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The Last Theorem of Fermat for n=3

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Abstract: In this paper on FLT, one solves the case n=3 in elementary way, extensible to n odd. The author works only through the sole factorization in factors and with the proceeding for absurd, that is if x, y, z are prime among them, under the hypothesis that (x, y, z) are a solution, one obtains that the first and the second term of an equivalent relation are odd (the first) and even (the second).

§ 1.The case n=3, that is the Last Theorem of Pierre de Fermat

Here is the
Theorem of Pierre de Fermat and Andrew Wiles for n=3: The equation

$$x^3 + y^3 = z^3 \tag{1}$$

has no integer positive solution.
Proof: We could suppose that (x, y, z) is solution with x, y, z prime among them, in particular x and y cannot be both multiple of 3: let y not be multiple of 3.
Let

$z = x + k$ with k prime with x.

It follows that

$$y^3=z^3-x^3=(z-x)(z^2+zx+x^2)=k(3x^2+3xk+k^2)$$

Every factor of k must divide y^3, then every factor of k will be a power of exponent 3, otherwise it should divide also $3x^2$, that is x or 3, against the hypothesis; so k will be in the form of

$$k=u^3$$

and also

$$v^3=3x^2 +3xk=k^2$$

that is

$$v^3=3x^2 + 3xu^3+u^6 \tag{2}$$

where ***u*** and ***v*** are prime among them.
This result, that is

$$y = uv \text{ and } k=u^3$$

one can obtain also by the relations of Waring.
We observe that v is always odd because ***x*** and ***z*** cannot be even at the same time.
By the relation (2) , working out the term

$$3x^2 = v^3 - 3xu^3 - u^6$$

we obtain that

if **x** is even, then **u** is odd;
if **x** is odd, then **u** is either even or odd.
In other words one has two cases:
the first one: **x** even, **y** and **z** odd,
the second: **x** and **y** odd, **z** even.

Therefore

$$z = x + u^3$$

is either odd or even.
At the same time, we have

$$z^3 = x^3 + y^3 = (x+y)(x^2 - xy + y^2).$$

Edward Waring, professor of Cambridge University (Shrewsbury, Shropshire, 1734-Pleasley, 1798) worked out

$$x^2 - xy + y^2$$

in a polynomial of ***x+y*** and ***xy***

We remember that ***x*** and ***y*** are prime among them, so ***x+y*** is odd if **x** even and divides z^3 then the factors of **z**, that divide ***x+y*** , cannot divide ***xy***, that is

$$x^2 - xy + y^2$$

such factor of $x^3 + y^3$ we indicate with

$$p^3$$

One concludes:

$$x + y = p^3 \text{ and } z = pq$$

that is

$z^3 = p^3 q^3$ with **p** and **q** prime among them and **q** odd

and

$$q^3 = x^2 - xy + y^2$$

Finally, we have

$$x = pq - u^3 \text{ and } x = p^3 - uv.$$

We remember that when **z** is odd, then **p** and **q** are odd. Therefore it is

$$z + y = pq + uv = x + y + z - x = p^3 + u^3 = (p+u)(p^2 - pu + u^2).$$

The factors ***p, q, v*** and ***u*** are prime among them and ***p+u*** divides ***z+y*** , so

$$\exists c : c = \frac{z+y}{u+p} \text{ with } v < c < q$$

or

$$pq+uv=c(u+p)$$

that is

$$u(c-v)=p(q-c),$$

but u and p are prime among them, so exists a such that

$$c\text{-}v=ap \text{ and } q\text{-}c=au$$

or

$$c = v + ap = q - au;$$

where

$$a = \frac{q-v}{u+p}$$

and

$$q - au = v + ap = u^2 - up + p^2 ;$$

By those relations, one has:

$$\text{(I)}\quad p^2 - q = u(p-u-a) = ul$$

and

$$\text{(II)}\quad v - u^2 = p(p-u-a) = pl .$$

We observe that in the relation (I) the first term is even when p is odd, in that case u l is even and also a is even.

