Reliability for Lindley Distribution with an Outlier
Hossein Jabbari Khamnei
Department of Statistics, Faculty of mathematical sciences, University of Tabriz, Tabriz, Iran

Keywords. Lindley Distribution, Maximum Likelihood Estimator, Newton-Raphson Method, Outlier.

Abstract. In this paper, we consider the problem of estimating \( R = P(Y < X) \), when \( Y \) has lindley distribution with parameter \( a \) and \( x \) has lindley distribution with presence of one outlier with parameters \( b \) and \( c \), such that \( X \) and \( Y \) are independent. The maximum likelihood estimator of \( R \) is derived and some results of simulation studies are presented.

1 Introduction
In reliability context inferences about \( R = P(Y < X) \), when \( X \) and \( Y \) are independently distributed, are a subject of interest. For example in mechanical reliability of a system if \( X \) is the strength of a component which is subject to stress \( Y \), then \( R \) is a measure of system performance. The system fails, if at any time the applied stress is greater than its strength. Stress-strength reliability has been discussed in Kapur and Lamberson (1977). Sathe and Dixit (2001) have done estimation of \( R \) in the negative binomial distribution. Baklizi and Dayyeh (2003) have done shrinkage estimation of \( R \) in exponential case, and recently Deiri (2011) has done estimation of \( R \) with presence of two outliers in the exponential and gamma cases, respectively. Jafari (2011) has obtained the moment, maximum likelihood and mixture estimators of \( R \) in Rayleigh distribution in the presence of one outlier and Jabbari, Abolhasani and Fathipour (2012) have discussed the estimation of \( R \) in the six parameter generalized Burr XII distribution with transformation method.

In this paper, we obtain the maximum likelihood estimator of \( R \) for lindley distribution with presence of one outlier generated from the same distribution.

The probability density function of the lindley distribution with parameter of \( a \) is given by:

\[
f(y; a) = \frac{a^2}{1+a} (1 + y) e^{-ay}, x > 0, a > 0.
\]

In this paper we assume that the random variables \((Y_1, Y_2, ..., Y_m)\) have lindley distribution with parameter \( a \) and the random variables \((X_1, X_2, ..., X_n)\) are such that one of them is from lindley distribution with parameter \( c \) and the remaining \((n-1)\) random variables are from lindley distribution with parameter \( b \).

The paper is organized as follows:
In section 2, we obtain the joint distribution of \((X_1, X_2, ..., X_n)\) in the presence of one outlier. Section 3 and section 4 discusses the method of maximum likelihood estimators of parameters and the MLE of \( R \) respectively. In section 5 simulation studies are presented and the results are summarized in section 6.

2 Joint distribution of \(X_1, X_2, ..., X_n \) in presence of an outlier
Assume \((X_1, X_2, ..., X_n)\) are such that one of them is distributed with p.d.f \( g(x, c) \) as lindley\((c) \) and remaining \((n-1)\) of them are distributed with p.d.f \( f(x, b) \) as lindley\((b) \). The joint distribution of \((X_1, X_2, ..., X_n)\) can be expressed as

\[
f(x_1, x_2, ..., x_n; b, c) = \frac{(n-1)!}{n!} \prod_{i=1}^{n} f(x_i, b) \sum_{i=1}^{n} g(x_i; c) \frac{f(x_i b)}{f(x_i b)}
\]
\[ \frac{(n-1)!}{n!} \frac{b^{2n}}{(1+b)^n} \prod_{i=1}^{n} \frac{(1+x_i)e^{-b\sum_{i=1}^{n} x_i}}{(1+b)^{\sum_{i=1}^{n} x_i}} \]

\[ \frac{(n-1)!}{n!} \frac{b^{2n-2}}{(1+b)^{n-1}} \frac{c^2}{1+c} \prod_{i=1}^{n} \frac{(1+x_i)e^{-b\sum_{i=1}^{n} x_i}}{(1+b)^{\sum_{i=1}^{n} x_i}} + (1+x_i)e^{x_i(b-c)} \]

(1)

See Dixit (1989), Dixit and Nasiri (2001), and Nasiri and Pazira (2009). From (1), the marginal distribution of \( X \) is

\[ f(x; b, c) = \frac{c^2}{n+1+c} (1+x)e^{-cx} + \frac{n-1}{n} \frac{b^2}{1+b} (1+x)e^{-bx}; \ x, b, c > 0 \]

