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Left Cancellative and Right Cancellative Elements of Semigroup of Relations

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Abstract: Let $R(X)$ denote the set of all relations on a set X. Then $R(X)$ becomes a semigroup under composition. The purpose of this paper is to describe the element of $R(X)$ which is left cancellative and right cancellative.

1. Introduction

An element a of a semigroup S is said to be left cancellative if for all $b,c \in S$, $ab = ac$ implies that $b = c$ and *right cancellative* if for all $b,c \in S$, $ba = ca$ implies that $b=c$

For a nonempty set X a subset a of X x X is called a (binary) *relation on*. A binary operation on the set $R(X)$ consisting of all relations on is defined by

$$\alpha\beta = \{(x,y) \in X \times X \mid (x,z) \in \alpha \text{ and } (z,y) \in \beta\}$$

Then $R(X)$ becomes a semigroup under this multiplication. We call $R(X)$ the *semigroup of relations on X.* For $\alpha \in R(X)$, we denote the domain of a by dom a that is,

$$\operatorname{dom} \alpha = \{x \in X \mid (x,y) \in \alpha \text{ for some } y \in X\}$$

and the *range of* α by ran α, that is,

$$\operatorname{ran} \alpha = \{y \in X \mid (x,y) \in \alpha \text{ for some } x \in X\}$$

If $x \in X$, we define

$$x\alpha = \{y \in X \mid (x,y) \in \alpha\} \text{ and}$$
$$\alpha x = \{y \in X \mid (y,x) \in \alpha\}.$$

As usual, $P(X)$ denote the *partial transformation on X*, the set of all mappings α whose dom α and ran α are subsets of X and $T(X)$ denote the *full transformation on X*, the set of all $\alpha \in P(X)$ with dom $\alpha = X$. Both $P(X)$ and $T(X)$ are subsemigroups of $R(X)$. Our main purpose, we give a necessary and sufficient condition for element of $R(X)$ to be left cancellative and right cancellative.

For the remainder of this paper, we assume that X is a nonempty set and $|X| \geq 2$.

2. Main Results

For $\alpha \in R(X)$, denote

$$X_\alpha = \{x \in X \mid |x\alpha| = 1\} \text{ and}$$
$$\alpha_* = \{(x,y) \in \alpha \mid x \in X_\alpha\}$$

The following result is to characterize a left cancellative element of $R(X)$.

Theorem 1.1 Let $\alpha \in R(X)$. Then a is left cancellative if and only if ran $\alpha_* = X$.

Proof Suppose that α is a left cancellative element of $R(X)$. Claim that ran $\alpha = X$, suppose not. Then there exists $x \in X \setminus \operatorname{ran} \alpha$. Let z and w be two distinct elements in X. Define $\beta,\gamma \in R(X)$ by $\beta = \{(x,w)\}$ and $\gamma = \{(x,z)\}$. Then $\alpha\beta = \varnothing = \alpha\gamma$. Since α is left cancellative, we deduce that $\beta = \gamma$, a contradiction. So we have the claim. If $X_\alpha = X$, then ran $\alpha_* = X$. Assume that $X_\alpha \neq X$. To show that ran $\alpha_* = X$, suppose not. Then there exists $z \in X \setminus \operatorname{ran} \alpha_*$. Since ran $\alpha = X$, there exists $x \in X$, $(x,z) \in \alpha$. Since $z \notin \operatorname{ran} \alpha_*$, it follows that $x \notin X_\alpha$. Then there exists $w \in X$ such that $w \neq z$ and $(x,w) \in \alpha$. Let $y \in X$ be such that $(y,z) \in \alpha$. Similarly, we obtain that there exists $u \in X$ such that $u \neq z$ and $(y,u) \in \alpha$. Let $\beta,\gamma \in R(X)$ be such that $\beta = X \times \{v\}$ and $\gamma = \beta \setminus \{(z,v)\}$ for some $v \in X$. It is not difficult to verify that $\alpha\beta = \alpha\gamma$. By assumption, we have $\beta = \gamma$ which is a contradiction. Hence ran $\alpha_* = X$.

Conversely, suppose that ran $\alpha_* = X$. Let $\beta,\gamma \in R(X)$ be such that $\alpha\beta = \alpha\gamma$. Let $(x,y) \in \beta$. Since ran $\alpha_* = X$, there exists $z \notin X_\alpha$ such that $(z,x) \in \alpha$. Thus $(z,y) \in \alpha\beta$ so $(z,y) \in \alpha\gamma$. We then have that $(z,w) \in \alpha$ and $(w,y) \in \gamma$ for some $w \in X$. Since $z \in X_\alpha$, we deduce that $x = w$. Thus $(x,y) \in \gamma$, so $\beta \subseteq \gamma$. In a similarly way one can verify that $\gamma \subseteq \beta$. Therefore $\beta = \gamma$ as desired. □

The following examples illustrate the applications of Theorem 1.1.