Likewise, placed

$$t = p^2 - q = ul \text{ and } s = v - u^2 = pl ,$$

we have:

$$x + y - z = p^3 - pq = p(p^2 - q) = pt = pul = us =$$
$$= y - (z - x) = uv - u^3 = u(v - u^2) = ulp$$

We observe that ***l*** is prime with ***u, p, v*** and ***q***; looking ***l***:
l is prime with ***u*** and ***p***;
l is prime with ***v***:
in fact, if ***l*** were not prime with **v**, we would have a common factor of **v**, but

$$u(v) - pu(l) = y - (x + y - z) = z - x = u^3 ,$$

and u is prime with v;
l is prime with ***q***: so, if l were not prime with ***q***, they would have a common factor of ***q***, but it is

$$p(q) + up(l) = z + (x + y - z) = x + y = p^3$$

and ***p*** is prime with ***q***.
We observe that all the factors of ***u, v, p, q, uv, up, uq, vp, vq, pq*** do not divide ***uv+pq*** and ***u+v, p+q***.
For example, we proof that ***up*** does not divide ***uv+pq***.
In fact, although it is obvious, if ***up*** divides ***uv+pq***, then

$$p(uv + pq) - v(up) = p^2 q \quad \text{and} \quad u(uv + pq) - q(up) = u^2 v ,$$

that is ***p²q*** and ***u²v*** must have a common factor, against the hypothesis that ***pq*** and ***uv*** are prime among them.

We observe that ***l*** is the factor of **x** and of **z-y** in fact it is

$$ulp = x + y - z = x - (z - y) .$$

Then we have ***l*<x**: in fact, if it were ***l*=x**, we would have the absurd:

$$z - y = x - upx = x(1 - up) \leq 0.$$

Now, we take:

$$x^3 = z^3 - y^3 = (z - y)(z^2 + zy + y^2);$$

a factor of ***z-y***, (for Waring), cannot divide ***zy***, because ***z*** and ***y*** are prime among them; therefore, we have:

$$z - y = l^3 \text{ and } x = lf \text{, so } f^3 = z^2 + zy + y^2;$$

with *l* and *f* prime among them and *f* is odd.

At the same time , we have:

$$x^3 + z^3 = (x + z)(x^2 - xz + z^2)$$

and

$$x + z = x + y + z - y = p^3 + l^3 = lf + pq = (p + l)(p^2 - pl + l^2),$$

from here, we obtain that

$$p + l \text{ divides } lf + pq,$$

it is:

$$\frac{lf + pq}{p + l} = q - bl = f + bp, \text{ with } b = \frac{q - f}{p + l},$$

because

$$c(p+l)=lf+pq$$

or

$$l(c-f)=p(q-c)$$

in which ***p*** and ***l*** are prime among them;
then

$$q - bl = f + bp = p^2 - pl + l^2.$$

We have:

$$\textbf{(III)} \quad p^2 - q = l(p - l - b) = li = lu,$$

$$\textbf{(IV)} \quad f - l^2 = p(p - l - b) = pi = pu,$$

but, $i=u$ because

$$x+y-z=lf-l^3=l(f-l^2)=upl=ipl=y-(z-x)=uv-u^3=u(v-u^2)=ulp$$

We observe that in the relation **(III)**, the first term is even when **p** is odd, in that case ***lu*** is even moreover also **b**, as **a**, is even.
We conclude:

$$x=lf\,,\ \ y=uv\,,\ z=pq$$

and

$$lu=p^2-q\,,\ pu=f-l^2\,,\ pl=v-u^2\,. \qquad (3)$$

Now we consider

$$x-y=z-y-(z-x)=l^3-u^3=lf-uv=(l-u)(l^2+lu+u^2)\,.$$

It is:

$$\frac{lf-uv}{l-u}=f-du=v-dl \text{ with } d=\frac{v-f}{l-u}\,,$$

because

$$c(l-u)=lf-uv$$

or

$$l(c-f)=u(c-v).$$

Then, we have:

