Reliability for Lindley Distribution with an Outlier

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Abstract. In this paper, we consider the problem of estimating $R = P(Y < X)$, when $Y$ has Lindley distribution with parameter $a$ and $X$ has Lindley distribution with presence of one outlier with parameters $b$ and $c$, such that $X$ and $Y$ are independent. The maximum likelihood estimator of $R$ is derived and some results of simulation studies are presented.

1 Introduction

In reliability context inferences about $R = P(Y < X)$, when $X$ and $Y$ are independently distributed, are a subject of interest. For example in mechanical reliability of a system if $X$ is the strength of a component which is subject to stress $Y$, then $R$ is a measure of system performance. The system fails, if at any time the applied stress is greater than its strength. Stress-strength reliability has been discussed in Kapur and Lamberson (1977). Sathe and Dixit (2001) have done estimation of $R$ in the negative binomial distribution. Baklizi and Dayyeh (2003) have done shrinkage estimation of $R$ in exponential case, and recently Deiri (2011) has done estimation of $R$ with presence of two outliers in the exponential and gamma cases, respectively. Jafari (2011) has obtained the moment, maximum likelihood and mixture estimators of $R$ in Rayleigh distribution in the presence of one outlier and Jabbari, Abolhasani and Fathipour (2012) have discussed the estimation of $R$ in the six parameter generalized Burr XII distribution with transformation method.

In this paper, we obtain the maximum likelihood estimator of $R$ for Lindley distribution with presence of one outlier generated from the same distribution.

The probability density function of the Lindley distribution with parameter of $a$ is given by:

$$f(y; a) = \frac{a^2}{1+a} (1 + y) e^{-ay}, x > 0, a > 0.$$ 

In this paper we assume that the random variables $(Y_1, Y_2, ..., Y_m)$ have Lindley distribution with parameter $a$ and the random variables $(X_1, X_2, ..., X_n)$ are such that one of them is from Lindley distribution with parameter $c$ and the remaining $(n-1)$ random variables are from Lindley distribution with parameter $b$.

The paper is organized as follows:

In section 2, we obtain the joint distribution of $(X_1, X_2, ..., X_n)$ in the presence of one outlier. Section 3 and section 4 discuss the method of maximum likelihood estimators of parameters and the MLE of $R$ respectively. In section 5 simulation studies are presented and the results are summarized in section 6.

2 Joint distribution of $X_1, X_2, ..., X_n$ in presence of an outlier

Assume $(X_1, X_2, ..., X_n)$ are such that one of them is distributed with p.d.f $g(x, c)$ as Lindley$(c)$ and remaining $(n-1)$ of them are distributed with p.d.f $f(x, b)$ as Lindley$(b)$. The joint distribution of $(X_1, X_2, ..., X_n)$ can be expressed as

$$f(x_1, x_2, ..., x_n; b, c) = \frac{(n-1)!}{n!} \prod_{i=1}^{n} f(x_i, b) \sum_{i=1}^{n} g(x_i; c)$$
\[
\frac{(n-1)!}{n!} \frac{b^{2n}}{(1+b)^n} \prod_{i=1}^{n} (1+x_i) e^{-b \sum_{i=1}^{n} x_i} \sum_{i=1}^{n} \frac{c^2}{1+c} \frac{(1+x_i)e^{-cx_i}}{b^2 + 1}(1+x_i)e^{-bx_i} \\
= \frac{(n-1)!}{n!} \frac{b^{2n-2}}{(1+b)^{n-1}} \frac{c^2}{1+c} \prod_{i=1}^{n} (1+x_i) e^{-b \sum_{i=1}^{n} x_i} \sum_{i=1}^{n} (1+x_i)e^{x_i(b-c)}
\]

(1)

See Dixit (1989), Dixit and Nasiri (2001), and Nasiri and Pazira (2009). From (1), the marginal distribution of \(X\) is

\[
f(x; b, c) = \frac{c^2}{n+1+c} (1+x) e^{-cx} + \frac{n-1}{n} \frac{b^2}{1+b} (1+x) e^{-bx}; x, b, c > 0
\]

(2)

We will use (2) to obtain \(R = \Pr(Y < X)\)

### 3 Maximum likelihood estimators of parameters

Let \(Y_1, Y_2, \ldots, Y_m\) be a random sample for \(Y\) with pdf,

\[
f(y; a) = \frac{a^2}{1+a} (1+y)e^{-ay}; x, a > 0
\]

the log likelihood function is given by

\[
L(a) = 2mlna - mln(1+a) + \sum_{i=1}^{m} \ln(1+y_i) - a \sum_{i=1}^{m} y_i
\]

Taking the derivative with respect to \(a\) and equating to 0, we obtain the MLE of \(a\) as

\[
\hat{a} = \frac{m-\sum_{i=1}^{m} y_i \pm \sqrt{(\sum_{i=1}^{m} y_i - m)^2 + 8m \sum_{i=1}^{m} y_i}}{2 \sum_{i=1}^{m} y_i}
\]

(3)

Now let \(X_1, X_2, \ldots, X_n\) be a random sample for \(X\) with presence of one outlier with pdf,

\[
f(x; b, c) = \frac{c^2}{n+1+c} (1+x) e^{-cx} + \frac{n-1}{n} \frac{b^2}{1+b} (1+x) e^{-bx}; x, b, c > 0.
\]

