_0(\tau), u(\tau))d\tau, u(s)\right), \theta\right]ds \\ &\leq 2L_{fg}M |t - t_0| + 2L_{fgk}M \frac{(t - t_0)^2}{2!} \\ &\leq 2MM_1 \left(\frac{|t - t_0|}{1!} + \frac{(t - t_0)^2}{2!} \right) \end{aligned}$$

with $M = D_0[x_0, \theta] + B$, $L_{fg} = \max\{L_f, L_g\}$, $L_{fgk} = \max\{L_g L_k, L_h\}$ and $M_1 = \max\{L_{fg}, L_{fgk}\}$. Now assume that $\{x_{n-1}\}$ is levelwise continuous on $|t - t_0| \leq \sigma$ and that

$$D_0[x_{n-1}(t), x_0] \leq 2MM_1 \left(\frac{|t - t_0|}{1!} + \frac{(t - t_0)^2}{2!} \right)$$

Let us prove there a fuzzy set-valued sequences $\{x_n(t)\}$ such that $D_0[x_n(t), x(t)] \rightarrow 0$ uniformly on $|t - t_0| \leq \sigma$ as $n \rightarrow \infty$. For $n = 2$, we have

$$\begin{aligned}
 D_0[x_2(t), x_1(t)] &= \\
 D_0 \left[\begin{aligned} &x_0 + \int_{t_0}^t f(s, x_1(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_1(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g\left(s, x_1(s), \int_{t_0}^s k(s, \tau, x_1(\tau), u(\tau))d\tau, u(s)\right)ds, \\ &x_0 + \int_{t_0}^t f(s, x_0(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_0(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g\left(s, x_0(s), \int_{t_0}^s k(s, \tau, x_0(\tau), u(\tau))d\tau, u(s)\right)ds \end{aligned} \right] \\
 &\leq \int_{t_0}^t D_0[f(s, x_1(s), u(s)), f(s, x_0(s), u(s))]ds + \int_{t_0}^t \int_{t_0}^s D_0[h(s, \tau, x_1(\tau), u(\tau)), h(s, \tau, x_0(\tau), u(\tau))]d\tau ds \\
 &\quad + \int_{t_0}^t D_0\left[g\left(s, x_1(s), \int_{t_0}^s k(s, \tau, x_1(\tau), u(\tau))d\tau, u(s)\right), g\left(s, x_0(s), \int_{t_0}^s k(s, \tau, x_0(\tau), u(\tau))d\tau, u(s)\right)\right]ds \\
 &\leq \int_{t_0}^t D_0[x_1(s), x_0]ds + \int_{t_0}^t \int_{t_0}^s D_0[x_1(\tau), x_0]d\tau ds + \int_{t_0}^t (D_0[x_1(s), x_0] + \int_{t_0}^s D_0[x_1(\tau), x_0]d\tau)ds \\
 &\leq 2 \int_{t_0}^t D_0[x_1(s), x_0]ds + 2 \int_{t_0}^t \int_{t_0}^s D_0[x_1(\tau), x_0]d\tau ds \\
 &\leq 4MM_1 \left(\frac{(t-t_0)^2}{2.1!} + 2 \frac{(t-t_0)^3}{3.2!} + \frac{(t-t_0)^4}{4.3.2!} \right) \\
 &\leq 4MM_1 \left(\frac{\sigma^2}{2.1!} + 2 \frac{\sigma^3}{3.2!} + \frac{\sigma^4}{4.3.2!} \right)
 \end{aligned}$$

Observe that for $1 \leq n = 1, 2, 3, \dots$, one has

$$\begin{aligned}
 x_n(t) &= x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau))d\tau ds \\
 &\quad + \int_{t_0}^t g\left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s)\right)ds \\
 x_{n+1}(t) &= x_0 + \int_{t_0}^t f(s, x_n(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_n(\tau), u(\tau))d\tau ds \\
 &\quad + \int_{t_0}^t g\left(s, x_n(s), \int_{t_0}^s k(s, \tau, x_n(\tau), u(\tau))d\tau, u(s)\right)ds \\
 &\dots
 \end{aligned}$$

We have

$$\begin{aligned}
D_0[x_{n+1}(t), x_n(t)] &= \\
D_0 \left[\begin{aligned} &x_0 + \int_{t_0}^t f(s, x_n(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_n(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x_n(s), \int_{t_0}^s k(s, \tau, x_n(\tau), u(\tau))d\tau, u(s) \right) ds, \\ &x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s) \right) ds \end{aligned} \right] \\
&\leq \int_{t_0}^t D_0[f(s, x_n(s), u(s)), f(s, x_{n-1}(s), u(s))]ds + \int_{t_0}^t \int_{t_0}^s D_0[h(s, \tau, x_n(\tau), u(\tau)), h(s, \tau, x_{n-1}(\tau), u(\tau))]d\tau ds \\
&\quad + \int_{t_0}^t D_0 \left[g \left(s, x_n(s), \int_{t_0}^s k(s, \tau, x_n(\tau), u(\tau))d\tau, u(s) \right), g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s) \right) \right] ds \\
&\leq 2^{n+1} MM_1 \int_{t_0}^t D_0[x_n(s), x_{n-1}(s)]ds + 2^{n+1} MM_1 \int_{t_0}^t \int_{t_0}^s D_0[x_n(s), x_{n-1}(s)]d\tau ds
\end{aligned}$$

and consequently,

$$D_0[x_{n+1}(t), x_n(t)] \leq 2^{n+1} MM_1 \left(\sum_{i=1}^n \frac{1}{i!} \right) \left[\frac{\sigma^{n+1}}{n!} + 2 \frac{\sigma^{n+1}}{(n+1)} + \frac{\sigma^{n+2}}{(n+2)!} \right] \quad (3.4)$$