$$f-du=v-dl=l^2+lu+u^2$$

and

$$\textbf{(V)}\ \ f-l^2=u(l+u+d)=ue=up\,,$$
$$\textbf{(VI)}\ \ v-u^2=l(l+u+d)=le=lp\,,$$

because

$$x+y-z=lf-l^3=l(f-l^2)=lue=u(v-u^2)=ulp\,.$$

We observe that **d** is even if **u** is odd and **p** even.
Finally, we have:

$$x=p^3-uv\,,\ y=uv\,,\ z=(x+y)-(x+y-z)=p^3-ulp\,,$$

which verify the **(2)**, that is

$$(p^3-uv)^3+(uv)^3=(p^3-ulp)^3\,,$$

or

$$(p^3-uv+uv)(x^2-xy+y^2)=(p^3-upl)^3\,,$$

dividing for p^3 , we have:

$$x^2-xy+y^2=(p^2-ul)^3\,,$$

that is

$$q^3=q^3\,.$$

From this verification, we are sure on the developments.
For the conclusion of theorem we take again the relations:

$$\frac{uv+pq}{u+p}=v+ap=q-au \text{ with } a=\frac{q-v}{u+p}; \tag{4}$$

$$\frac{lf+pq}{p+l}=q-bl=f+bp \text{ with } b=\frac{q-f}{p+l}; \tag{5}$$

$$\frac{lf-uv}{l-u}=f-du=v-dl \text{ with } d=\frac{v-f}{l-u}; \tag{6}$$

where between ***a* , *b*, *d*** at least two are even.
Now, we remember:

$$ulp = x + y - z$$

is even, so among ***u*, *l* , *p*** , only one is even.
By (I), (III), (V) we have

$$\begin{gathered} p\text{-}u\text{-}a=l \\ p\text{-}l\text{-}b=u \\ l+u+d=p \end{gathered}$$

or

$$\begin{gathered} a=p\text{-}l\text{-}u \\ b=p\text{-}u\text{-}l \\ d=p\text{-}l\text{-}u \end{gathered} \tag{7}$$

Then we have **a, b, d** even and

$$\frac{q-v}{u+p}=\frac{q-f}{p+l}=\frac{v-f}{l-u}.$$

Now we prefer to work in this way: by

$$z-y=l^3,\ z-x=u^3,\ x+y=p^3,\ x=z-y+(x+y-z)=l^3+ulp,$$
$$y=z-x+(x+y-z),\ z=x+y-(x+y-z),$$

we have

$$\begin{cases} x=lf=2^{-1}(l^3-u^3+p^3)=l^3+upl \\ y=uv=2^{-1}(-l^3+u^3+p^3)=u^3+upl \\ z=pq=2^{-1}(l^3+u^3+p^3)=p^3-upl \\ x+y-z=upl=2^{-1}(-l^3-u^3+p^3) \end{cases} \tag{8}$$

By the relations (8), we have

$$l \text{ divides } p^3 - u^3$$
$$u \text{ divides } p^3 - l^3$$
$$p \text{ divides } u^3 + l^3$$

and exist c_1, c_2, c_3 such that

$p^3 - u^3 = lc_1$, $p^3 - l^3 = uc_2$, $u^3 + l^3 = pc_3$, with c_2 and c_3 odd, if l even;

c_1 and c_2 odd if p even.

It turns out:

$$c_1 = l^2 + 2up \ , \ c_2 = u^2 + 2lp \ , \ c_3 = p^2 - 2ul \ ,$$

with $c_1 = 2C_1$ and C_1 odd, if l is even, because $c_1 = 2(\frac{l^2}{2} + up)$;

with $c_3 = 2C_3$ and C_3 odd, if p is even, because $c_3 = 2(\frac{p^2}{2} - ul)$.

