Ablowitz-Kaup-Newel-Segur Formalism and N-Soliton Solutions of Generalized Shallow Water Wave Equation

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Abstract. We apply Ablowitz-Kaup-Newel-Segur hierarchy to derive the generalized shallow water wave equation and we also investigate N -soliton solutions of the derived equation using Inverse Scattering Transform method and Hirota's bilinear method.

Introduction

Studying integrable systems by the methods introduced by Zakharov and Shabat (ZS) [1] and by Ablowitz-Kaup-Newel-Segur (AKNS) [2] has been the focus of considerable attention through the past many years. These approaches basically deal with inverse scattering transform (IST) [3, 7, 8] and Lax-type conditions. By a simple extension of the Lax integrability condition the desired nonlinear evolution equation is generated. In recent times it is an accepted fact that existence of a Lax-pair is indeed a decisive trademark of integrable systems. ZS introduced this method to study Nonlinear Schrödinger equation and soon there after AKNS showed that one relatively minor modification of the ZS approach recovers the theory of the famous KdV equation [2], while another leads to an IST analysis for a well-known evolution equation, the Sine-Gordon equation [4].

The purpose of this work is to extend the well-known generalized shallow water wave (GSWW) equation in the variable coefficient form using the AKNS formalism and to investigate the novel N -soliton solutions of the derived equations with constant coefficients using IST and Hirota's bilinear method.

The usual GSWW equation (that is with constant coefficient) reads [5, 6]

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - \mu u_{xt} - \eta u_{xx} = 0, \quad (1)$$

where $\alpha, \beta \in \mathbb{R} - \{0\}$. Clarkson and Mansfield [6] showed some time ago, by investigating the Painlevé property that equation (1) is completely integrable if and only if $(\alpha - 2\beta)(\alpha - \beta) = 0$. These correspond to the specific types

$$\alpha = \beta : u_{xxt} + \alpha u_x u_t - u_t - u_x + f(t) = 0, \quad (2)$$

$$\alpha = 2\beta : u_{xxxt} + \alpha u_x u_{xt} + \frac{\alpha}{2} u_t u_{xx} - \mu u_{xt} - \eta u_{xx} = 0, \quad (3)$$

which we call GSWWI equation and GSWWII equation respectively. In (2), $f(t)$ is an arbitrary function of time t .

Our concern will be to introduce variable coefficients in (2) and (3) in the manner

$$u_{xxt} + \alpha u_x u_t - \eta(t) u_x - \gamma u_t + f(t) = 0 \quad (4)$$

and

$$u_{xxxt} + \alpha(t) u_x u_{xt} + \frac{\alpha(t)}{2} u_t u_{xx} - \mu(t) u_{xt} - \eta(t) u_{xx} = 0, \quad (5)$$

where $\alpha(t)$, $\beta(t)$, $\mu(t)$ and $\eta(t)$ are variable functions of time.

The multi soliton solution of (4) and (5) will be derived using IST for (3) and Hirota's bilinear method for (2).

GSWW Equation with Variable Coefficients using AKNS Hierarchy and Explicit Soliton Solutions

Case-I ($\alpha = 2\beta$) : The linear eigenvalue problem is given by

$$L[u]\psi(x, t) = \lambda\psi(x, t), \quad (6)$$

where $\lambda = \text{eigenvalue}$ and $\psi(x, t)$ evolve with time in a prescribed manner determined by

$$\partial_t \psi(x, t) = A[u]\psi(x, t). \quad (7)$$

Consider the eigenvalue problem analogous to (6)

$$\psi_x = M\psi; \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}; \quad M = \begin{bmatrix} -i\xi & q(x, t) \\ r(x, t) & i\xi \end{bmatrix}.$$

The time dependence of ψ analogous to (7) is given by

$$\psi_t = N\psi; \quad N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The compatibility condition $\psi_{xxt} = \psi_{txx}$ gives

$$N_{xx} - M_{xt} + 2N_x M - M M_t - M_t M + [N, M_x] + [N, M M] = \Theta, \quad (8)$$

where $[N, M] = NM - MN$ and Θ is the null matrix of order 2×2 .

