Reliability for Lindley Distribution with an Outlier
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Abstract. In this paper, we consider the problem of estimating \( R = P(Y < X) \), when \( Y \) has lindley distribution with parameter \( a \) and \( x \) has lindley distribution with presence of one outlier with parameters \( b \) and \( c \), such that \( X \) and \( Y \) are independent. The maximum likelihood estimator of \( R \) is derived and some results of simulation studies are presented.

1 Introduction

In reliability context inferences about \( R = P(Y < X) \), when \( X \) and \( Y \) are independently distributed, are a subject of interest. For example in mechanical reliability of a system if \( X \) is the strength of a component which is subject to stress \( Y \), then \( R \) is a measure of system performance. The system fails, if at any time the applied stress is greater than its strength. Stress-strength reliability has been discussed in Kapur and Lamberson (1977). Sathe and Dixit (2001) have done estimation of \( R \) in the negative binomial distribution. Baklizi and Dayyeh (2003) have done shrinkage estimation of \( R \) in exponential case, and recently Deiri (2011) has done estimation of \( R \) with presence of two outliers in the exponential and gamma cases, respectively. Jafari (2011) has obtained the moment, maximum likelihood and mixture estimators of \( R \) in Rayleigh distribution in the presence of one outlier and Jabbari, Abolhasani and Fathipour (2012) have discussed the estimation of \( R \) in the six parameter generalized Burr XII distribution with transformation method.

In this paper, we obtain the maximum likelihood estimator of \( R \) for lindley distribution with presence of one outlier generated from the same distribution.

The probability density function of the lindley distribution with parameter \( a \) is given by:
\[
f(y; a) = \frac{a^2}{1+a}(1+y)e^{-ay}, x > 0, a > 0.
\]
In this paper we assume that the random variables \( (Y_1, Y_2,...,Y_m) \) have lindley distribution with parameter \( a \) and the random variables \( (X_1, X_2,...,X_n) \) are such that one of them is from lindley distribution with parameter \( c \) and the remaining \((n-1)\) random variables are from lindley distribution with parameter \( b \).

The paper is organized as follows:
In section 2, we obtain the joint distribution of \( (X_1, X_2,...,X_n) \) in the presence of one outlier. Section 3 and section 4 discusses the method of maximum likelihood estimators of parameters and the MLE of \( R \) respectively. In section 5 simulation studies are presented and the results are summarized in section 6.

2 Joint distribution of \( X_1, X_2,...,X_n \) in presence of an outlier

Assume \( (X_1, X_2,...,X_n) \) are such that one of them is distributed with p.d.f \( g(x,c) \) as lindley\((c)\) and remaining \((n-1)\) of them are distributed with p.d.f \( f(x,b) \) as lindley\((b)\). The joint distribution of \( (X_1, X_2,...,X_n) \) can be expressed as
\[
f(x_1, x_2,...,x_n; b, c) = \frac{(n-1)!}{n!} \prod_{i=1}^{n} f(x_i, b) \sum_{i=1}^{n} g(x_i; c) f(x_i;b)
\]
\[
\frac{(n-1)!}{n!} \frac{b^{2n}}{(1+b)^n} \prod_{i=1}^{n} (1+x_i)e^{-b\Sigma_{i=1}^{n}x_i} \sum_{i=1}^{n} \frac{c^2}{b^2} \frac{(1+x_i)}{1+b}(1+x_i)e^{-bx_i} = \frac{(n-1)!}{n!} \frac{b^{2n-2}}{(1+b)^{n-1}+c} \prod_{i=1}^{n} (1+x_i)e^{-b\Sigma_{i=1}^{n}x_i} \sum_{i=1}^{n} (1+x_i)e^{-x_i(b-c)}
\]

(1)

See Dixit (1989), Dixit and Nasiri (2001), and Nasiri and Pazira (2009). From (1), the marginal distribution of \(X\) is

\[ f(x; b, c) = \frac{1}{n+1+c}(1+x)e^{-cx} + \frac{n-1}{n+1} \frac{b^2}{1+b}(1+x)e^{-bx}; x, b, c > 0 \]

(2)

We will use (2) to obtain \(R=P(Y<X)\)

3 Maximum likelihood estimators of parameters

Let \((Y_1, Y_2, \ldots, Y_m)\) be a random sample for \(Y\) with pdf,

\[ f(y; a) = \frac{a^2}{1+a} (1+y)e^{-ay}, x, a > 0 \]

the log likelihood function is given by

\[ L(a) = 2m\ln a - m\ln(1+a) + \sum_{i=1}^{m} \ln(1+y_i) - a \sum_{i=1}^{m} y_i \]

Taking the derivative with respect to \(a\) and equating to \(0\), we obtain the MLE of \(a\) as

\[ \hat{a} = \frac{m-\sum_{i=1}^{m} y_i \pm \sqrt{(\sum_{i=1}^{m} y_i - m)^2 + 8m \sum_{i=1}^{m} y_i}}{2 \sum_{i=1}^{m} y_i} \]

(3)

Now let \(X_1, X_2, \ldots, X_n\) be a random sample for \(X\) with presence of one outlier with pdf,

\[ f(x; b, c) = \frac{1}{n+1+c} \frac{c^2}{1+x}e^{-cx} + \frac{n-1}{n+1} \frac{b^2}{1+b} \frac{1}{1+x}e^{-bx}; x, b, c > 0. \]

