

Left Cancellative and Right Cancellative Elements of Semigroup of Relations

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Abstract: Let $R(X)$ denote the set of all relations on a set X . Then $R(X)$ becomes a semigroup under composition. The purpose of this paper is to describe the element of $R(X)$ which is left cancellative and right cancellative.

1. Introduction

An element a of a semigroup S is said to be left cancellative if for all $b, c \in S$, $ab = ac$ implies that $b = c$ and right cancellative if for all $b, c \in S$, $ba = ca$ implies that $b = c$

For a nonempty set X a subset a of $X \times X$ is called a (binary) relation on X . A binary operation on the set $R(X)$ consisting of all relations on X is defined by

$$\alpha\beta = \{(x, y) \in X \times X \mid (x, z) \in \alpha \text{ and } (z, y) \in \beta\}$$

Then $R(X)$ becomes a semigroup under this multiplication. We call $R(X)$ the semigroup of relations on X . For $\alpha \in R(X)$, we denote the domain of α by $\text{dom } \alpha$ that is,

$$\text{dom } \alpha = \{x \in X \mid (x, y) \in \alpha \text{ for some } y \in X\}$$

and the range of α by $\text{ran } \alpha$, that is,

$$\text{ran } \alpha = \{y \in X \mid (x, y) \in \alpha \text{ for some } x \in X\}$$

If $x \in X$, we define

$$x\alpha = \{y \in X \mid (x, y) \in \alpha\} \text{ and} \\
\alpha x = \{y \in X \mid (y, x) \in \alpha\}.$$

As usual, $P(X)$ denote the partial transformation on X , the set of all mappings α whose $\text{dom } \alpha$ and $\text{ran } \alpha$ are subsets of X and $T(X)$ denote the full transformation on X , the set of all $\alpha \in P(X)$ with $\text{dom } \alpha = X$. Both $P(X)$ and $T(X)$ are subsemigroups of $R(X)$. Our main purpose, we give a necessary and sufficient condition for element of $R(X)$ to be left cancellative and right cancellative.

For the remainder of this paper, we assume that X is a nonempty set and $|X| \geq 2$.

2. Main Results

For $\alpha \in R(X)$, denote

$$X_\alpha = \{x \in X \mid |x\alpha| = 1\} \text{ and} \\
\alpha_* = \{(x, y) \in \alpha \mid x \in X_\alpha\}$$

The following result is to characterize a left cancellative element of $R(X)$.

Theorem 1.1 Let $\alpha \in R(X)$. Then α is left cancellative if and only if $\text{ran } \alpha_* = X$.

Proof Suppose that α is a left cancellative element of $R(X)$. Claim that $\text{ran } \alpha_* = X$, suppose not. Then there exists $x \in X \setminus \text{ran } \alpha_*$. Let z and w be two distinct elements in X . Define $\beta, \gamma \in R(X)$ by $\beta = \{(x, w)\}$ and $\gamma = \{(x, z)\}$. Then $\alpha\beta = \emptyset = \alpha\gamma$. Since α is left cancellative, we deduce that $\beta = \gamma$, a contradiction. So we have the claim. If $X_\alpha = X$, then $\text{ran } \alpha_* = X$. Assume that $X_\alpha \neq X$. To show that $\text{ran } \alpha_* = X$, suppose not. Then there exists $z \in X \setminus \text{ran } \alpha_*$. Since $\text{ran } \alpha = X$, there exists $x \in X$, $(x, z) \in \alpha$. Since $z \notin \text{ran } \alpha_*$, it follows that $x \notin X_\alpha$. Then there exists $w \in X$ such that $w \neq z$ and $(x, w) \in \alpha$. Let $y \in X$ be such that $(y, z) \in \alpha$. Similarly, we obtain that there exists $u \in X$ such that $u \neq z$ and $(y, u) \in \alpha$. Let $\beta, \gamma \in R(X)$ be such that $\beta = X \times \{y\}$ and $\gamma = \beta \setminus \{(z, y)\}$ for some $y \in X$. It is not difficult to verify that $\alpha\beta = \alpha\gamma$. By assumption, we have $\beta = \gamma$ which is a contradiction. Hence $\text{ran } \alpha_* = X$.

Conversely, suppose that $\text{ran } \alpha_* = X$. Let $\beta, \gamma \in R(X)$ be such that $\alpha\beta = \alpha\gamma$. Let $(x, y) \in \beta$. Since $\text{ran } \alpha_* = X$, there exists $z \in X_\alpha$ such that $(z, x) \in \alpha$. Thus $(z, y) \in \alpha\beta$ so $(z, y) \in \alpha\gamma$. We then have that $(z, w) \in \alpha$ and $(w, y) \in \gamma$ for some $w \in X$. Since $z \in X_\alpha$, we deduce that $x = w$. Thus $(x, y) \in \gamma$, so $\beta \subseteq \gamma$. In a similarly way one can verify that $\gamma \subseteq \beta$. Therefore $\beta = \gamma$ as desired. \square

The following examples illustrate the applications of Theorem 1.1.