In fact, it is

$$p^3 - u^3 - l^3 = (x+y) - (z-x) - (z-y) =$$
$$= 2(x+y-z) = 2ulp$$

and

$$p^3 - u^3 = l(l^2 + 2up) \ , \ p^3 - l^3 = u(u^2 + 2lp) \ , \ u^3 + l^3 = p(p^2 - 2ul) \ .$$

In addition, we conclude that since, from **(8)**, $f = l^2 + up$, $v = u^2 + lp$, $q = p^2 - ul$

$$c_1 = f + up, \ c_2 = v + lp, \ c_3 = q - ul \ . \tag{9}$$

We observe that

$$(2^\alpha x_1 + 1)^2 = 2^{2\alpha} x_1^{\ 2} + 2^{\alpha+1} x_1 + 1 = 2^{\alpha+1} x_2 + 1$$

and

$$(2^\alpha x_1 + 1)^3 = 2^{3\alpha} x_1^{\ 3} + 3 \cdot 2^{2\alpha} x_1^{\ 2} + 3 \cdot 2^\alpha x_1 + 1 = 2^\alpha x_3 + 1$$

with x^1, x^2 and x^3 odd and we use this symbology, on the following transformations.

The problem of Fermat is symmetric with respect to x and y, therefore, for simplicity, we consider 3 two cases:

the first: **p** odd and among ***l*** and **u**,

l even;

the second: only **p** even

p even

We go on in the first case and we use the above-mentioned symbology on the following transformations.

Let

$$l = 2^{\alpha} L \qquad l^2 = 2^{2\alpha} L^2 \qquad l^3 = 2^{3\alpha} L^3$$
$$u = 2^{\beta} U_1 + 1 \qquad u^2 = 2^{\beta+1} U_2 + 1 \qquad u^3 = 2^{\beta} U_3 + 1$$
$$p = 2^{\gamma} P_1 + 1 \qquad p^2 = 2^{\gamma+1} P_2 + 1 \qquad p^3 = 2^{\gamma} P_3 + 1$$
$$f = 2^{\delta} F_1 + 1 \qquad v = 2^{\varepsilon} V_1 + 1 \qquad q = 2^{\rho} Q_1 + 1$$
$$c_1 = 2C_1 \qquad c_2 = 2^{\tau} C_2 + 1 \qquad c_3 = 2^{\sigma} C_3 + 1$$

with $L, U_1, U_2, U_3, P_1, P_2, P_3, F_1, V_1, Q_1, C_1, C_2, C_3$ odd

We begin to examine the first of (8)

$$2^{\alpha} L(2^{\delta} F_1 + 1) = 2^{-1}(2^{3\alpha} L^3 - 2^{\beta} U_3 + 2^{\gamma} P_3) = 2^{3\alpha} L^3 + 2^{\alpha} L(2^{\beta} U_1 + 1)(2^{\gamma} P_1 + 1)$$

or

$$2^{\delta} F_1 + 1 = 2^{2\alpha-1} L^2 + L^{-1}(2^{\gamma-\alpha-1} P_3 - 2^{\beta-\alpha-1} U_3) = 2^{2\alpha} L^2 + (2^{\beta} U_1 + 1)(2^{\gamma} P_1 + 1)$$

we have two possibilities:

(a) $\gamma - \alpha - 1 = 0$ if $\gamma < \beta$
(b) $\beta - \alpha - 1 = 0$ if $\beta < \gamma$

that is

(a) $\gamma = \alpha + 1,\ \delta = \gamma = \alpha + 1$ if $\gamma < \beta$
(b) $\beta = \alpha + 1,\ \delta = \gamma = \alpha + 1$ if $\beta < \gamma$

Note down: the case $\gamma = \beta$ is impossible because, in such hypothesis

$2^{\delta} F_1 + 1$ is odd and

$2^{\gamma-\alpha-1} P_3 - 2^{\beta-\alpha-1} U_3 = 2^{\beta-\alpha-1}(P_3 - U_3)$ is even.

