

# Left Cancellative and Right Cancellative Elements of Semigroup of Relations

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**Abstract:** Let  $R(X)$  denote the set of all relations on a set  $X$ . Then  $R(X)$  becomes a semigroup under composition. The purpose of this paper is to describe the element of  $R(X)$  which is left cancellative and right cancellative.

## 1. Introduction

An element  $a$  of a semigroup  $S$  is said to be left cancellative if for all  $b, c \in S$ ,  $ab = ac$  implies that  $b = c$  and right cancellative if for all  $b, c \in S$ ,  $ba = ca$  implies that  $b = c$

For a nonempty set  $X$  a subset  $a$  of  $X \times X$  is called a (binary) relation on  $X$ . A binary operation on the set  $R(X)$  consisting of all relations on  $X$  is defined by

$$\alpha\beta = \{(x,y) \in X \times X \mid (x,z) \in \alpha \text{ and } (z,y) \in \beta\}$$

Then  $R(X)$  becomes a semigroup under this multiplication. We call  $R(X)$  the semigroup of relations on  $X$ . For  $\alpha \in R(X)$ , we denote the domain of  $\alpha$  by  $\text{dom } \alpha$  that is,

$$\text{dom } \alpha = \{x \in X \mid (x,y) \in \alpha \text{ for some } y \in X\}$$

and the range of  $\alpha$  by  $\text{ran } \alpha$ , that is,

$$\text{ran } \alpha = \{y \in X \mid (x,y) \in \alpha \text{ for some } x \in X\}$$

If  $x \in X$ , we define

$$x\alpha = \{y \in X \mid (x,y) \in \alpha\} \text{ and} \\
\alpha x = \{y \in X \mid (y,x) \in \alpha\}.$$

As usual,  $P(X)$  denote the partial transformation on  $X$ , the set of all mappings  $\alpha$  whose  $\text{dom } \alpha$  and  $\text{ran } \alpha$  are subsets of  $X$  and  $T(X)$  denote the full transformation on  $X$ , the set of all  $\alpha \in P(X)$  with  $\text{dom } \alpha = X$ . Both  $P(X)$  and  $T(X)$  are subsemigroups of  $R(X)$ . Our main purpose, we give a necessary and sufficient condition for element of  $R(X)$  to be left cancellative and right cancellative.

For the remainder of this paper, we assume that  $X$  is a nonempty set and  $|X| \geq 2$ .

## 2. Main Results

For  $\alpha \in R(X)$ , denote

$$X_\alpha = \{x \in X \mid |x\alpha| = 1\} \text{ and} \\
\alpha_* = \{(x,y) \in \alpha \mid x \in X_\alpha\}$$

The following result is to characterize a left cancellative element of  $R(X)$ .

**Theorem 1.1** Let  $\alpha \in R(X)$ . Then  $\alpha$  is left cancellative if and only if  $\text{ran } \alpha_* = X$ .

**Proof** Suppose that  $\alpha$  is a left cancellative element of  $R(X)$ . Claim that  $\text{ran } \alpha_* = X$ , suppose not. Then there exists  $x \in X \setminus \text{ran } \alpha_*$ . Let  $z$  and  $w$  be two distinct elements in  $X$ . Define  $\beta, \gamma \in R(X)$  by  $\beta = \{(x,w)\}$  and  $\gamma = \{(x,z)\}$ . Then  $\alpha\beta = \emptyset = \alpha\gamma$ . Since  $\alpha$  is left cancellative, we deduce that  $\beta = \gamma$ , a contradiction. So we have the claim. If  $X_\alpha = X$ , then  $\text{ran } \alpha_* = X$ . Assume that  $X_\alpha \neq X$ . To show that  $\text{ran } \alpha_* = X$ , suppose not. Then there exists  $z \in X \setminus \text{ran } \alpha_*$ . Since  $\text{ran } \alpha = X$ , there exists  $x \in X$ ,  $(x,z) \in \alpha$ . Since  $z \notin \text{ran } \alpha_*$ , it follows that  $x \notin X_\alpha$ . Then there exists  $w \in X$  such that  $w \neq z$  and  $(x,w) \in \alpha$ . Let  $y \in X$  be such that  $(y,z) \in \alpha$ . Similarly, we obtain that there exists  $u \in X$  such that  $u \neq z$  and  $(y,u) \in \alpha$ . Let  $\beta, \gamma \in R(X)$  be such that  $\beta = X \times \{y\}$  and  $\gamma = \beta \setminus \{(z,v)\}$  for some  $v \in X$ . It is not difficult to verify that  $\alpha\beta = \alpha\gamma$ . By assumption, we have  $\beta = \gamma$  which is a contradiction. Hence  $\text{ran } \alpha_* = X$ .