Equation (8) can be split into the following equations

$$A_{xx} = q_x C - r_x B - 2r B_x + (qr)_t + 2i\xi A_x, \quad (9)$$

$$B_{xx} = q_x D - q_x A - 2q A_x - 2i\xi B_x + q_{xt}, \quad (10)$$

$$C_{xx} = r_x A - r_x D - 2r D_x + 2i\xi C_x + r_{xt}, \quad (11)$$

$$D_{xx} = -q_x C + r_x B - 2q C_x + (qr)_t - 2i\xi D_x. \quad (12)$$

Expanding A , B , C and D as

$$A(x, t) = \sum_{j=0}^1 A^{(j)}(x, t) \xi^j, \quad B(x, t) = \sum_{j=0}^1 B^{(j)}(x, t) \xi^j,$$

$$C(x, t) = \sum_{j=0}^1 C^{(j)}(x, t) \xi^j, \quad D(x, t) = \sum_{j=0}^1 D^{(j)}(x, t) \xi^j,$$

we have from equations (9) to (12)

$$\begin{aligned} A_{xx}^{(j)} - 2iA_x^{(j-1)} &= q_x C^{(j)} - r_x B^{(j)} - 2r B_x^{(j)} + \delta_{j0}(qr)_t; \quad j = 0, 1, \\ B_{xx}^{(1)} + 2iB_x^{(0)} &= q_x D^{(1)} - q_x A^{(1)} - 2q A_x^{(1)}, \\ C_{xx}^{(1)} - 2iC_x^{(0)} &= r_x A^{(1)} - r_x D^{(1)} - 2r D_x^{(1)}, \\ A_{xx}^{(j)} + 2iA_x^{(j-1)} &= -q_x C^{(j)} + r_x B^{(j)} - 2q C_x^{(j)} + \delta_{j0}(qr)_t; \quad j = 0, 1. \end{aligned}$$

For $j = 0$ equations (10) and (11) lead to the following equations

$$\begin{aligned} q_{xt} &= B_{xx}^{(0)} - q_x D^{(0)} + q_x A^{(0)} + 2q A_x^{(0)}, \\ r_{xt} &= C_{xx}^{(0)} - r_x A^{(0)} + r_x D^{(0)} + 2r D_x^{(0)}. \end{aligned}$$

To find GSWW equation ($\alpha = 2\beta$) let us assume

$$A^{(1)} = w(x, t) = -D^{(1)}, \quad B^{(1)} = C^{(1)} = 0$$

using which, the above equations give

$$A^{(0)} = -\frac{i}{2}w_x + a_0(t), \quad B^{(0)} = iwq, \quad C^{(0)} = iwr, \quad D^{(0)} = -\frac{i}{2}w_x + d_0(t),$$

where $a_0(t)$ and $d_0(t)$ are arbitrary functions of time t .

For $j = 0$, the above equations indicate

$$w_{xxx} - 4w_xqr - 2w(qr)_x - 2i(qr)_t = 0, \quad (13)$$

$$q_{xt} - iwq_{xx} - 2iw_xq_x + (d_0 - a_0)q_x = 0, \quad (14)$$

$$r_{xt} - iwr_{xx} - 2iw_xr_x - (d_0 - a_0)r_x = 0. \quad (15)$$

With

$$w_x = u_t(x, t), \quad q = u_x(x, t) + q_0(t), \quad r = -\frac{1}{4}\alpha(t),$$

$$q_0 = -\frac{\mu(t)}{\alpha(t)}, \quad a_0 - d_0 = \frac{2i\eta(t) - \alpha_t(t)}{\alpha(t)},$$

the equations (13)-(15) are reduced to the GSWWII equation (5) with variable coefficients.

The Lax pair associated with the GSWWII equation (3) is

$$\psi_{xx} + (\lambda - v(x, t))\psi = 0, \quad (16)$$

$$\psi_t + \frac{2 - \alpha u_x}{2(1 + 4\lambda)}\psi_x + \frac{\alpha u_{xt}}{4(1 + 4\lambda)}\psi = 0, \quad (17)$$

where $v(x, t) = -\frac{\alpha}{4}u_x$ is the potential and $\lambda = \xi^2 - \frac{\mu}{4}$. This linear pair is compatible i.e. $\psi_{xxt} = \psi_{txx}$ when $u(x, t)$ satisfies the GSWWII equation (3) and λ is isospectral i.e. $\frac{\partial \lambda}{\partial t} = 0$.