Now we examine the second of **(8)**

$$(2^{\beta} U_1 + 1)(2^{\varepsilon} V_1 + 1) = 2^{-1}(-2^{3\alpha} L^3 + 2^{\beta} U_3 + 2^{\gamma} P_3 + 2) = 2^{\beta} U_3 + 1 +$$
$$+ 2^{\alpha} L(2^{\beta} U_1 + 1)(2^{\gamma} P_1 + 1)$$

or

$$(2^{\beta} U_1 + 1)(2^{\varepsilon} V_1 + 1) = (-2^{3\alpha-1} L^3 + 2^{\beta-1} U_3 + 2^{\gamma-1} P_3 + 1) = 2^{\beta} U_3 + 1 +$$
$$+ 2^{\alpha} L(2^{\beta} U_1 + 1)(2^{\gamma} P_1 + 1)$$

that is

$$(2^{\beta} U_1 + 1)(2^{\varepsilon} V_1 + 1) = 2^{-1}[(2^{\beta} U_1 + 1)^3 + (2^{\beta} U_1 + 1)(2^{\tau} C_2 + 1)] =$$
$$= (2^{\beta} U_1 + 1)^3 + 2^{\alpha} L(2^{\beta} U_1 + 1)(2^{\gamma} P_1 + 1)$$

or

$$2^{\varepsilon} V_1 + 1 = 2^{-1}(2^{\beta+1} U_2 + 1 + 2^{\tau} C_2 + 1) = 2^{\beta+1} U_2 + 1 + 2^{\alpha} L(2^{\gamma} P_1 + 1)$$

$$2^{\varepsilon} V_1 + 1 = 2^{\beta} U_2 + 2^{\tau-1} C_2 + 1 = 2^{\beta+1} U_2 + 1 + 2^{\alpha} L(2^{\gamma} P_1 + 1)$$

and

We have

(a) $\gamma = \alpha + 1,\ \delta = \alpha + 1,\ \varepsilon = \gamma - 1 = \alpha,\ \tau - 1 = \alpha$
(b) $\beta = \alpha + 1,\ \delta = \alpha + 1,\ \varepsilon = \beta - 1 = \alpha,\ \tau - 1 = \alpha.$

The problem is symmetric with respect to **γ** and **β** , therefore on the cases **(a), (b)**, we study only the case **(a)**.

We obtain, in the case **(a)**,

$$
\begin{array}{lll}
c_1 = 2C_1 & c_2 = 2^{\alpha+1}C_2 + 1 & c_3 = 2^{\alpha+1}C_3 + 1 \\
l = 2^{\alpha}L & u = 2^{\beta}U_1 + 1 & p = 2^{\alpha+1}P_1 + 1 \\
f = 2^{\alpha+1}F_1 + 1 & v = 2^{\alpha}V_1 + 1 & q = 2^{\alpha}Q_1 + 1
\end{array}
$$

Now we consider the hypothesis (a) $\beta \succ \gamma = \alpha + 1$ for relations (9)

Now we consider the hypothesis **(a)** $\beta \succ \gamma = \alpha + 1$, for relations **(9)**

$$2C_1 = 2^{\alpha+1}F_1 + 1 + (2^{\beta}U_1 + 1)(2^{\alpha+1}P_1 + 1),$$
$$2^{\alpha+1}C_2 + 1 = 2^{\alpha}V_1 + 1 + 2^{\alpha}L(2^{\alpha+1}P_1 + 1),$$
$$2^{\alpha+1}C_3 + 1 = 2^{\alpha}Q_1 + 1 - (2^{\beta}U_1 + 1)2^{\alpha}L.$$

that is

$$C_1 = 2^{\alpha}F_1 + 1 + 2^{\alpha+\beta}U_1P_1 + 2^{\beta-1}U_1 + 2^{\alpha}P_1,$$
$$2C_2 = V_1 + L(2^{\alpha+1}P_1 + 1),$$
$$2C_3 = Q_1 - (2^{\beta}U_1 + 1)L.$$

We conclude that we can write $C_1 = 2^{\alpha}\Gamma_1 + 1$, with Γ_1 odd. By $c_1 = f + up$, we obtain

$$2^{\alpha+1}\Gamma_1 + 2 = (2^{\alpha+1}F_1 + 1) + (2^{\beta}U_1 + 1)(2^{\alpha+1}P_1 + 1),$$

that is

$$2^{\alpha+1}\Gamma_1 = 2^{\alpha+1}F_1 + 2^{\alpha+\beta+1}U_1P_1 + 2^{\alpha+1}P_1 + 2^{\beta}U_1$$

or

$$\Gamma_1 = F_1 + 2^{\beta}U_1P_1 + P_1 + 2^{\beta-\alpha-1}U_1;$$

this relation is an absurd because the first term, $\forall \alpha \succ 0$ and $\forall \beta \succ \alpha + 1$ is odd and the second is even.