Example 1. Let $\alpha \in R(\mathrm{Z})$ be defined by

$$\alpha = \{(2n,n),(2n + 1,n),(2n + 1, n + 1) \mid n \in Z\}.$$

We have that $\alpha_* = \{(2n,n) \mid n \in Z\}$ and ran $\alpha_* = X$. By theorem 1.1, we deduce that α is a left cancellative element of $R(\mathrm{Z})$. □

Example 2. Let $\alpha \in R(\mathrm{N})$ be defined by

$$\alpha = \{(n,n),(n,n + 1) \mid n \in Z\}.$$

It is easy to see that α is onto, but $\alpha_* = \varnothing$. Then α is not a left cancellative element of $R(\mathrm{N})$. □

If $\alpha \in R(X)$ is a mapping, we have X_α = dom α and $\alpha_* = \alpha$. The following corollaries are immediately consequence of Theorem 1.1.

Corollary 1.2 Let $\alpha \in P(X)$. Then α is left cancellative if and only if α is onto.

Corollary 1.3 Let $\alpha \in T(X)$. Then α is left cancellative if and only if α is onto.

For $\alpha \in R(\mathrm{N})$, denote

$$X^\alpha = \{y \in X \mid |\alpha y| = 1\} \text{ and}$$
$$\alpha^* = \{(x,y) \in \alpha \mid y \in X^\alpha\}$$

Next, to characterize the right cancellative of element of $R(X)$.

Theorem 1.4 Let $\alpha \in R(X)$. Then α is right cancellative if and only if dom $\alpha^* = X$.

Proof Suppose that α is a right cancellative element of $R(X)$. Claim that dom α=X, suppose not. Then there exists $x \in X \setminus$ dom α. Let z and w be two distinct elements in X. Define $\beta,\gamma \in R(X)$ by $\beta = \{(w,x)\}$ and $\gamma = \{(z,x)\}$. Then $\beta\alpha = \varnothing = \gamma\alpha$. Since α is left cancellative, we deduce that $\beta = \gamma$, a contradiction. Thus we have the claim. If $X^\alpha = X$, then dom α^* = dom $\alpha = X$. Assume that $X^\alpha \neq X$. To show that dom $\alpha^* = X$, suppose not. Then there exists $z \in X \setminus$ dom α^*. Since dom $\alpha = X$, there exists $x \in X$, $(z,x) \in \alpha$. Since $z \notin$ dom α^*, $(z,x) \notin \alpha_*$. We then have $x \notin X^\alpha$, so there exists $w \in X$ such that $w \neq z$ and $(w,x) \in \alpha$. Let $y \in X$ be such that $(z,y) \in \alpha$. Similarly, we obtain that there exists $u \in X$ such that $u \neq z$ and $(u,y) \in \alpha$. Define $\beta,\gamma \in R(X)$ by $\beta = \{v\} \times X$ and $\gamma = \beta \setminus \{(v, z)\}$ for some $v \in X$. It's easy to verify that $\beta\alpha = \gamma\alpha$. By assumption, we have $\beta = \gamma$ which is a contradiction. Therefore dom $\alpha^* = X$.

Conversely, suppose that dom $\alpha^* = X$. Let be such that $\beta,\gamma \in R(X)$. Let $(x,y) \in \beta$. Since dom $\alpha^* = X$, there exists $z \in X^\alpha$ such that $(y,z) \in \alpha^*$. Thus $(x,z) \in \beta\alpha$, so $(x,z) \in \gamma\alpha$. We then have that $(w,z) \in \alpha$ and $(x,w) \in \gamma$ for some $w \in X$. Since $z \in X^\alpha$, we deduce that $w = y$. Thus $(x,y) \in \gamma$, so $\beta \subseteq \gamma$. In a similarly way one can verify that $\gamma \subseteq \beta$. Hence $\beta = \gamma$. The proof is now complete. □

We now illustrate the use of Theorem 1.4.

Example 3. Let $\alpha \in R(\mathrm{Z})$ be such that

$$\alpha = \{(n,n + 1),(n,1) \mid n \in Z\}$$

We then have $\alpha_* = \{(n,n + 1) \mid n \in Z\}$ and dom $\alpha^* = X$ which satisfy Theorem 1.4.
Hence α is a right cancellative element of $R(\mathrm{Z})$. □

The following corollaries are clearly obtained from Theorem 1.4.

Corollary 1.5 Let $\alpha \in P(X)$. Then α is right cancellative if and only if α is one-to-one and dom $\alpha = X$.

Corollary 1.6 Let $\alpha \in T(X)$. Then α is right cancellative if and only if α is one-to-one.

For each $\alpha \in R(X)$ we define α^{-1} the *inverse of* α by

$$\alpha^{-1} = (x,y) \in X \times X \mid (y,z) \in \alpha \}.$$

It now follows easily that $x\alpha = \alpha^{-1}x$ for all $x \in X$. We have

Corollary 1.7 Let $\alpha \in R(X)$. Then α is left cancellative if and only if α^{-1} is right cancellative.

References

[1] Higgins, P. M. 1992. Techniques of Semigroup Theory, New York, Oxford University Press.

[2] Howie, J. M. 1995. Fundamentals of Semigroup Theory, Clarendon Press, Oxford.