The series $\sum_{i=1}^{\infty} \frac{1}{i!}$ is convergent. From (3.4) it follows that $D_0[x_{n+1}(t), x_n(t)] \rightarrow 0$ uniformly on $|t - t_0| \leq \sigma$ as $n \rightarrow \infty$. Now, we want to show that limit process is solution to (3.1). Indeed, we show that $\{x_n(t)\}$ satisfies (3.1), we have

$$\begin{aligned}
&D_0 \left[\begin{aligned} &x(t), x_0 + \int_{t_0}^t f(s, x(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau))d\tau \right) ds \end{aligned} \right] \\
&\leq D_0[x(t), x_n(t)] + D_0 \left[\begin{aligned} &x_n(t), x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s) \right) ds \end{aligned} \right] \\
&\quad + D_0 \left[\begin{aligned} &\int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s) \right) ds, \\ &\int_{t_0}^t f(s, x(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau))d\tau \right) ds \end{aligned} \right]
\end{aligned}$$

We see $D_0[x_n(t), x(t)] \rightarrow 0$ uniformly converges to zero and

$$D_0 \left[\begin{aligned} &x_n(t), x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau))d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau))d\tau, u(s) \right) ds \end{aligned} \right] \text{ is equal to zero } 0.$$

We consider

$$\begin{aligned}
 & D_0 \left[\begin{aligned} & \int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau)) d\tau ds \\ & + \int_{t_0}^t g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau)) d\tau, u(s) \right) ds, \\ & \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau)) d\tau ds \\ & + \int_{t_0}^t g \left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau)) d\tau \right) ds \end{aligned} \right] \\
 & \leq D_0 \left[\int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds, \int_{t_0}^t f(s, x(s), u(s)) ds \right] \\
 & + D_0 \left[\int_{t_0}^t \int_{t_0}^s h(s, \tau, x_{n-1}(\tau), u(\tau)) d\tau ds, \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau)) d\tau ds \right] \\
 & + D_0 \left[\int_{t_0}^t g \left(s, x_{n-1}(s), \int_{t_0}^s k(s, \tau, x_{n-1}(\tau), u(\tau)) d\tau, u(s) \right) ds, \int_{t_0}^t g \left(s, x(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau)) d\tau, u(s) \right) ds \right] \\
 & \leq 2L_{f\bar{g}} \frac{|t-t_0|}{1!} D_0[x_n(t), x(t)] + 2L_{f\bar{g}k} \frac{(t-t_0)^2}{2!} D_0[x_n(t), x(t)] \\
 & \leq 2M_1 \left(\frac{\sigma}{1!} + \frac{\sigma^2}{2!} \right) D_0[x_n(t), x(t)] \rightarrow 0
 \end{aligned}$$

Therefore,

$$D_0 \left[\begin{aligned} & x(t), x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau)) d\tau ds \\ & + \int_{t_0}^t g \left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau)) d\tau \right) ds \end{aligned} \right] = 0$$

with any fixed $t \in T$. Hence the existence of solutions is proved.

If we have two solutions $x(t), y(t)$ of (3.1) with condition $x_0=y_0$. We have to prove that $D_0[x(t), y(t)] = 0$. We have

$$\begin{aligned}
 x(t) &= x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau)) d\tau ds \\
 &+ \int_{t_0}^t g \left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau)) d\tau \right) ds \\
 y(t) &= y_0 + \int_{t_0}^t f(s, y(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, y(\tau), u(\tau)) d\tau ds \\
 &+ \int_{t_0}^t g \left(s, y(s), u(s), \int_{t_0}^s k(s, \tau, y(\tau), u(\tau)) d\tau \right) ds
 \end{aligned}$$

By using assumption above, we infer that

$$\begin{aligned}
D_0[x(t), y(t)] &= D_0 \left[\begin{aligned} &x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, x(\tau), u(\tau)) d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, x(s), u(s), \int_{t_0}^s k(s, \tau, x(\tau), u(\tau)) d\tau \right) ds, \\ &y_0 + \int_{t_0}^t f(s, y(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s h(s, \tau, y(\tau), u(\tau)) d\tau ds \\ &\quad + \int_{t_0}^t g \left(s, y(s), u(s), \int_{t_0}^s k(s, \tau, y(\tau), u(\tau)) d\tau \right) ds \end{aligned} \right] \\
&\leq 2D_0[x_0, y_0] + 2L_f \int_{t_0}^t D_0[x(s), y(s)] ds + 2L_h \int_{t_0}^t D_0[x(s), y(s)] ds \\
&\quad + 2L_g \left(\int_{t_0}^t \left(D_0[x(s), y(s)] + L_k \int_{t_0}^s D_0[x(\tau), y(\tau)] d\tau \right) ds \right) \\
&\leq 2D_0[x_0, y_0] + 4(L_{fgh} + \sigma L_{gk}) \int_{t_0}^t D_0[x(s), y(s)] ds
\end{aligned}$$

Using Gronwall's inequality (see [2]), we get

$$D_0[x(t), y(t)] \leq 2D_0[x_0, y_0] \exp\{4(L_{fgh} + \sigma L_{gk})t\}$$

By $x_0=y_0$ then $D_0[x_0, y_0]=0$. It implies $D_0[x(t), y(t)] = 0$ Hence, the unique of solutions is proves. $\square$

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