We consider the Schrödinger spectral problem from (16) for $t = 0$,

$$\psi_{xx} + (\lambda - v(x, 0))\psi = 0, \quad (18)$$

where $v \rightarrow 0$ as $|x| \rightarrow \infty$. Equation (18) admits

(i) a finite number of bound states with the eigenvalues $\lambda = -k_n^2$; $n = 1, 2, 3, \dots, N$ and the normalization constants $C_n(0)$ of the associated eigenstates and

(ii) a continuum or scattering states with the continuous eigenvalues $\lambda = k^2$, $|k| < \infty$.

As $|x| \rightarrow \infty$, the time evolution equation (17) reduces to

$$\psi_t = -\frac{1}{4\lambda}\psi_x. \quad (19)$$

We impose on $\psi(x, t)$ the boundary conditions

$$\psi(x, t) \approx a_+(k, t)e^{ikx} + a_-(k, t)e^{-ikx}, \quad x \rightarrow -\infty, \quad (20)$$

$$\approx b_+(k, t)e^{ikx} + b_-(k, t)e^{-ikx}, \quad x \rightarrow \infty. \quad (21)$$

Equations (19), (20) and (21) implies

$$\frac{da_{\pm}}{dt} = \mp \frac{ik}{1 + 4\lambda}a_{\pm}, \quad \frac{db_{\pm}}{dt} = \mp \frac{ik}{1 + 4\lambda}b_{\pm},$$

which gives

$$a_{\pm}(k, t) = a_{\pm}(k, 0)e^{\mp \frac{ik}{1+4\lambda}t}, \quad b_{\pm}(k, t) = b_{\pm}(k, 0)e^{\mp \frac{ik}{1+4\lambda}t}.$$

For the standard type of scattering solutions we take

$$\begin{aligned} \frac{1}{a_+(k, t)} \psi &\approx e^{ikx} + R(k, t)e^{-ikx}, \quad x \rightarrow -\infty, \\ &\approx T(k, t)e^{ikx}, \quad x \rightarrow \infty, \end{aligned}$$

where the reflection coefficient $R(k, t)$ is given by

$$R(k, t) = \frac{a_-(k, t)}{a_+(k, t)} = R(k, 0)e^{\frac{2ik}{1+4\lambda}t},$$

the transmission coefficient $T(k, t)$ is taken as

$$T(k, t) = \frac{b_+(k, t)}{a_+(k, t)} = T(k, 0)$$

and $b_-(k, t) = 0$. The coefficient of reflection $R(k, t)$ and the coefficient of transmission $T(k, t)$ can be shown to satisfy $|R(k, t)|^2 + |T(k, t)|^2 = 1$.

As we have taken λ is isospectral, the eigenvalues $\lambda = -k_n^2$; $n = 1, 2, 3, \dots, N$, do not change with time i.e.

$$\lambda(t) = \lambda(0) \Rightarrow k_n(t) = k_n(0). \quad (22)$$

The eigenfunctions $\psi_n(x, t)$, $n = 1, 2, 3, \dots, N$, corresponding to the eigenvalues (22) satisfy the time evolution equations

$$\psi_{n,t} = -\frac{1}{1-4k_n^2} \psi_{n,x}, \quad |x| \rightarrow \infty. \quad (23)$$

We have

$$\psi_n(x, t) \approx e^{k_n x}, \quad x \rightarrow -\infty, \quad (24)$$

$$\approx C_n(t)e^{-k_n x}, \quad x \rightarrow \infty, \quad (25)$$

with the normalization condition

$$\int_{-\infty}^{\infty} |\psi_n(x, t)|^2 dx = 1.$$

The normalization constants $C_n(t)$ can be evaluated from (23), (24) and (25) as

$$\frac{dC_n}{dt} = \frac{k_n}{1-4k_n^2} C_n \Rightarrow C_n(t) = C_n(0)e^{\frac{k_n}{1-4k_n^2}t}.$$

The scattering data $S(t)$ corresponding to the potential $v(x, t)$ evolves from $S_n(0)$ of the initial data $v(x, 0)$ is given as

$$\begin{aligned} S(t) &= \{k_n(t) = k_n(0), C_n(t) = C_n(0)e^{\frac{k_n}{1-4k_n^2}t}, n = 1, 2, 3, \dots, N, \\ &R(k, t) = R(k, 0)e^{\frac{2ik}{1-4k^2}t}, |k| < \infty\}. \end{aligned}$$