Conversely, suppose that  $\text{ran } \alpha_* = X$ . Let  $\beta, \gamma \in R(X)$  be such that  $\alpha\beta = \alpha\gamma$ . Let  $(x, y) \in \beta$ . Since  $\text{ran } \alpha_* = X$ , there exists  $z \in X_\alpha$  such that  $(z, x) \in \alpha$ . Thus  $(z, y) \in \alpha\beta$  so  $(z, y) \in \alpha\gamma$ . We then have that  $(z, w) \in \alpha$  and  $(w, y) \in \gamma$  for some  $w \in X$ . Since  $z \in X_\alpha$ , we deduce that  $x = w$ . Thus  $(x, y) \in \gamma$ , so  $\beta \subseteq \gamma$ . In a similarly way one can verify that  $\gamma \subseteq \beta$ . Therefore  $\beta = \gamma$  as desired.  $\square$

The following examples illustrate the applications of Theorem 1.1.

**Example 1.** Let  $\alpha \in R(Z)$  be defined by

$$\alpha = \{(2n, n), (2n + 1, n), (2n + 1, n + 1) \mid n \in Z\}.$$

We have that  $\alpha_* = \{(2n, n) \mid n \in Z\}$  and  $\text{ran } \alpha_* = X$ . By theorem 1.1, we deduce that  $\alpha$  is a left cancellative element of  $R(Z)$ .  $\square$

**Example 2.** Let  $\alpha \in R(N)$  be defined by

$$\alpha = \{(n, n), (n, n + 1) \mid n \in Z\}.$$

It is easy to see that  $\alpha$  is onto, but  $\alpha_* = \emptyset$ . Then  $\alpha$  is not a left cancellative element of  $R(N)$ .  $\square$

If  $\alpha \in R(X)$  is a mapping, we have  $X_\alpha = \text{dom } \alpha$  and  $\alpha_* = \alpha$ . The following corollaries are immediately consequence of Theorem 1.1.

**Corollary 1.2** Let  $\alpha \in P(X)$ . Then  $\alpha$  is left cancellative if and only if  $\alpha$  is onto.

**Corollary 1.3** Let  $\alpha \in T(X)$ . Then  $\alpha$  is left cancellative if and only if  $\alpha$  is onto.

For  $\alpha \in R(N)$ , denote

$$X^\alpha = \{y \in X \mid |\alpha y| = 1\} \text{ and } \alpha^* = \{(x, y) \in \alpha \mid y \in X^\alpha\}$$

Next, to characterize the right cancellative of element of  $R(X)$ .

**Theorem 1.4** Let  $\alpha \in R(X)$ . Then  $\alpha$  is right cancellative if and only if  $\text{dom } \alpha^* = X$ .

