NOTE ON PERFECT NUMBERS AND THEIR EXISTENCE

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ABSTRACT: This paper will address the interesting results on perfect numbers. As we know that, perfect number ends with 6 or 8 and perfect numbers had some special relation with primes. Here one can understand that the reasons of relation with primes and existence of odd perfect numbers. If exists, the structures of odd perfect numbers in modulo.

1. INTRODUCTION

A perfect number is a whole number in which the term itself is equal to the sum of all its factors. Given below are some example based on perfect numbers.
Lets consider the number 28
Factors of 28 are 1, 2, 4, 7, 14 and 28
of this the proper factors are 1, 2, 4, 7, 14 and 28
Some of the proper factors = 1+2+4+7+14 = 28
Hence, 28 is a perfect number.
28 is also considered as the only even perfect number.
A perfect number can be defined as an integer, which is a non-zero number. A perfect number can be obtained by adding all the factors which are less than that number. All perfect numbers are even. There is no odd perfect number.
It is easy to find unusual properties of small numbers that would characterize these into their peculiarity. The number six (6) has a unique property in which it has both the sum and the product of all its smaller factors; 6 = 1+2+3 or 1 x 2 x 3. By the divisors of a number we mean the factors including unity are less than the number. If the sum of its proper divisors or aliquot divisors is less than the number then we call them as deficient (as in case of 8). If the total sum of the proper divisors or aliquot divisors exceeds the number as in case of 12, the number is then called abundant. The early Hebrews considered 6 to be a perfect number and Philo Judeus (1st century AD) also regarded 6 to be a perfect number.
There are two main types of perfect numbers and they are even perfect numbers [4] which would follow $2^{n-1}(2^n-1)$ and odd perfect numbers [2].

EVEN Perfect numbers
Euclid knew that $2^{n-1}(2^n-1)$ was perfect if 2n-1 is prime.
Euclid proved that, if and when 'p' = 1 + 2 + 22+ ....2n is a prime then 2^n p is referred as a perfect number.
2^np is divisible by 1, 2, ...2^n , p, 2p ..2^{n-1} p is the number less than itself and so the sum of these divisors is 2n p.

ODD Perfect numbers
Eular showed that these numbers have dimensions of p^a m^2 , where 'p' is prime and p = a = 1 or mod 4.
If 'n' is an odd number with s (n) = an, then n < (4d) ^{4K}, where 'd' is the denominator of a and 'K' is the number of distinct prime factors on 'n'.
If n_K is an odd number with k distinct prime factors then n < 4^{4K}
Let us discuss our main problems in the following section.

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1. PROBLEMS ON PERFECTNESS

1. If $p(2^{r-1})$ is even perfect number, then p must be a prime and in the form of $2^r - 1$.

$$\sum_{d|p,2^{r-1}} d = 1 + 2 + 2^2 + \ldots + 2^{r-1} + p(1 + 2 + 2^2 + \ldots + 2^{r-1})$$

PROOF: $= 2^2 - 1 + p(2^r - 1)$

$\qquad = (p+1)(2^r - 1).$

As we want it to be perfect number, we have

$= (p+1)(2^r - 1) = 2 \times p.2^{r-1} = p.2^r$

$= p.2^r - p + 2^r - 1$

$= p.2^r$

Hence $p = 2^r - 1$. ◊

2. Every even perfect numbers ends with 6 or 28.

PROOF: It is not hard to show that [1] all even perfect numbers of the form $(2^r - 1)(2^{r-1})$. Here

$2^r - 1$ is a prime. Since $2^r - 1$ is a prime, it ends with 1, 3 or 7.

$\Rightarrow 2^r \equiv 2,4,8 \pmod{10}$

$\Rightarrow 2^{r-1} \equiv 6,2,4 \pmod{10}$

Hence, $\quad 2^r - 1(2^{r-1}) \equiv 6 \times 1, 2 \times 3, 4 \times 7 \pmod{10}$
$\qquad\qquad \equiv 6,8 \pmod{10}.$

Hence, even perfect numbers ends in 6 or 8. If it ends in 8, we want to show that it is in fact ends in 28. We have that if $2^r \equiv 8 \pmod{10}$, then $r \equiv 3 \pmod 4$.

$\Rightarrow 2^{2r-1} - 2^{r-1} = 2^{8k+5} - 2^{4k+2}$

$= 32.256^k - 4.16^k.$

Now we have;

$$256^k \equiv \begin{cases} 56 \pmod{100} \rightarrow k \equiv 1 \pmod 5 \\ 36 \pmod{100} \rightarrow k \equiv 2 \pmod 5 \\ 16 \pmod{100} \rightarrow k \equiv 3 \pmod 5 \\ 96 \pmod{100} \rightarrow k \equiv 4 \pmod 5 \\ 76 \pmod{100} \rightarrow k \equiv 0 \pmod 5 \end{cases}$$

$$16^k \equiv \begin{cases} 16 \pmod{100} \rightarrow k \equiv 1 \pmod 5 \\ 56 \pmod{100} \rightarrow k \equiv 2 \pmod 5 \\ 96 \pmod{100} \rightarrow k \equiv 3 \pmod 5 \\ 36 \pmod{100} \rightarrow k \equiv 4 \pmod 5 \\ 76 \pmod{100} \rightarrow k \equiv 0 \pmod 5 \end{cases}$$

Hence, $32.256^k - 4.16^k \equiv 28 \pmod{100} \forall k \in Z^+$.

Therefore, every even perfect number ends in 6 or 8. Also, if it ends in 8, then in fact ends in 28. ◊

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3. If $a^k - 1$ is a prime for $a > 0$ and $k \geq 2$. If we fix $a = 2$, then k becomes prime.

