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Reliability for Lindley Distribution with an Outlier

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Abstract. In this paper, we consider the problem of estimating $R=P(Y<X)$, when Y has lindley distribution with parameter a and x has lindley distribution with presence of one outlier with parameters b and c, such that X and Y are independent. The maximum likelihood estimator of R is derived and some results of simulation studies are presented.

1 Introduction

In reliability context inferences about $R = P(Y<X)$, when X and Y are independently distributed, are a subject of interest.For example in mechanical reliability of a system if X is the strength of a component which is subject to stress Y , then R is a measure of system performance. The system fails, if at any time the applied stress is greater than its strength. Stress-strength reliability has been discussed in Kapur and Lamberson (1977). Sathe and Dixit (2001) have done estimation of R in the negative binomial distribution. Baklizi and Dayyeh (2003) have done shrinkage estimation of R in exponential case, and recently Deiri (2011) has done estimation of R with presence of two outliers in the exponential and gamma cases, respectively.Jafari (2011) has obtained the moment, maximum likelihood and mixture estimators of R in Rayleigh distribution in the presence of one outlier and Jabbari, Abolhasani and Fathipour (2012) have discussed the estimation of R in the six parameter generalized Burr XII distribution with transformation method.
In this paper, we obtain the maximum likelihood estimator of R for lindley distribution with presence of one outlier generated from the same distribution.

The probability density function of the lindley distribution with parameter of a is given by:
$f(y;a) = \frac{a^2}{1+a}(1+y)e^{-ay}, x > 0, a > 0.$
In this paper we assume that the random variables $(Y_1,Y_2,\ldots,Y_m)$ have lindley distribution with parameter a and the random variables $(X_1,X_2,\ldots,X_n)$ are such that one of them is from lindley distribution with parameter c and the remaining $(n-1)$ random variables are from lindley distribution with parameter b

The paper is organized as follows:
In section 2, we obtain the joint distribution of $(X_1,X_2,\ldots,X_n)$ in the presence of one outlier. Section 3 and section 4 discusses the method of maximum likelihood estimators of parameters and the MLE of R respectively. In section 5 simulation studies are presented and the results are summarized in section 6.

2 Joint distribution of *X1,X2,...,Xn* in presence of an outlier

Assume $(X_1,X_2,\ldots,X_n)$ are such that one of them is distributed with p.d.f $g(x,c)$ as lindley(c) and remaining $(n-1)$ of them are distributed with p.d.f $f(x,b)$ as lindley(b). The joint distribution of $(X_1,X_2,\ldots,X_n)$ can be expressed as

$$f(x_1,x_2,\ldots,x_n;b,c) = \frac{(n-1)!}{n!}\prod_{i=1}^{n} f(x,b)\sum_{i=1}^{n}\frac{g(x_i;c)}{f(x_i b)}$$

$$= \frac{(n-1)!}{n!}\frac{b^{2n}}{(1+b)^n}\prod_{i=1}^{n}(1+x_i)e^{-b\sum_{i=1}^{n}x_i}\sum_{i=1}^{n}\frac{\frac{c^2}{1+c}(1+x_i)e^{-cx_i}}{\frac{b^2}{1+b}(1+x_i)e^{-bx_i}}$$

$$= \frac{(n-1)!}{n!}\frac{b^{2n-2}}{(1+b)^{n-1}}\frac{c^2}{1+c}\prod_{i=1}^{n}(1+x_i)e^{-b\sum_{i=1}^{n}x_1}\sum_{i=1}^{n}(1+x_i)e^{x_i(b-c)} \quad (1)$$

See Dixit (1989), Dixit and Nasiri (2001), and Nasiri and Pazira (2009). From (1) , the marginal distribution of *X* is

$$f(x;b,c) = \frac{1}{n}\frac{c^2}{1+c}(1+x)e^{-cx} + \frac{n-1}{n}\frac{b^2}{1+b}(1+x)e^{-bx}; x,b,c > 0 \quad (2)$$