From the scattering data $S(t)$ in (26) at time t , one can invert the data and obtain uniquely $v(x, t)$ of the Schrödinger type spectral problem (16), by solving the Gelfand-Levitan-Marchenko (GLM) integral equation [9, 10]

$$K(x, y, t) + F(x + y, t) + \int_x^\infty F(y + z, t)K(x, z, t) dz = 0, \quad y > x \quad (26)$$

where the time variable t enters only as a parameter and

$$F(x+y, t) = \sum_{n=1}^N C_n^2(t) e^{-k_n(x+y)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k, t) e^{ik(x+y)} dk. \quad (27)$$

Solving the GLM equation (26), one can recover the potential $v(x, t)$ from K as

$$v(x, t) = -\frac{\alpha}{4} u_x(x, t) = -2 \left[\frac{\partial K(x, y, t)}{\partial x} \Big|_{y=x} + \frac{\partial K(x, y, t)}{\partial y} \Big|_{y=x} \right]. \quad (28)$$

One-soliton solution (N=1)

To evaluate one-soliton solution of GSWWII equation (3) we consider the special case of reflectionless potential ($R(k, t) = 0$) with only one bound state, $N = 1$, specified by

$$k_1 = k, \quad C_1(t) = C(t) = C_0 e^{\frac{k}{1-4k^2}t}, \quad C_0 = C(0).$$

The GLM integral equation (26) for this case becomes

$$K(x, y, t) + C_0^2 e^{-k(x+y) + \frac{k}{1-4k^2}t} + C_0^2 e^{-ky + \frac{k}{1-4k^2}t} \int_x^{\infty} e^{-kz} K(x, z, t) dz = 0. \quad (29)$$

The GLM integral equation (29) can be put as

$$\frac{\partial K}{\partial y} + kK = 0,$$

which indicates

$$K(x, y, t) = e^{-ky} h(x, t). \quad (30)$$

Substituting (30) into the GLM integral equation (29), we have

$$K(x, y, t) = -\frac{2k e^{-k(x+y) + \frac{k}{1-4k^2}t}}{\frac{2k}{C_0^2} + e^{-2kx + \frac{k}{1-4k^2}t}}.$$

Equation (28) gives the one-soliton solution of (3)

$$u(x, t) = \frac{8k}{\alpha} \tanh \left[k \left(x - \frac{t}{2(1-4k^2)} \right) + \delta \right]; \quad \delta = \frac{1}{2} \ln \left(\frac{2k}{C_0^2} \right).$$

Two-soliton solution (N=2)

To evaluate two-soliton solution of the equation (3), we need to consider again the special case of reflectionless potential ($R(k, t) = 0$) with two bound states, $N = 2$, specified by the discrete eigenvalues k_1 and k_2 and the corresponding normalization constants $C_j(t) = C_{j0} e^{\frac{k_j}{1-4k_j^2}t}$, $C_{j0} = C_j(0)$, $j = 1, 2$. In this case the GLM integral equation (26) can be put as

$$K(x, y, t) + \sum_{j=1}^2 [C_{j0}^2 e^{-k_j(x+y) + \frac{k_j}{1-4k_j^2}t} + C_{j0}^2 e^{-k_j y + \frac{k_j}{1-4k_j^2}t} \int_x^{\infty} e^{-k_j z} K(x, z, t) dz] = 0. \quad (31)$$

To solve GLM integral equation (31), we take $K(x, y, t)$ as

$$K(x, y, t) = \sum_{j=1}^2 e^{-k_j y} h_j(x, t).$$

Substituting this into the GLM integral equation (31) and equating the coefficients of $e^{-k_1 y}$ and $e^{-k_2 y}$ from both sides separately, we obtain

$$h_1(x, t) = \frac{H_1^{(1)} H_2^{(2)} - H_3^{(1)} H_1^{(2)}}{H_4^{(1)} H_4^{(2)} - H_3^{(1)} H_3^{(2)}}, \quad h_2(x, t) = \frac{H_4^{(1)} H_1^{(2)} - H_1^{(1)} H_3^{(2)}}{H_4^{(1)} H_4^{(2)} - H_3^{(1)} H_3^{(2)}},$$

where

$$\begin{aligned} H_1^{(j)} &= -C_{j0}^2 \exp\left(-k_j x + \frac{k_j}{1 - 4k_j^2} t\right), \quad j = 1, 2, \\ H_2^{(j)} &= 1 + \frac{C_{j0}^2}{2k_j} \exp\left(-k_j x + \frac{k_j}{1 - 4k_j^2} t\right), \quad j = 1, 2, \\ H_3^{(j)} &= \frac{C_{j0}^2}{k_1 + k_2} \exp\left[-(k_1 + k_2)x + \frac{k_j}{1 - 4k_j^2} t\right], \quad j = 1, 2, \\ H_4^{(j)} &= 1 + \frac{C_{j0}^2}{2k_j} \exp\left(-2k_j x + \frac{k_j}{1 - 4k_j^2} t\right), \quad j = 1, 2. \end{aligned}$$