From (1), the log likelihood function is given by

\[ L(b, c) = \ln \left( \frac{(n-1)!}{n!} \right) + (2n-2)\ln d - (n-1) \ln(1+b) + 2\ln c - \ln(1+c) + \sum_{i=1}^{n} \ln(1+x_i) - b \sum_{i=1}^{n} x_i + \ln \left( \sum_{i=1}^{n} e^{-x_i(b-c)} \right) \]

Taking the derivatives with respect to \(b\) and \(c\) and equating the results to \(0\), we obtain the normal equations as

\[ \frac{\partial L(b, c)}{\partial b} = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-x_i(b-c)} \]

(4)

\[ \frac{\partial L(b, c)}{\partial c} = \frac{2}{c} - \frac{1}{1+c} - \sum_{i=1}^{n} x_i e^{-x_i(b-c)} \]

(5)

There is no closed-form solution to this system of equations, so we will solve for \(\hat{b}\) and \(\hat{c}\) iteratively, using the Newton-Raphson method. In our case we will estimate \(\hat{b} = (\hat{b}, \hat{c})\) iteratively:

\[ \hat{b}_{i+1} = \hat{b}_i - G^{-1}g \]

(6)

where \(g\) is the vector of normal equations for which we want

\[ g = [g_1 \ g_2] \]

With

\[ g_1 = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-x_i(b-c)} \]

\[ g_2 = \frac{2}{c} - \frac{1}{1+c} - \sum_{i=1}^{n} x_i e^{-x_i(b-c)} \]

\[ \sum_{i=1}^{n} e^{-x_i(b-c)} \]

\[ \sum_{i=1}^{n} x_i e^{-x_i(b-c)} \]
and $G$ is the matrix of second derivatives

$$G = \begin{bmatrix}
\frac{dg_1}{db} & \frac{dg_1}{dc} \\
\frac{dg_2}{db} & \frac{dg_2}{dc}
\end{bmatrix}
$$

where

$$\frac{dg_1}{db} = \frac{2 - 2n}{b^2} + \frac{n - 1}{(1 + b)^2} + \sum_{i=1}^{n} x_i^2 e^{x_i(b-c)} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_1}{dc} = -\frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} + \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_2}{db} = -2 + \frac{1}{c^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

The Newton-Raphson algorithm converges, as our estimate of $b$ and $c$ change by less than a tolerated amount with each successive iteration, to $\hat{b}$ and $\hat{c}$.

4 The maximum likelihood estimator of $R$

Let $Y \sim \text{lindley}(a)$ with pdf $h(y; a)$ and $X$ be distributed with pdf $f(x; b, c)$ given in (2). The parameter $R$ we want to estimate is

$$R = P(Y < X) = \int_0^\infty \int_0^X h(y; a) f(x; b, c) dy dx$$

$$= \frac{1}{b} \int_0^\infty \int_0^x a^2 \frac{1}{1 + a} (1 + y) e^{-ay} \frac{c^2}{1 + c} (1 + x) e^{-cx} dy dx$$

$$+ \frac{n - 1}{n} \int_0^\infty \int_0^x a^2 \frac{1}{1 + a} (1 + y) e^{-ay} \frac{b^2}{1 + b} (1 + x) e^{-bx} dy dx$$

$$= \frac{1}{n} \left[ \frac{c^2(c(1 + c) + (1 + c)(3 + c)a + (3 + 2c)a^2 + a^3)}{(1 + c)(1 + a)(c + a)^3} \right]$$

$$+ \frac{n - 1}{n} \left[ \frac{b^2(b(1 + b) + (1 + b)(3 + b)a + (3 + 2b)a^2 + a^3)}{(1 + b)(1 + a)(b + a)^3} \right]$$

Thus, by invariant property for MLEs, the MLE of $R$ is

$$\hat{R} = \frac{1}{2} \left[ \frac{\hat{c}^2(\hat{c}^2(1 + \hat{c}) + (1 + \hat{c})(3 + \hat{c})\hat{a} + (3 + 2\hat{c})\hat{a}^2 + \hat{a}^3)}{(1 + \hat{c})(1 + \hat{a})(\hat{c} + \hat{a})^3} \right]$$

$$+ \frac{n - 1}{n} \left[ \frac{\hat{b}^2(\hat{b}(1 + \hat{b}) + (1 + \hat{b})(3 + \hat{b})\hat{a} + (3 + 2\hat{b})\hat{a}^2 + \hat{a}^3)}{(1 + \hat{b})(1 + \hat{a})(\hat{b} + \hat{a})} \right]$$

where $\hat{a}, \hat{b},$ and $\hat{c}$ can be obtained from (3) and (6).

5 Simulation Study

In this section we generate random numbers from lindley distribution (with and without outlier) with accept-reject method by Maple software. Using these samples and the Newton-Raphson method we obtain the maximum likelihood estimators of parameters $a, b$ and $c$. Then we use them to calculate the MLE of $R$. The values of biases and MSEs of these estimates are presented in table 1, for $a=1, b=2$ and $c=1.6,1.7,1.8,1.9,2.1,2.2,3.2,4.2,5.3,4$ and in table 2, for $a=1, b=2$, and the same values of $c$. All the results are based on 100 replications.

6 Conclusion

According to the results of simulation, when the value of parameters $b$ and $c$ are close to each other, the biases and MSEs are often around zero and when the difference between $b$ and $c$ is greater than 1, the biases and MSEs increase.
Table 1: Biases and (MSE)s of the MLEs of $R$, for $a=1$, $b=2$, and different values of $c$

<table>
<thead>
<tr>
<th>$(m,n)$</th>
<th>$c$</th>
<th>Biases</th>
<th>(MSE)</th>
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References