We will use (2) to obtain $R=P(Y<X)$

3 Maximum likelihood estimators of parameters

Let $(Y_1,Y_2,\ldots,Y_m)$ be a random sample for *Y* with pdf,

$$f(y;a) = \frac{a^2}{1+a}(1+y)e^{-ay}, x, a > 0$$

the log likelihood function is given by

$$L(a) = 2mlna - mln(1+a) + \sum_{i=1}^{m}\ln(1+y_i) - a\sum_{i=1}^{m}y_i$$

Taking the derivative with respect to *a* and equating to *0*, we obtain the MLE of *a* as

$$\hat{a} = \frac{m-\sum_{i=1}^{m}y_i \pm \sqrt{(\sum_{i=1}^{m}y_i-m)^2+8m\sum_{i=1}^{m}y_i}}{2\sum_{i=1}^{m}y_i} \quad (3)$$

Now let *X1,X2,...,Xn* be a random sample for *X* with presence of one outlier with pdf,

$$f(x;b,c) = \frac{1}{n}\frac{c^2}{1+c}(1+x)e^{-cx} + \frac{n-1}{n}\frac{b^2}{1+b}(1+x)e^{-bx}; x,b,c > 0.$$

From (1), the log likelihood function is given by

$$L(b,c) = \ln\left(\frac{(n-1)!}{n!}\right) + (2n-2)lnd - (n-1)\ln(1+b) + 2lnc - \ln(1+c)$$
$$+\sum_{i=1}^{n}\ln(1+x_i) - b\sum_{i=1}^{n}x_i + \ln(\sum_{i=1}^{n}e^{x_i(b-c)}$$

Taking the derivatives with respect to *b* and *c* and equating the results to *0*, we obtain the normal equations as

$$\frac{\partial L(b,c)}{\partial b} = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n}x_i + \frac{\sum_{i=1}^{n}x_ie^{x_i(b-c)}}{\sum_{i=1}^{n}e^{x_i(b-c)}} \quad (4)$$

$$\frac{\partial L(b,c)}{\partial c} = \frac{2}{c} - \frac{1}{1+c} - \frac{\sum_{i=1}^{n}x_ie^{x_i(b-c)}}{\sum_{i=1}^{n}e^{x_i(b-c)}} \quad (5)$$

There is no closed-form solution to this system of equations, so we will solve for $\hat{b}$ and $\hat{c}$ iteratively, using the Newton-Raphson method. In our case we will estimate $\hat{\beta} = (\hat{b},\hat{c})$ iteratively:

$$\hat{\beta}_{i+1} = \hat{\beta}_i - G^{-1}g \quad (6)$$

where g is the vector of normal equations for which we want

$$g = [g_1 g_2]$$

With

$$g_1 = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n}x_i + \frac{\sum_{i=1}^{n}x_ie^{x_i(b-c)}}{\sum_{i=1}^{n}e^{x_i(b-c)}}$$

$$g_2 = \frac{2}{c} - \frac{1}{1+c} - \frac{\sum_{i=1}^{n}x_ie^{x_i(b-c)}}{\sum_{i=1}^{n}e^{x_i(b-c)}}$$

and G is the matrix of second derivatives

$$G = \begin{bmatrix} \frac{dg_1}{db} & \frac{dg_1}{dc} \\ \frac{db_2}{db} & \frac{dg_2}{dc} \end{bmatrix}$$

where

$$\frac{dg_1}{db} = \frac{2-2n}{b^2} + \frac{n-1}{(1+b)^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_1}{db} = -\frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} + \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

$$\frac{dg_1}{db} = \frac{-2}{c^2} + \frac{1}{(1+c)^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \left(\frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}}\right)^2$$

The Newton-Raphson algorithm converges, as our estimate of b and c change by less than a tolerated amount with each successive iteration, to $\hat{b}$ and $\hat{c}$.

4 The maximum likelihood estimator of *R*

Let $Y \sim lindley(a)$ with pdf $h(y;a)$ and X be distributed with pdf $f(x;b,c)$ given in (2). The parameter R we want to estimate is

$$R = P(Y < X) = \int_0^\infty \int_0^x h(y;a) f(x;b,c)\,dydx$$

$$= \frac{1}{b} \int_0^\infty \int_0^x \frac{a^2}{1+a}(1+y)e^{-ay} \frac{c^2}{1+c}(1+x)e^{-cx} dydx$$

$$+ \frac{n-1}{n} \int_0^\infty \int_0^x \frac{a^2}{1+a}(1+y)e^{-ay} \frac{b^2}{1+b}(1+x)e^{-bx} dydx$$

$$= \frac{1}{n}\left[\frac{c^2(c(1+c) + (1+c)(3+c)a + (3+2c)a^{2+}a^3}{(1+c)(1+a)(c+a)^3}\right]$$

$$+ \frac{n-1}{n}\left[\frac{b^2(b(1+b) + (1+b)(3+b)a + (3+2b)a^2 + a^3}{(1+b)(1+a)(b+a)^3}\right] \tag{7}$$