(2)

We will use (2) to obtain \( R=P(Y<X) \)

3 Maximum likelihood estimators of parameters

Let \( (Y_1, Y_2, ..., Y_m) \) be a random sample for \( Y \) with pdf,

\[ f(y; a) = \frac{a^2}{1+a} (1+y)e^{-ay}, \ x, a > 0 \]

the log likelihood function is given by

\[ L(a) = 2mlna - mln(1+a) + \sum_{i=1}^{m} \ln(1+y_i) - a \sum_{i=1}^{m} y_i \]

Taking the derivative with respect to \( a \) and equating to \( \theta \), we obtain the MLE of \( a \) as

\[ \hat{a} = \frac{m - \sum_{i=1}^{m} y_i \pm \sqrt{(\sum_{i=1}^{m} y_i - m)^2 + 8m \sum_{i=1}^{m} y_i}}{2 \sum_{i=1}^{m} y_i} \]

(3)

Now let \( X_1, X_2, ..., X_n \) be a random sample for \( X \) with presence of one outlier with pdf,

\[ f(x; b, c) = \frac{1}{n+1+c} \frac{c^2}{1+x} e^{-cx} + \frac{n-1}{n} \frac{b^2}{1+b} (1+x)e^{-bx}; \ x, b, c > 0. \]

From (1), the log likelihood function is given by

\[ L(b, c) = \ln \left( \frac{(n-1)!}{n!} \right) + (2n-2)lnb - (n-1) \ln(1+b) + 2ln(c) - \ln(1+c) + \sum_{i=1}^{n} \ln(1+x_i) - b \sum_{i=1}^{n} x_i + \ln \left[ \sum_{i=1}^{n} e^{x_i(b-c)} \right] \]

Taking the derivatives with respect to \( b \) and \( c \) and equating the results to \( \theta \), we obtain the normal equations as

\[ \frac{\partial L(b, c)}{\partial b} = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \]

(4)

\[ \frac{\partial L(b, c)}{\partial c} = \frac{2}{c} - \frac{1}{1+c} - \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \]

(5)

There is no closed-form solution to this system of equations, so we will solve for \( \hat{b} \) and \( \hat{c} \) iteratively, using the Newton-Raphson method. In our case we will estimate \( \hat{b} = (\hat{b}, \hat{c}) \) iteratively:

\[ \hat{b}_{i+1} = \hat{b}_i - G^{-1}g \]

(6)

where \( g \) is the vector of normal equations for which we want

\[ g = [g_1 \ g_2] \]

With

\[ g_1 = \frac{2n-2}{b} - \frac{n-1}{1+b} - \sum_{i=1}^{n} x_i + \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \]

\[ g_2 = \frac{2}{c} - \frac{1}{1+c} - \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \]
and $G$ is the matrix of second derivatives

$$G = \begin{bmatrix}
dg_1 & dg_1 \\
db & dc \\
db_2 & dg_2 \\
db & dc
\end{bmatrix}$$

where

$$
\begin{align*}
dg_1 &= \frac{2 - 2n}{b^2} + \frac{n - 1}{(1 + b)^2} + \sum_{i=1}^{n} x_i^2 e^{x_i(b-c)} - \left( \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \right)^2 \\
dg_1 &= - \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} + \left( \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \right)^2 \\
dg_1 &= - \frac{2}{c^2} + \frac{1}{(1 + c)^2} + \frac{\sum_{i=1}^{n} x_i^2 e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} - \left( \frac{\sum_{i=1}^{n} x_i e^{x_i(b-c)}}{\sum_{i=1}^{n} e^{x_i(b-c)}} \right)^2
\end{align*}
$$

The Newton-Raphson algorithm converges, as our estimate of $b$ and $c$ change by less than a tolerated amount with each successive iteration, to $\hat{b}$ and $\hat{c}$.