We conclude that, in the cases **(a)** and **(b)**, no integer positive primitive solution of **(1)** is possible, so no integer positive primitive solution of **(1)** is possible.

Now we consider the second case: it is similar to first one, exchanging the roles of ***l*** and **p.**

We deal the second case using the showed symbology on the following transformations.

Let

$$l = 2^{\alpha} L_1 + 1 \qquad l^2 = 2^{\alpha+1} L_2 + 1 \qquad l^3 = 2^{\alpha} L_3 + 1$$
$$u = 2^{\beta} U_1 + 1 \qquad u^2 = 2^{\beta+1} U_2 + 1 \qquad u^3 = 2^{\beta} U_3 + 1$$
$$p = 2^{\gamma} P \qquad p^2 = 2^{2\gamma} P^2 \qquad p^3 = 2^{3\gamma} P^3$$
$$f = 2^{\delta} F_1 + 1 \qquad v = 2^{\varepsilon} V_1 + 1 \qquad q = 2^{\rho} Q_1 + 1$$
$$c_1 = 2^{\tau} C_1 + 1 \qquad c_2 = 2^{\sigma} C_2 + 1 \qquad c_3 = 2C_3$$

with $L, U_1, U_2, U_3, P_1, P_2, P_3, F_1, V_1, Q_1, C_1, C_2, C_3$ odd

We begin to examine the third of **(8)**

$$2^{\gamma} P(2^{\rho} Q_1 + 1) = 2^{-1}(2^{\alpha} L_3 + 2^{\beta} U_3 + 2^{3\gamma} P^3 + 2) = 2^{3\gamma} P^3 - 2^{\gamma} P(2^{\beta} U_1 + 1)(2^{\alpha} L_1 + 1)$$

or

$$2^{\rho} Q_1 + 1 = 2^{2\gamma-1} P^2 + P^{-1} 2^{-\gamma}(2^{\alpha-1} L_3 + 2^{\beta-1} U_3 + 1) = 2^{2\gamma} P^2 - (2^{\beta} U_1 + 1)(2^{\alpha} L_1 + 1);$$

we have two possibilities:

(a) $\alpha - \gamma - 1 = 0$ if $\alpha < \beta$

(b) $\beta - \gamma - 1 = 0$ if $\beta < \alpha$

that is

(a) $\alpha = \gamma + 1,\ \rho = \alpha = \gamma + 1$ if $\alpha < \beta$

(b) $\beta = \gamma + 1,\ \rho = \beta = \gamma + 1$ if $\beta < \alpha$

Note down: the case $\alpha = \beta$ is impossible because, in such hypothesis

$2^{\delta} Q_1 + 1$ is odd and $2^{\alpha-\gamma-1} L_3 - 2^{\beta-\gamma-1} U_3 + 1 = 2^{\alpha-\gamma-1}(L_3 + U_3) + 1$ could not be divisible for **P** even.

Now we examine the second of **(8)**

$$(2^{\beta} U_1 + 1)(2^{\varepsilon} V_1 + 1) = 2^{-1}(-2^{\alpha} L_3 + 2^{\beta} U_3 + 2^{3\gamma} P^3) = 2^{\beta} U_3 + 1 +$$
$$+ (2^{\alpha} L_1 + 1)(2^{\beta} U_1 + 1)(2^{\gamma} P)$$

or

$$(2^{\beta} U_1 + 1)(2^{\varepsilon} V_1 + 1) = (2^{3\gamma-1} P^3 + 2^{\beta-1} U_3 - 2^{\alpha-1} L_3) = 2^{\beta} U_3 + 1 +$$
$$+ (2^{\alpha} L_1 + 1)(2^{\beta} U_1 + 1)(2^{\gamma} P)$$