Equation (28) gives the two-soliton solution of (3)

$$u(x, t) = \frac{8k}{\alpha} (k_2^2 - k_1^2) \int \frac{k_1^2 \sinh^2 \zeta_2 + k_2^2 \cosh^2 \zeta_1}{(k_1 \sinh \zeta_1 - k_2 \cosh \zeta_2)^2} dx,$$

where

$$\zeta_j = k_j x - \frac{k_j}{2(1 - 4k_j^2)} t - \delta_j, \quad \delta_j = \frac{1}{2} \ln\left(\frac{C_{j0}^2 (k_2 - k_1)}{2k_j (k_2 + k_1)}\right), \quad j = 1, 2.$$

N -soliton solution

Now we investigate N -soliton solution of the equation (3) and for which we need reflectionless potentials ($R(k, t) = 0$) with N -bound states as earlier i.e. from (27) we express F in separation of variables form as

$$F(x + y, t) = \sum_{n=1}^N C_n^2(t) e^{-k_n(x+y)} = \sum g_n(x, t) g_n(y, t), \quad (32)$$

where

$$g_n(x, t) = C_n(t) e^{-k_n x}.$$

We also take K in separation of variables form

$$K(x, y, t) = \sum_{n=1}^N \omega_n(x, t) g_n(y, t). \quad (33)$$

Substituting (32) and (33) into the GLM integral equation (26), we obtain

$$\omega_m(x, t) + g_m(x, t) + \sum_{n=1}^N \omega_n(x, t) \int_x^\infty g_m(z, t) g_n(z, t) dz = 0. \quad (34)$$

Defining the matrices

$$Q_{mn}(x, t) = \delta_{mn} + \int_x^\infty g_m(z, t)g_n(z, t) dz, \quad (35)$$

$$\begin{aligned} \omega(x, t) &= (\omega_1(x, t), \omega_2(x, t), \dots, \omega_N(x, t))^T, \\ g(x) &= (g_1(x, t), g_2(x, t), \dots, g_N(x, t))^T, \end{aligned}$$

equation (34) can be recast as the matrix equation

$$Q(x, t)\omega(x, t) = -g(x, t) \Rightarrow \omega(x, t) = -Q^{-1}(x, t)g(x, t).$$

From equation (33), one may get

$$K(x, x, t) = g^T(x, t)\omega(x, t) = -g^T(x, t)Q^{-1}(x, t)g(x, t). \quad (36)$$

Equation (35) can be put as

$$\frac{dQ_{mn}}{dx} = -g_m(x, t)g_n(x, t). \quad (37)$$

Equations (36) and (37) together imply

$$\begin{aligned} K(x, x, t) &= -\text{tr}(g_m Q_{mn}^{-1} g_n) = \text{tr}(Q^{-1} \frac{dQ}{dx}) = \sum_l \sum_m \frac{Q_{ml}}{|Q|} \frac{dQ_{lm}}{dx} \\ &= \frac{1}{|Q|} \frac{d|Q|}{dx} = \frac{d}{dx} \ln |Q|, \end{aligned} \quad (38)$$

where Q_{ml} is the cofactor matrix and $|Q|$ is the determinant of Q .

Finally, from equations (28) and (38) we get the N -soliton solution of the equation (3) as

$$u(x, t) = \frac{8}{\alpha} \frac{d}{dx} \ln |Q|.$$

Case-II ($\alpha = \beta$) : The linear eigenvalue problem is given by

$$L[u]\psi(x, t) = \lambda\psi(x, t), \quad (39)$$

where $\lambda = \text{eigenvalue}$ and $\psi(x, t)$ evolve with time in a prescribed manner determined by

$$\partial_t \psi(x, t) = A[u]\psi(x, t). \quad (40)$$

Consider the eigenvalue problem analogous to (39) [13]

$$\psi_x = M\psi; \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}; \quad M = \begin{bmatrix} 0 & 1 & q(x, t) \\ 0 & 0 & \xi \\ r(x, t) & 0 & 0 \end{bmatrix},$$

where $q(x, t)$ and $r(x, t)$ are potentials and ξ is the eigenvalue.