Thus, by invariant property for MLEs, the MLE of R is

$$\hat{R} = \frac{1}{2}\left[\frac{\hat{c}^2(\hat{c}(1+\hat{c}) + (1+\hat{c})(3+\hat{c})\hat{a} + (3+2\hat{c})\hat{a}^2 + \hat{a}^3)}{(1+\hat{c})(1+\hat{a})(\hat{c}+\hat{a})^3}\right]$$

$$+ \frac{n-1}{n}\left[\frac{\hat{b}^2(\hat{b}(1+\hat{b}) + (1+\hat{b})(3+\hat{b})\hat{a} + (3+2\hat{b})\hat{a}^2 + \hat{a}^3}{(1+\hat{b})(1+\hat{a})(\hat{b}+\hat{a})}\right]$$

where $\hat{a}, \hat{b}$, and $\hat{c}$can be obtained from (3) and (6).

5 Simulation Study

In this section we generate random numbers from lindley distribution (with and without outlier) with accept-reject method by Maple software. Using these samples and the Newton-Raphson method we obtain the maximum likelihood estimators of parameters *a,b and c*. Then we use them to calculate the MLE of *R*. The values of biases and MSEs of these estimates are presented in table1, for *a=1*, *b=2* and c=1.6,1.7,1.8,1.9,2.1,2.2,2.3,2.4,2.5,3,4 and in table 2, for *a=1*, *b=2* , and the same values of *c*. All the results are based on *100* replications.

6 Conclusion

Acording to the results of simulation, when the value of parameters b and c are close to each other, the biases and MSEs are often around zero and when the difference between b and c is greater than 1, the biases and MSEs increase.

Table1: Biases and (MSE)s of the MLEs of R, for $a=1$, $b=2$, and different values of c

(m,n)→ c↓	(30,30)	(30,20)	(30,10)	(20,30)	(20,20)	(20,10)	(10,30)	(10,20)	(10,10)
1.6	0.00160 (0.00344)	0.00188 (0.00352)	-0.00469 (0.00538)	-0.00303 (0.00386)	0.00715 (0.00517)	-0.00410 (0.00468)	-0.00496 (0.00942)	-0.02129 (0.08441)	0.00736 (0.01113)
1.7	-0.00195 (0.00263)	0.00293 (0.00377)	0.00646 (0.00733)	-0.00757 (0.00332)	-0.00113 (0.00414)	-0.00419 (0.00516)	0.00658 (0.00659)	-0.01512 (0.05739)	0.01178 (0.00888)
1.8	-0.00652 (0.01033)	0.00396 (0.00405)	0.01069 (0.00631)	-0.00538 (0.00307)	0.01011 (0.00508)	0.01939 (0.00712)	0.01411 (0.00673)	0.00935 (0.05318)	-0.01832 (0.00788)
1.9	0.00395 (0.00300)	-0.00454 (0.00308)	-0.00186 (0.00523)	0.00632 (0.00357)	0.00683 (0.00541)	0.01388 (0.00833)	-0.00120 (0.00644)	0.00747 (0.00683)	-0.00133 (0.00749)
2.1	-0.00002 (0.00427)	-0.02159 (0.10328)	-0.00800 (0.00623)	0.00874 (0.00452)	-0.00094 (0.00466)	0.00568 (0.00664)	0.00894 (0.00669)	0.00107 (0.00513)	0.01708 (0.00795)
2.2	0.00566 (0.00297)	0.00333 (0.00283)	0.00021 (0.00646)	0.00780 (0.00377)	-0.00053 (0.00394)	-0.02459 (0.04191)	0.00300 (0.00629)	-0.00361 (0.02297)	-0.02646 (0.06832)
2.3	-0.00818 (0.00579)	0.00560 (0.00273)	0.00986 (0.00676)	-0.02177 (0.04988)	-0.00261 (0.00402)	0.01025 (0.00720)	-0.00400 (0.01227)	0.01435 (0.00781)	-0.00508 (0.00990)
2.4	-0.00104 (0.00338)	-0.00325 (0.00358)	-0.01584 (0.02352)	0.00427 (0.00506)	-0.03511 (0.12531)	-0.02792 (0.02757)	0.01112 (0.00656)	-0.00714 (0.05474)	-0.03669 (0.11735)
2.5	-0.02042 (0.03452)	0.00051 (0.00329)	-0.05331 (0.11499)	0.00488 (0.00371)	-0.02950 (0.04784)	-0.00573 (0.00599)	0.00688 (0.00477)	-0.00505 (0.00516)	0.01175 (0.00928)
3	-0.04401 (0.02940)	-0.02920 (0.01602)	-0.04264 (0.11258)	-0.03270 (0.04416)	-0.01497 (0.02869)	-0.05624 (0.13216)	-0.06668 (0.18385)	0.05340 (0.06810)	-0.02963 (0.04569)
4	-0.08250 (0.03120)	-0.02272 (0.01323)	-0.01788 (0.04550)	-0.01816 (0.05063)	-0.03934 (0.13930)	-0.05107 (0.09691)	0.02253 (0.11411)	-0.01600 (0.07030)	-0.04071 (0.10216)