**Proof** Suppose that  $\alpha$  is a right cancellative element of  $R(X)$ . Claim that  $\text{dom } \alpha^* = X$ , suppose not. Then there exists  $x \in X \setminus \text{dom } \alpha^*$ . Let  $z$  and  $w$  be two distinct elements in  $X$ . Define  $\beta, \gamma \in R(X)$  by  $\beta = \{(w, x)\}$  and  $\gamma = \{(z, x)\}$ . Then  $\beta\alpha = \emptyset = \gamma\alpha$ . Since  $\alpha$  is left cancellative, we deduce that  $\beta = \gamma$ , a contradiction. Thus we have the claim. If  $X^\alpha = X$ , then  $\text{dom } \alpha^* = \text{dom } \alpha = X$ . Assume that  $X^\alpha \neq X$ . To show that  $\text{dom } \alpha^* = X$ , suppose not. Then there exists  $z \in X \setminus \text{dom } \alpha^*$ . Since  $\text{dom } \alpha = X$ , there exists  $x \in X$ ,  $(z, x) \in \alpha$ . Since  $z \notin \text{dom } \alpha^*$ ,  $(z, x) \notin \alpha^*$ . We then have  $x \notin X^\alpha$ , so there exists  $w \in X$  such that  $w \neq z$  and  $(w, x) \in \alpha$ . Let  $y \in X$  be such that  $(z, y) \in \alpha$ . Similarly, we obtain that there exists  $u \in X$  such that  $u \neq z$  and  $(u, y) \in \alpha$ . Define  $\beta, \gamma \in R(X)$  by  $\beta = \{v\} \times X$  and  $\gamma = \beta \setminus \{(v, z)\}$  for some  $v \in X$ . It's easy to verify that  $\beta\alpha = \gamma\alpha$ . By assumption, we have  $\beta = \gamma$  which is a contradiction. Therefore  $\text{dom } \alpha^* = X$ .

Conversely, suppose that  $\text{dom } \alpha^* = X$ . Let be such that  $\beta, \gamma \in R(X)$ . Let  $(x, y) \in \beta$ . Since  $\text{dom } \alpha^* = X$ , there exists  $z \in X^\alpha$  such that  $(y, z) \in \alpha^*$ . Thus  $(x, z) \in \beta\alpha$ , so  $(x, z) \in \gamma\alpha$ . We then have that  $(w, z) \in \alpha$  and  $(x, w) \in \gamma$  for some  $w \in X$ . Since  $z \in X^\alpha$ , we deduce that  $w = y$ . Thus  $(x, y) \in \gamma$ , so  $\beta \subseteq \gamma$ . In a similarly way one can verify that  $\gamma \subseteq \beta$ . Hence  $\beta = \gamma$ . The proof is now complete.  $\square$

We now illustrate the use of Theorem 1.4.

**Example 3.** Let  $\alpha \in R(Z)$  be such that

$$\alpha = \{(n, n + 1), (n, 1) \mid n \in Z\}$$

We then have  $\alpha_* = \{(n, n + 1) \mid n \in Z\}$  and  $\text{dom } \alpha^* = X$  which satisfy Theorem 1.4.

Hence  $\alpha$  is a right cancellative element of  $R(Z)$ .  $\square$

The following corollaries are clearly obtained from Theorem 1.4.

**Corollary 1.5** Let  $\alpha \in P(X)$ . Then  $\alpha$  is right cancellative if and only if  $\alpha$  is one-to-one and  $\text{dom } \alpha = X$ .

**Corollary 1.6** Let  $\alpha \in T(X)$ . Then  $\alpha$  is right cancellative if and only if  $\alpha$  is one-to-one.

For each  $\alpha \in R(X)$  we define  $\alpha^{-1}$  the inverse of  $\alpha$  by

$$\alpha^{-1} = \{(x, y) \in X \times X \mid (y, z) \in \alpha\}.$$

It now follows easily that  $x\alpha = \alpha^{-1}x$  for all  $x \in X$ . We have

**Corollary 1.7** Let  $\alpha \in R(X)$ . Then  $\alpha$  is left cancellative if and only if  $\alpha^{-1}$  is right cancellative.

**References**

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