The time dependence of ψ analogous to (40) is given by

$$\psi_t = N\psi; \quad N = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

The compatibility condition $\psi_{xt} = \psi_{tx}$ gives

$$N_x - M_t + [N, M] = \Theta_{3 \times 3}, \quad (41)$$

where $[N, M] = NM - MN$ and Θ is the null matrix of order 3×3 .

Equation (41) can be split into the following equations

$$A_{11,x} + rA_{13} - A_{21} - qA_{31} = 0 \quad (42)$$

$$A_{12,x} + A_{11} - A_{22} - qA_{32} = 0 \quad (43)$$

$$A_{13,x} + qA_{11} - A_{23} - qA_{33} + \xi A_{12} = q_t \quad (44)$$

$$A_{21,x} + rA_{23} - \xi A_{31} = 0 \quad (45)$$

$$A_{22,x} + A_{21} - \xi A_{32} = 0 \quad (46)$$

$$A_{23,x} + qA_{21} + \xi A_{22} - \xi A_{33} = 0 \quad (47)$$

$$A_{31,x} + rA_{33} - rA_{11} = r_t \quad (48)$$

$$A_{32,x} + A_{31} - rA_{12} = 0 \quad (49)$$

$$A_{33,x} + qA_{31} + \xi A_{32} - rA_{13} = 0. \quad (50)$$

To find GSWWI equation (4) with variable coefficient let us assume

$$A_{jj} = 0, \quad j = 1, 2, 3, \quad A_{23} = -\xi A_{12} = a(x, t),$$

$$A_{21} = A_{13} = \xi A_{32} = b(x, t), \quad A_{31} = 0, \quad r = 1,$$

using which, the equations (42)-(50) can be put as

$$a_x(x, t) + b(x, t)q(x, t) = 0 \quad (51)$$

$$b_x(x, t) + a(x, t) = 0 \quad (52)$$

$$q_t(x, t) + 2a(x, t) - b_x(x, t) = 0. \quad (53)$$

With

$$b(x, t) = v_t(x, t) + \frac{1}{3}\eta(t) - g_t(t), \quad v(x, t) = -\frac{\alpha}{3}u(x, t) + \frac{\gamma}{3}x + g(t), \quad f(t) = \frac{\gamma}{\alpha}\eta(t),$$

equations (51)-(53) reduce to the GSWWI equation (4) with variable coefficients.

The scattering problem for the GSWWI equation (2) is

$$\psi_{xxx} + (\alpha u_x - 1)\psi = \lambda\psi, \quad (54)$$

where $\lambda = \xi$.

Clearly the scattering problem (54) is not of Schrödinger type. To evaluate explicit soliton solutions of the GSWWI equation (2) we use the Hirota's bilinear [11] form of equation (2).

We introduce the transformation to a new dependent variable W by

$$u(x, t) = \frac{\partial W}{\partial x}, \quad (55)$$

Substituting (55) in (2) and integrating once gives

$$W_{xxx} + \alpha W_{xx}W_{xt} - W_{xx} - W_{xt} = 0.$$

Defining

$$W = \frac{6}{\alpha} \ln F, \quad \text{i.e.} \quad u(x, t) = \frac{6}{\alpha} (\ln F)_x, \quad (56)$$

equation (2) can be transformed to the bilinear form

$$(D_x^3 D_t - D_x^2 - D_x D_t) F \cdot F = 0, \quad (57)$$

where the D -operator is defined as ($n, m \geq 0$)

$$D_x^n D_t^m G(x, t).F(x, t) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m G(x, t) F(x', t')|_{x'=x, t'=t}.$$

To derive explicit soliton solutions of equation (2), we need the following bilinear identities

$$D_x^n D_t^m F.1 = D_x^n D_t^m 1.F = D_x^n D_t^m F, \quad m + n = 2N, \quad (58)$$

$$D_x^n D_t^m e^{\eta_1}.e^{\eta_2} = (k_1 - k_2)^n (w_1 - w_2)^m e^{\eta_1 + \eta_2}, \quad (59)$$

where F is an arbitrary continuous function of independent variables x, t and $\eta_j = k_j x + w_j t$, $j = 1, 2$ is the dispersion relation.