that is

$$(2^{\beta} U_1 + 1)(2^{\varepsilon} V_1 + 1) = 2^{-1}[(2^{\beta} U_1 + 1)^3 + (2^{\beta} U_1 + 1)(2^{\sigma} C_2 + 1)] =$$
$$= (2^{\beta} U_1 + 1)^3 + (2^{\alpha} L_1 + 1)(2^{\beta} U_1 + 1)(2^{\gamma} P)$$

or

$$2^{\varepsilon} V_1 + 1 = 2^{-1}(2^{\beta+1} U_2 + 1 + 2^{\sigma} C_2 + 1) = 2^{\beta+1} U_2 + 1 + (2^{\alpha} L_1 + 1)(2^{\gamma} P)$$

and

$$2^{\varepsilon} V_1 + 1 = 2^{\beta} U_2 + 2^{\sigma-1} C_2 + 1 = 2^{\beta+1} U_2 + 1 + (2^{\alpha} L_1 + 1)(2^{\gamma} P)$$

We have

(a) $\alpha = \gamma + 1$, $\rho = \gamma + 1$, $\varepsilon = \gamma = \alpha - 1$, $\sigma - 1 = \gamma$
(b) $\beta = \gamma + 1$, $\rho = \gamma + 1$, $\varepsilon = \gamma = \beta - 1$, $\sigma - 1 = \gamma$.

Now we examine the first of (8)

$$(2^\alpha L_1 + 1)(2^\delta F_1 + 1) = 2^{-1}(2^\alpha L_3 - 2^\beta U_3 + 2^{3\gamma} P^3) = 2^\alpha L_3 + 1 + \\ + (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)$$

or

$$(2^\alpha L_1 + 1)(2^\delta F_1 + 1) = 2^{3\gamma - 1} P^3 - 2^{\beta - 1} U_3 + 2^{\alpha - 1} L_3 + 1 = \\ = 2^\alpha L_3 + 1 + 2^\gamma P(2^\beta U_1 + 1)(2^\alpha L_1 + 1)$$

that is

$$(2^\alpha L_1 + 1)(2^\delta F_1 + 1) = 2^{-1}[(2^\alpha L_1 + 1)^3 + (2^\tau C_1 + 1)(2^\alpha L_1 + 1)] = \\ = (2^\alpha L_1 + 1)^3 + (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)$$

or

$$2^\delta F_1 + 1 = 2^{-1}(2^{\alpha+1} L_2 + 1 + 2^\tau C_1 + 1) = 2^{\alpha+1} L_2 + 1 + 2^\gamma P(2^\beta U_1 + 1)$$

and

$$2^\delta F_1 + 1 = 2^\alpha L_2 + 2^{\tau-1} C_1 + 1 = 2^{\alpha+1} L_2 + 1 - 2^\gamma P(2^\beta U_1 + 1).$$

We have

(a) $\alpha = \gamma + 1$, $\rho = \gamma + 1$, $\varepsilon = \gamma$, $\sigma = \gamma + 1$, $\delta = \gamma$, $\tau = \gamma + 1$, $(\alpha < \beta)$
(b) $\beta = \gamma + 1$, $\rho = \gamma + 1$, $\varepsilon = \gamma$, $\sigma = \gamma + 1$, $\delta = \gamma$, $\tau = \gamma + 1$.

