

A Study of the Zarisky Topology through Functors

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Abstract: The aim of this paper is to study the **Zarisky topology** through a functor. In this paper we construct

- i) **Zarisky topology** in K^n and a functor associated with this topology ;
- ii) $\text{Spec}(R)$ is a **Zarisky topology** in a ring R , where $\text{Spec}(R)$ denotes the set of all prime ideals of R and construct a functor '**Spec**' associated with **Zarisky topology**; and finally
- iii) we study the functor '**Spec**' associated with **Zarisky topology**

1. Introduction

The concept of affine algebraic sets plays an important role to construct Zarisky topology. In this section we give some basic definitions.

Definition 1.1 Let $K[x_1, x_2, x_3, \dots, x_n]$ denotes the polynomial ring over an arbitrary field K in n -variables and $f_1, f_2, f_3, \dots, f_m \in K[x_1, x_2, x_3, \dots, x_n]$. The function $V: S \rightarrow K^n$ defined by $V(S) = \{(x) \in K^n : f(x) = 0, \forall f \in S\}$, where $S = \{f_1, f_2, f_3, \dots, f_m\} \subset K[x_1, x_2, x_3, \dots, x_n]$ and $(x) = (x_1, x_2, x_3, \dots, x_n) \in K^n$ is called the algebraic set i.e., the algebraic set $V(S)$ is the set of solutions in K^n of the system of equations:

$$f_1(x_1, x_2, x_3, \dots, x_n) = 0, f_2(x_1, x_2, x_3, \dots, x_n) = 0, f_3(x_1, x_2, x_3, \dots, x_n) = 0, \dots, f_m(x_1, x_2, x_3, \dots, x_n) = 0.$$

Definition 1.2 A subset A in K^n is called an affine algebraic set if $A = V(S)$ for some $S \subset K[x]$. Thus any algebraic set A is defined by a finite set of polynomials in $K[x]$.

Definition 1.3 An algebraic set A in K^n is called irreducible or an affine variety iff $A \neq B \cup C$, where B and C are algebraic sets in K^n and $A \neq B, A \neq C$.

Definition 1.4 Let k be a subfield of K . If A is an affine algebraic set in K^n admits a set of generators in $k[x_1, x_2, x_3, \dots, x_n] \subset K[x_1, x_2, x_3, \dots, x_n]$, then A is called an affine (K, k) algebraic set and k is called the field of definition of A . Thus an affine (K, k) algebraic set A is a subset in K^n consisting of all common zeros of a subset of polynomials in $k[x_1, x_2, x_3, \dots, x_n]$.

If $k = K$, we call A is an absolute affine algebraic set in K^n .

Definition 1.5 A category \mathcal{C} consists of

- a) a class of objects X, Y, \dots , denoted by $\text{Ob}(\mathcal{C})$;
- b) for each ordered pair of objects X, Y a set of morphisms with domain X and range Y denoted by $\mathcal{C}(X, Y)$;
- c) for each ordered triple of objects X, Y and Z and a pair of morphisms; $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, their composite is denoted by $gf : X \rightarrow Z$, satisfying the two axioms:
 - i) associativity
 - ii) identity

Definition 1.6 Let \mathcal{C} and \mathcal{D} be two categories. A contravariant functor T from \mathcal{C} to \mathcal{D} consists of

- a) an object function which assigns to every object X of \mathcal{C} and object $T(X)$ of \mathcal{D} ; and
- b) a morphism function which assigns to every morphism $f : X \rightarrow Y$ in \mathcal{C} , a morphism $T(f) : T(Y) \rightarrow T(X)$ in \mathcal{D} such that

- i) $T(fg) = T(f)T(g)$,
- ii) $T(gf) = T(f)T(g)$, for $g : Y \rightarrow Z$ in \mathcal{C}

Definition 1.7 Let \mathcal{C} and \mathcal{D} be categories. Suppose T_1 and T_2 are both contravariant functors from \mathcal{C} to \mathcal{D} . A natural transformation Φ from T_1 to T_2 is a function from the objects of \mathcal{C} to the morphisms of \mathcal{D} such that for every morphism $f : X \rightarrow Y$ in \mathcal{C} the following condition holds:

$$\Phi(X) T_1(f) = T_2(f) \Phi(Y)$$

2. Elementary properties

In this section we give some basic results which are essential in the sequel.

Lemma 2.1 A subset U of K^n is an open set iff $K^n - U$ is an affine k – algebraic set. Thus an affine k -algebraic sets in K^n are the closed sets in K^n .

Proof: Using [4] and [5], it follows.

Lemma 2.2 Any subset A of K^n is an affine k -variety, then a subset A is closed in K^n iff it is an affine k -algebraic set.

Proof: Using **Definition 1.2**, **Definition 1.4** and **Lemma 2.1**, it follows.

Lemma 2.3 Any finite subset of K^n is an algebraic set.

Proof: Using [4, art.5.5], it follows.

Lemma 2.4 An algebraic set A in K^n is affine variety iff $I(A)$ is a prime ideal.

Proof: Using [4, art.5.5], it follows.