The dispersion relation can be obtained by substituting

$$u(x, t) = e^{\zeta_j}, \quad \zeta_j = k_j(x - c_j t), \quad j = 1, 2, \dots, N,$$

into the linear terms of equation (2) as

$$\zeta_j = k_j \left(x - \frac{1}{1 - k_j^2} t\right), \quad |k_j| \neq 1, \quad j = 1, 2, \dots, N.$$

One-soliton solution

To obtain one-soliton solution of equation (2), the seed solutions of the corresponding bilinear equation (57) are taken as

$$F_0 = 1, \quad F_1 = e^{\zeta_1}, \quad \zeta_1 = k_1 \left(x - \frac{1}{1 - k_1^2} t\right).$$

Using (58), it is clear that F_0 and F_1 satisfies

$$(D_x^3 D_t - D_x^2 - D_x D_t) F_0.F_1 = 0. \quad (60)$$

Taking $F = F_0 + F_1$ in (57) and using (60), we have

$$(D_x^3 D_t - D_x^2 - D_x D_t)(F_0 + F_1).(F_0 + F_1) = 0.$$

With the help of the bilinear transformation (56), the one-soliton solution of equation (2) can be put as

$$u(x, t) = \frac{6}{\alpha} \cdot \frac{k_1 e^{\zeta_1}}{1 + e^{\zeta_1}}.$$

Two-soliton solution

With in mind the identities (58) and (59), we start with the derived solution $F_0 + F_1$ above as new seed solution and choosing another seed solution $F_2 = e^{\zeta_2}$, $\zeta_2 = k_2 \left(x - \frac{1}{1 - k_2^2} t\right)$ of equation (57), then on substitution of

$$F = F_0 + F_1 + F_2 + \Lambda_{12} F_1 F_2$$

in left hand side of equation (57) gives

$$\Delta F.F = 2\delta e^{\zeta_1}.e^{\zeta_2} + 2\Lambda_{12}\delta 1.e^{\zeta_1 + \zeta_2} + 2\Lambda_{12}\delta e^{\zeta_1}.e^{\zeta_1 + \zeta_2} + 2\Lambda_{12}\delta e^{\zeta_2}.e^{\zeta_1 + \zeta_2}, \quad (61)$$

where $\Delta = D_x^3 D_t - D_x^2 - D_x D_t$ and the expression given in (61) vanishes for

$$\Lambda_{12} = \frac{(k_1^2 - k_1 k_2 + k_2^2 - 3)(k_1 - k_2)^2}{(k_1^2 + k_1 k_2 + k_2^2 - 3)(k_1 + k_2)^2}. \quad (62)$$

With the help of the bilinear transformation (56), the two-soliton solution of equation (2) can be written as

$$u(x, t) = \frac{6}{\alpha} \cdot \frac{k_1 e^{\zeta_1} + k_2 e^{\zeta_2} + \Lambda_{12}(k_1 + k_2)e^{\zeta_1 + \zeta_2}}{1 + e^{\zeta_1} + e^{\zeta_2} + \Lambda_{12}e^{\zeta_1 + \zeta_2}},$$

where Λ_{12} is given in (62).

N –soliton solution

In general the similar process can be done continuously and will yield a series of explicit soliton solutions. To obtain N –soliton solution, we choose

$$F = \sum_{\mu_i=0,1; 1 \leq i \leq N} \exp \left[\sum_{1 \leq i < j \leq N} \phi(i, j) \mu_i \mu_j + \sum_{i=1}^N \mu_i \zeta_i \right], \quad (63)$$

where

$$\Lambda_{ij} = \exp(\phi(i, j)) = \frac{(k_i^2 - k_i k_j + k_j^2 - 3)(k_i - k_j)^2}{(k_i^2 + k_i k_j + k_j^2 - 3)(k_i + k_j)^2}, \quad 1 \leq i < j \leq N.$$

The N –soliton solution of equation (2) can be obtained from

$$u(x, t) = \frac{6}{\alpha} (\ln F)_x,$$

where F is given in (63).

Summary

In this work we have employed the AKNS scheme to derive the generalized shallow water wave equation for the complete integrable cases. We also obtained N –soliton solutions using IST and Hirota's bilinear method.

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