The problem is symmetric with respect to α and β, therefore on the cases **(a), (b),** we study only the case **(a).**

We obtain, in the case **(a),**

$$\begin{array}{lll} c_1 = 2^{\gamma+1} C_1 + 1 & c_2 = 2^{\gamma+1} C_2 + 1 & c_3 = 2C_3 \\ l = 2^{\gamma+1} L_1 + 1 & u = 2^\beta U_1 + 1 & p = 2^\gamma P \\ f = 2^{\gamma+1} F_1 + 1 & v = 2^\gamma V_1 + 1 & q = 2^{\gamma+1} Q_1 + 1 \end{array}$$

Now we consider the hypothesis (a) $\beta \succ \alpha = \gamma + 1$, for relations (9)

$$2C_3 = 2^{\gamma+1} Q_1 + 1 - (2^\beta U_1 + 1)(2^{\gamma+1} L_1 + 1),$$
$$2^{\gamma+1} C_1 + 1 = 2^\gamma F_1 + 1 + 2^\gamma P(2^\beta U_1 + 1),$$
$$2^{\gamma+1} C_2 + 1 = 2^\gamma V_1 + 1 + (2^\beta U_1 + 1)2^\gamma P.$$

that is

$$C_3 = 2^{\gamma} Q_1 + 1 - 2^{\gamma+\beta} U_1 L_1 - 2^{\beta-1} U_1 - 2^{\gamma} L_1,$$

$$2C_1 = F_1 + P(2^{\beta} U_1 + 1),$$

$$2C_2 = V_1 + (2^{\beta} U_1 + 1)P.$$

We conclude that we can write

$$C_3 = 2^{\gamma} \Gamma_1 + 1 \text{, with } \Gamma_1 \text{ odd.}$$

By $c_3 = q - ul$, we obtain

$$2^{\gamma+1} \Gamma_1 + 2 = (2^{\gamma+1} Q_1 + 1) - (2^{\beta} U_1 + 1)(2^{\gamma+1} L_1 + 1),$$

that is

$$2^{\gamma+1} \Gamma_1 + 2 = 2^{\gamma+1} Q_1 - 2^{\gamma+\beta+1} U_1 L_1 - 2^{\gamma+1} L_1 - 2^{\beta} U_1$$

or

$$2^{\gamma} \Gamma_1 + 1 = 2^{\gamma} Q_1 - 2^{\gamma+\beta} U_1 L_1 - 2^{\gamma} L_1 - 2^{\beta-1} U_1 ;$$

this relation is an absurd because the first term, $\forall \gamma \succ 0$ and $\forall \beta \succ \gamma + 1$, is odd and the second is even.

We conclude that, in the cases **(a)** and **(b)**, no integer positive primitive solution of **(1)** is possible, so no integer positive primitive solution of **(1)** is possible.

We conclude that, in both cases, ***l*** even and ***p*** even, no integer positive primitive solution of **(1)** is possible, so no integer positive primitive solution of **(1)** is possible.

q.e.d.

Final note: also the equation $p^3 - u^3 - l^3 = 2upl$ solves the problem.

References

[1] Nicola Fragnito. "EULER BRICK". Pioneer Journal of Algebra, Number Theory and its Applications, ISSN 2231-1831, Vol. 2 , Issue 1, September 2011, pp.37-43, available online at http://www.pspchv.com/content_1_PJANTA_vol_2_1.html.

[2] Nicola Fragnito. "A solution of the Goldbach conjecture." JP Journal of Algebra, Number Theory and Applications, Volume 20, Number 2, 2011, pp. 147-211; available online at http://pphmj.com/journals/jpanta.htm.

[3] Nicola Fragnito. " The distribution of infinite primes of the first and second type, according to the Riemann Hypothesis, of infinite pairs of twin primes and the calculus of the finite number of the consecutive pairs of twin primes." JP Journal of Algebra, Number Theory and Applications, Volume 19, Number 1, 2010, pp. 13-119; available online at http://pphmj.com/journals/jpanta.htm ; http://www.pphmj.com/abstract/5380.htm.

[4] Nicola Fragnito**.** "A solution of Collatz conjecture." Pioneer Journal of Mathematics and Mathematical Sciences, ISSN 2230-9829, Vol. 5, 2012, Issue 1, pp. 23-53 available at http://www.pspchv.com/Abstract-pdf-PJMMS-Vol-5-Issue-1-pages-1-160/4.%20Abstract%20%20Pages%2023-

[5] Nicola Fragnito. "The last theorem of Fermat for every n", Annals of Mathematics, submitted.