### 4 The maximum likelihood estimator of $R$

Let $Y \sim \text{Lindley}(a)$ with pdf $h(y; a)$ and $X$ be distributed with pdf $f(x; b, c)$ given in (2). The parameter $R$ we want to estimate is

$$R = P(Y < X) = \int_0^\infty \int_0^x h(y; a) f(x; b, c) dy dx$$

$$= \frac{1}{b} \int_0^\infty \int_0^x \frac{a^2}{1 + a} (1 + y)e^{-ay} \frac{c^2}{1 + c} (1 + x)e^{-cx} dy dx$$

$$+ \frac{b^2}{n} \int_0^\infty \int_0^x \frac{a^2}{1 + a} (1 + y)e^{-ay} \frac{b^2}{1 + b} (1 + x)e^{-bx} dy dx$$

$$= \frac{1}{n} \left[ \frac{c^2(c(1 + c) + (1 + c)(3 + c)a + (3 + 2c)a^2 + a^3)}{(1 + c)(1 + a)(c + a)^3} \right]$$

$$+ \frac{n - 1}{n} \left[ \frac{b^2(b(1 + b) + (1 + b)(3 + b)a + (3 + 2b)a^2 + a^3)}{(1 + b)(1 + a)(b + a)^3} \right]$$

Thus, by invariant property for MLEs, the MLE of $R$ is

$$\hat{R} = \frac{1}{2} \left[ \frac{\hat{c}^2(\hat{c}(1 + \hat{c}) + (1 + \hat{c})(3 + \hat{c})\hat{a} + (3 + 2\hat{c})\hat{a}^2 + \hat{a}^3)}{(1 + \hat{c})(1 + \hat{a})(\hat{c} + \hat{a})^3} \right]$$

$$+ \frac{n - 1}{n} \left[ \frac{\hat{b}^2(\hat{b}(1 + \hat{b}) + (1 + \hat{b})(3 + \hat{b})\hat{a} + (3 + 2\hat{b})\hat{a}^2 + \hat{a}^3)}{(1 + \hat{b})(1 + \hat{a})(\hat{b} + \hat{a})^3} \right]$$

where $\hat{a}, \hat{b},$ and $\hat{c}$ can be obtained from (3) and (6).

### 5 Simulation Study

In this section we generate random numbers from Lindley distribution (with and without outlier) with accept-reject method by Maple software. Using these samples and the Newton-Raphson method we obtain the maximum likelihood estimators of parameters $a, b$ and $c$. Then we use them to calculate the MLE of $R$. The values of biases and MSEs of these estimates are presented in table1, for $a=1, b=2$ and $c=1, 2, 6, 1, 7, 1, 1, 8, 1, 9, 2, 1, 2, 2, 2, 3, 2, 4, 2, 5, 3, 4$ and in table 2, for $a=1, b=2$, and the same values of $c$. All the results are based on 100 replications.

### 6 Conclusion

According to the results of simulation, when the value of parameters $b$ and $c$ are close to each other, the biases and MSEs are often around zero and when the difference between $b$ and $c$ is greater than 1, the biases and MSEs increase.
Table 1: Biases and (MSE) s of the MLEs of $R$, for $a=1, b=2$, and different values of $c$

<table>
<thead>
<tr>
<th>(m,n)</th>
<th>(30,30)</th>
<th>(50,50)</th>
<th>(100,100)</th>
<th>(20,20)</th>
<th>(20,20)</th>
<th>(20,20)</th>
<th>(20,20)</th>
<th>(20,20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>-0.0060</td>
<td>0.0088</td>
<td>-0.0069</td>
<td>0.0073</td>
<td>-0.0041</td>
<td>0.0049</td>
<td>-0.0029</td>
<td>0.0027</td>
</tr>
<tr>
<td>1.7</td>
<td>-0.0095</td>
<td>0.0093</td>
<td>-0.0097</td>
<td>0.0091</td>
<td>-0.0085</td>
<td>0.0085</td>
<td>-0.0079</td>
<td>0.0076</td>
</tr>
<tr>
<td>1.8</td>
<td>-0.0052</td>
<td>0.0046</td>
<td>-0.0051</td>
<td>0.0049</td>
<td>-0.0048</td>
<td>0.0050</td>
<td>-0.0045</td>
<td>0.0049</td>
</tr>
<tr>
<td>1.9</td>
<td>-0.0003</td>
<td>0.0004</td>
<td>-0.0003</td>
<td>0.0004</td>
<td>-0.0003</td>
<td>0.0004</td>
<td>-0.0003</td>
<td>0.0004</td>
</tr>
<tr>
<td>2.1</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.2</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.3</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.4</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.5</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.6</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.7</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
<tr>
<td>2.8</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-0.0002</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

References