PROOF: If $(2^k - 1)$ is a prime, then k has to be a prime. This can be proved easily by looking at the contra-positive of this statement. If k is composite, then $k = m \times n$, when $m, n > 1$.

This gives us,

$$\Rightarrow 2^k - 1 = 2^{mn} - 1 = (2^m)^n - 1$$
$$\Rightarrow (2^m - 1)(1 + 2^m + 2^{2m} + ... + 2^{(n-1)m})$$
$$\qquad 2^m - 1 > 1, \qquad\qquad 2^{mn} - 1.$$
$$\text{Here,} \qquad\qquad \text{divides}$$

$\Rightarrow 2^k - 1$ is composite if k is composite.

i.e., if $2^k - 1$ is a prime, then k is also prime.

Also, note that $a^k - 1 = (a - 1)(1 + a + a^2 + ... + a^{k-1})$, and since $a - 1 > 1$, we have that $a^k - 1$ is composite (for $a > 2$), irrespective of whether k is prime or not. ◊

4. If odd perfect numbers exists, then such numbers can be expressible in 12k + 1 or 324k + 81 or 468k + 117 for some positive integer k.

PROOF: If M is perfect, then $M = S - M$, where M is the sum of divisors (excluding M itself). i.e.,

$2M = S$. As M is perfect, so one of the factors of M is 3^k with $M \mid (3^0 + 3^1 + 3^2 + ... + 3^k)$. Let us fix k

$= 2$. Then M will be in 9 (mod 36) or $3^0 + 3^1 + 3^2 = 1 + 3 + 9 = 13$ must divide $2M$

$\Rightarrow M \equiv 0 (\mathrm{mod}\, 13)$

$\Rightarrow M \equiv 0 (\mathrm{mod}\, 13) \cup 9 (\mathrm{mod}\, 36)$ for some $l_1, l_2 \in N$.

$\Rightarrow M \equiv 13 l_1, M - 9 = 36 l_2$

$\therefore M \equiv 117 (\mathrm{mod}\, 468)$.

Instead of fixing the k value 2, 3, 4 and so on... let us take $k > 2$, then $M = 3^4 = 81$.

$\Rightarrow M \equiv 9 (\mathrm{mod}\, 36) \cup 0 (\mathrm{mod}\, 81)$

$\Rightarrow M \equiv 81 (\mathrm{mod}\, 324)$

Hence, M is odd perfect number then M can be expressible in the form of *1(mod 12)* or *117(mod 468)* or *81 (mod 324)*. ◊

5. If K is perfect and in the form of $\dfrac{4^n - 2^n}{2}$, then k should be $1^3 + 3^3 + ... + (2m - 1)^3$ for

      some *m*.

PROOF: We know that

$$\sum_{k-1}^{n} k^3 = \frac{n^2(n+1)^2}{4}, \text{ which leads to } 1^3 + 3^3 + ... + (2m - 1)^3$$

$$\Rightarrow \sum_{k-1}^{2m} k^3 - \sum_{k-1}^{m} (2k)^3$$

$$\Rightarrow \frac{(2m)^2(2m+1)^2}{4} - 8 \cdot \frac{m^2(m+1)^2}{4}$$

$$= m^2(2m^2 - 1).$$

We know that, every even perfect number K is in the form of $2^{p-1}(2^p - 1)$ with $p = 2n+1$ an odd prime and $2^p - 1$ a Mersenne prime [3], letting $m = 2^n$, we can see that;

$$K = \frac{4^n - 2^n}{2} = m^2(2m^2 - 1)$$

$$= 1^3 + 3^3 + \dots + (2m-1)^3. \Diamond$$

6. If n is perfect number then $(8n + 1)$ is always square.

PROOF: By Euler Theorem [5], every even perfect number is of the form $2^{p-1}(2^p - 1)$, where p isprime. Now just multiply by 8 and then add 1 to it, we get;

$$\Rightarrow 2^{2p+2} - 2(2^{p+1}) + 1, \text{ This is } (2^{p+1} - 1)^2 \Diamond$$

7. An odd perfect number cannot have only two distinct prime factors. (other than 1 itself)

PROOF: Let $N = p^a q^b$, where p and q are odd primes and $p < q$. Then the sum of the all divisors of N is:

$$(1 + p + \dots + p^a)(1 + q + \dots + q^b)$$

Now using sum of finite terms in Geometric series, we get;

$$N = \frac{p^{a+1} - 1}{p - 1} \cdot \frac{q^{b+1} - 1}{q - 1}$$

Now, divide by N, we get;

$$\frac{p - \frac{1}{p^a}}{p - 1} \cdot \frac{q - \frac{1}{q^b}}{q - 1}$$

$$\Rightarrow \frac{p - \frac{1}{p^a}}{p - 1} \cdot \frac{q - \frac{1}{q^b}}{q - 1} < \frac{p}{p - 1} \cdot \frac{q}{q - 1}$$

$$(1)$$

Now, we should justify $(1) < 2$. In particular, that shows that the product cannot be 2 as N is in fact *deficient*. It is easier to show that the reciprocal of expression $(1) > \frac{1}{2}$.

$$\Rightarrow \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right).$$

But $p \geq 3$ and $q \geq 5$

$$\Rightarrow 1 - \frac{1}{3} \geq \frac{2}{3} \text{ and } 1 - \frac{1}{q} \geq \frac{4}{5}.$$

$\therefore$ The product $\geq \frac{8}{15}$, which is $> \frac{1}{2}. \Diamond$

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