From (1), the log likelihood function is given by

\[
L(b, c) = \ln \left(\frac{(n-1)!}{n!}\right) + (2n-2)\ln b - (n-1) \ln (1+b) + 2ln c - \ln (1+c)
\]

\[+ \sum_{i=1}^{m} \ln (1+x_i) - b \sum_{i=1}^{m} x_i + \ln \left(\sum_{i=1}^{m} e^{x_i(b-c)}\right)
\]

Taking the derivatives with respect to \(b\) and \(c\) and equating the results to 0, we obtain the normal equations as

\[
\frac{\partial L(b, c)}{\partial b} = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}
\]

(4)

\[
\frac{\partial L(b, c)}{\partial c} = \frac{2}{c} - \frac{1}{1+c} - \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}
\]

(5)

There is no closed-form solution to this system of equations, so we will solve for \(\hat{b}\) and \(\hat{c}\) iteratively, using the Newton-Raphson method. In our case we will estimate \(\hat{b} = (\hat{b}, \hat{c})\) iteratively:

\[
\hat{b}_{i+1} = \hat{b}_i - G^{-1}g
\]

(6)

where \(g\) is the vector of normal equations for which we want

\[
g = [g_1\ g_2]
\]

With

\[
g_1 = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}
\]

\[
g_2 = \frac{2}{c} - \frac{1}{1+c} - \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}
\]
and $G$ is the matrix of second derivatives

$$G = \begin{bmatrix}
    \frac{dg_1}{db} & \frac{dg_1}{dc} \\
    \frac{db_2}{db} & \frac{dg_2}{dc}
\end{bmatrix}$$

where

$$\frac{dg_1}{db} = \frac{2 - 2n}{b^2} + \frac{n - 1}{(1 + b)^2} + \sum_{i=1}^{n} x_i^2 e^{x_i(b-c)} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_1}{dc} = -\frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} + \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_1}{db} = -2 + \frac{1}{c^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}^2$$

The Newton-Raphson algorithm converges, as our estimate of $b$ and $c$ change by less than a tolerated amount with each successive iteration, to $\hat{b}$ and $\hat{c}$.

## 4 The maximum likelihood estimator of $R$

Let $Y \sim \text{lindley}(a)$ with pdf $h(y; a)$ and $X$ be distributed with pdf $f(x; b, c)$ given in (2). The parameter $R$ we want to estimate is

$$R = P(Y < X) = \int_{0}^{\infty} \int_{0}^{x} h(y; a) f(x; b, c) dy dx$$

$$= \frac{1}{b} \int_{0}^{\infty} \int_{0}^{x} \frac{a^2}{1 + a} (1 + y)e^{-ay} \frac{e^y}{1 + c} (1 + x)e^{-cx} dy dx$$

$$+ \frac{n - 1}{n} \int_{0}^{\infty} \int_{0}^{x} \frac{a^2}{1 + a} (1 + y)e^{-ay} \frac{b^2}{1 + b} (1 + x)e^{-bx} dy dx$$

$$= \frac{1}{n} \left[ \frac{c^4 (c+1) + (1+c)(3+c)a + (3+2c)a^2 + a^3}{(1+c)(1+a)(c+a)^3} \right]$$

Thus, by invariant property for MLEs, the MLE of $R$ is

$$\hat{R} = \frac{1}{2} \frac{\hat{c}^2 (\hat{c} (1 + \hat{c}) + (1 + \hat{c})(3 + \hat{c})\hat{a} + (3 + 2\hat{c})\hat{a}^2 + \hat{a}^3)}{(1 + \hat{c})(1 + \hat{a})(\hat{c} + \hat{a})^3}$$

$$+ \frac{n - 1}{n} \frac{\hat{b}^2 (\hat{b} (1 + \hat{b}) + (1 + \hat{b})(3 + \hat{b})\hat{a} + (3 + 2\hat{b})\hat{a}^2 + \hat{a}^3)}{(1 + \hat{b})(1 + \hat{a})(\hat{b} + \hat{a})}$$

where $\hat{a}$, $\hat{b}$, and $\hat{c}$ can be obtained from (3) and (6).

## 5 Simulation Study

In this section we generate random numbers from lindley distribution (with and without outlier) with accept-reject method by Maple software. Using these samples and the Newton-Raphson method we obtain the maximum likelihood estimators of parameters $a, b$ and $c$. Then we use them to calculate the MLE of $R$. The values of biases and MSEs of these estimates are presented in table 1, for $a=1$, $b=2$ and $c=1.6,1.7,1.8,1.9,2.1,2.2,2.3,2.4,2.5,3,4$ and in table 2, for $a=1$, $b=2$ , and the same values of $c$. All the results are based on 100 replications.

## 6 Conclusion

According to the results of simulation, when the value of parameters $b$ and $c$ are close to each other, the biases and MSEs are often around zero and when the difference between $b$ and $c$ is greater than 1, the biases and MSEs increase.
Table 1: Biases and (MSE)s of the MLEs of $R$, for $a=1$, $b=2$, and different values of $c$

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References