Table1: Biases and (MSE)s of the MLEs of R, for $a=2$, $b=2$, and different values of c

(m,n)→ c↓	(30,30)	(30,20)	(30,10)	(20,30)	(20,20)	(20,10)	(10,30)	(10,20)	(10,10)
1.6	-0.00894 (0.00536)	-0.00051 (0.00505)	-0.02141 (0.00862)	0.02561 (0.06577)	0.00093 (0.00648)	-0.01142 (0.00968)	0.02449 (0.01180)	-0.00806 (0.00994)	-0.00067 (0.01311)
1.7	0.00223 (0.00428)	0.01084 (0.03039)	-0.01395 (0.00794)	0.02439 (0.04779)	0.01601 (0.01122)	0.00518 (0.02929)	0.00762 (0.00689)	0.02317 (0.02399)	-0.00405 (0.01476)
1.8	0.00525 (0.00383)	0.00700 (0.00484)	0.00998 (0.00881)	-0.03445 (0.00628)	0.00316 (0.00600)	0.02482 (0.10408)	0.01503 (0.00947)	0.00841 (0.01061)	0.02735 (0.01075)
1.9	0.00318 (0.00422)	0.01384 (0.00651)	-0.01125 (0.00699)	0.00115 (0.00511)	0.00500 (0.00574)	0.00859 (0.00863)	0.00155 (0.00855)	0.00756 (0.00968)	0.04176 (0.19672)
2.1	-0.00071 (0.00496)	-0.00698 (0.00519)	-0.00825 (0.01161)	0.00895 (0.00138)	-0.00316 (0.00630)	-0.01272 (0.01099)	0.02543 (0.00993)	0.00068 (0.01023)	-0.00387 (0.01605)
2.2	0.00174 (0.00601)	0.01184 (0.04628)	-0.00745 (0.00675)	0.01298 (0.01710)	0.01729 (0.02688)	0.03452 (0.10519)	0.02671 (0.01055)	0.02793 (0.04221)	0.00335 (0.01348)
2.3	0.04997 (0.07521)	-0.01035 (0.01416)	0.00490 (0.00883)	0.02085 (0.00340)	0.01264 (0.00601)	0.00419 (0.02833)	0.00366 (0.00843)	0.01446 (0.01465)	0.00144 (0.01268)
2.4	0.00578 (0.03098)	0.01086 (0.03569)	0.00681 (0.02232)	0.00961 (0.00912)	0.04411 (0.07843)	-0.00759 (0.00856)	0.02362 (0.08939)	0.01448 (0.01167)	-0.00517 (0.01309)
2.5	-0.00903 (0.00507)	0.05967 (0.12854)	-0.01382 (0.00842)	0.00244 (0.00552)	0.03839 (0.12965)	0.00148 (0.03331)	0.01188 (0.01048)	0.01131 (0.01996)	-0.00679 (0.01214)
3	-0.01090 (0.03660)	0.06576 (0.16607)	0.03458 (0.08306)	0.06911 (0.00613)	0.03340 (0.01719)	0.05936 (0.14885)	0.05115 (0.06400)	0.03324 (0.06344)	0.01505 (0.04383)
4	0.02155 (0.03128)	0.07522 (0.22516)	0.03231 (0.22089)	0.03820 (0.04950)	0.04210 (0.05290)	-0.01316 (0.02619)	0.02908 (0.12232)	0.02143 (0.03249)	0.01231 (0.14150)

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