Example 1. Let $\alpha \in R(Z)$ be defined by

$$\alpha = \{(2n, n), (2n + 1, n), (2n + 1, n + 1) \mid n \in Z\}.$$

We have that $\alpha_* = \{(2n, n) \mid n \in Z\}$ and $\text{ran } \alpha_* = X$. By theorem 1.1, we deduce that α is a left cancellative element of $R(Z)$. \square

Example 2. Let $\alpha \in R(N)$ be defined by

$$\alpha = \{(n, n), (n, n + 1) \mid n \in Z\}.$$

It is easy to see that α is onto, but $\alpha_* = \emptyset$. Then α is not a left cancellative element of $R(N)$. \square

If $\alpha \in R(X)$ is a mapping, we have $X_\alpha = \text{dom } \alpha$ and $\alpha_* = \alpha$. The following corollaries are immediately consequence of Theorem 1.1.

Corollary 1.2 Let $\alpha \in P(X)$. Then α is left cancellative if and only if α is onto.

Corollary 1.3 Let $\alpha \in T(X)$. Then α is left cancellative if and only if α is onto.

For $\alpha \in R(N)$, denote

$$X^\alpha = \{y \in X \mid |\alpha y| = 1\} \text{ and } \alpha^* = \{(x, y) \in \alpha \mid y \in X^\alpha\}$$

Next, to characterize the right cancellative of element of $R(X)$.

Theorem 1.4 Let $\alpha \in R(X)$. Then α is right cancellative if and only if $\text{dom } \alpha^* = X$.

Proof Suppose that α is a right cancellative element of $R(X)$. Claim that $\text{dom } \alpha^* = X$, suppose not. Then there exists $x \in X \setminus \text{dom } \alpha^*$. Let z and w be two distinct elements in X . Define $\beta, \gamma \in R(X)$ by $\beta = \{(w, x)\}$ and $\gamma = \{(z, x)\}$. Then $\beta\alpha = \emptyset = \gamma\alpha$. Since α is left cancellative, we deduce that $\beta = \gamma$, a contradiction. Thus we have the claim. If $X^\alpha = X$, then $\text{dom } \alpha^* = \text{dom } \alpha = X$. Assume that $X^\alpha \neq X$. To show that $\text{dom } \alpha^* = X$, suppose not. Then there exists $z \in X \setminus \text{dom } \alpha^*$. Since $\text{dom } \alpha = X$, there exists $x \in X$, $(z, x) \in \alpha$. Since $z \notin \text{dom } \alpha^*$, $(z, x) \notin \alpha^*$. We then have $x \notin X^\alpha$, so there exists $w \in X$ such that $w \neq z$ and $(w, x) \in \alpha$. Let $y \in X$ be such that $(z, y) \in \alpha$. Similarly, we obtain that there exists $u \in X$ such that $u \neq z$ and $(u, y) \in \alpha$. Define $\beta, \gamma \in R(X)$ by $\beta = \{v\} \times X$ and $\gamma = \beta \setminus \{(v, z)\}$ for some $v \in X$. It's easy to verify that $\beta\alpha = \gamma\alpha$. By assumption, we have $\beta = \gamma$ which is a contradiction. Therefore $\text{dom } \alpha^* = X$.

Conversely, suppose that $\text{dom } \alpha^* = X$. Let be such that $\beta, \gamma \in R(X)$. Let $(x, y) \in \beta$. Since $\text{dom } \alpha^* = X$, there exists $z \in X^\alpha$ such that $(y, z) \in \alpha^*$. Thus $(x, z) \in \beta\alpha$, so $(x, z) \in \gamma\alpha$. We then have that $(w, z) \in \alpha$ and $(x, w) \in \gamma$ for some $w \in X$. Since $z \in X^\alpha$, we deduce that $w = y$. Thus $(x, y) \in \gamma$, so $\beta \subseteq \gamma$. In a similarly way one can verify that $\gamma \subseteq \beta$. Hence $\beta = \gamma$. The proof is now complete. \square

We now illustrate the use of Theorem 1.4.

Example 3. Let $\alpha \in R(Z)$ be such that

$$\alpha = \{(n, n + 1), (n, 1) \mid n \in Z\}$$

We then have $\alpha_* = \{(n, n + 1) \mid n \in Z\}$ and $\text{dom } \alpha^* = X$ which satisfy Theorem 1.4.

Hence α is a right cancellative element of $R(Z)$. \square

The following corollaries are clearly obtained from Theorem 1.4.

Corollary 1.5 Let $\alpha \in P(X)$. Then α is right cancellative if and only if α is one-to-one and $\text{dom } \alpha = X$.

Corollary 1.6 Let $\alpha \in T(X)$. Then α is right cancellative if and only if α is one-to-one.

For each $\alpha \in R(X)$ we define α^{-1} the inverse of α by

$$\alpha^{-1} = \{(x, y) \in X \times X \mid (y, z) \in \alpha\}.$$

It now follows easily that $x\alpha = \alpha^{-1}x$ for all $x \in X$. We have

Corollary 1.7 Let $\alpha \in R(X)$. Then α is left cancellative if and only if α^{-1} is right cancellative.

References

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