Lemma 2.5 Let C be any category and T be a contravariant functor from C to S (category of sets and functions). Then for any object C in C , there is an equivalence

$\Omega: (h^C, T) \rightarrow T(C)$, where (h^C, T) is the class of natural transformations from the set valued functor h^C to the set valued functor T such that Ω is natural in C and T .

Proof: From [4, appendix B], it follows.

Lemma 2.6 For each subset P of a ring R , let $X = V(P)$ denotes the set of all prime ideals of R which contains P , then

i) $V(0) = X$ and $V(1) = \emptyset$,

ii) If $(P_i)_{i \in I}$ is any family of subsets of prime ideals of R , then

$$V\left(\bigcup_{i \in I} P_i\right) = \bigcap_{i \in I} V(P_i)$$

iii) $V(P_1) \cup V(P_2) = V(P_1 \cap P_2) = V(P_1 P_2)$.

Proof: Using [4, art.5.5] , it follows.

This shows that the set $V(P)$ satisfies all the axioms for closed sets in a topological space.

Lemma 2.7 (Yoneda’s Lemma) Let C be any category and S be the category of sets and functions.

Let $T : C \rightarrow S$ be a contravariant functor. Then for any C in C , there is an equivalence

$\Psi: (h^C, T) \rightarrow T(C)$, where (h^C, T) is the class of natural transformations from the set valued functor h^C to the set valued functor T such that Ψ is natural in C and T .

3. Functor associated with Zarisky topology

In this section we construct and investigate the **Zarisky topology**. To do this we prove the following:

Theorem 3.1 All affine k -algebraic sets in K^n are the closed sets in K^n . Then this sets form a topology in K^n . This topology is called the **Zarisky topology** in K^n .

Proof: Since empty set and whole set are closed sets; the intersection of any family of closed sets is a closed set; and the union of two closed set is a closed set. Using **Lemma 2.2**, it follows.

Let R be a ring. Define prime spectrum **Spec**(R) of R by $X = \text{Spec}(R) = \{P : P \text{ is prime ideal of } R\}$.

Theorem 3.2 Let **Spec**(R) denotes the set of all prime ideals of a ring R . All subsets in the set of all prime ideals of a ring R , form a topology **Spec**(R), is called **Zarisky topology**.

Proof: Using **Lemma 2.2**, **Lemma 2.5**, **Lemma 2.7** & **Theorem 3.1**, it follows.

Theorem 3.3 Let \mathbf{R} be the category of rings and ring homomorphisms and \mathbf{T} denote the category of sets and functions. Then **Spec**: $\mathbf{R} \rightarrow \mathbf{T}$ is a contravariant functor

Proof: Let R and T are rings in \mathbf{R} and $f: R \rightarrow T$ is a ring homomorphism in \mathbf{R} . Let $X = \text{Spec}(R)$ and $Y = \text{Spec}(T)$. Define $f^*: Y \rightarrow X$ by $f^*(Q) = f^{-1}(Q)$, $\forall Q \in Y$, then $f^{-1}(Q)$ is a prime ideal of R and hence $f^{-1}(Q) \in X$.

Let $f : R \rightarrow T$ and $g : T \rightarrow M$ be a ring homomorphisms in \mathbf{R} . Then $gf : R \rightarrow M$ is also a ring homomorphism.

Therefore **Spec**(gf) : **Spec**(M) \rightarrow **Spec**(R) by

$$\begin{aligned}\text{Spec}(gf)(Q) &= (gf)^{-1}(Q), \forall Q \in \text{Spec}(M) \\ &= (f^{-1}g^{-1})(Q) \\ &= \text{Spec}(f)(\text{Spec}(g)(Q))\end{aligned}$$

$$\Rightarrow \text{Spec}(gf)(Q) = (\text{Spec}(f) \text{Spec}(g))(Q), \forall Q \in \text{Spec}(M).$$

$$\Rightarrow \text{Spec}(gf) = \text{Spec}(f) \text{Spec}(g)$$

Also, $\text{Spec}(I_R) = I_{\text{Spec}(R)}$, where $I_R : R \rightarrow R$ in \mathbf{R} .

Using **Lemma 2.4**, we say that the set of all subvarieties of A has a bijective correspondence with the set of all prime ideals of the ring $\mathbf{R}[A]$, where $\mathbf{R}[A]$ denotes the set of all polynomial functions on A .

Let $\mathbf{h}^{\mathbf{R}[A]}(\mathbf{R}) = \text{Hom}(\mathbf{R}, \mathbf{R}[A])$.

We define for each $f: R \rightarrow T$ in \mathbf{R} , $\mathbf{h}^{\mathbf{R}[A]}(f) = \text{Hom}(T, \mathbf{R}[A]) \rightarrow \text{Hom}(R, \mathbf{R}[A])$ by
 $\mathbf{h}^{\mathbf{R}[A]}(f)(\alpha) = \alpha \circ f, \forall \alpha \in \text{Hom}(T, \mathbf{R}[A])$.

Theorem 3.4 $\mathbf{h}^{\mathbf{R}[A]}: \mathbf{R} \rightarrow \mathbf{T}$ is a contravariant functor.

Proof: From [5], it follows

Thus we have two contravariant functors **Spec** and $\mathbf{h}^{\mathbf{R}[A]}$ from the category \mathbf{R} to the category \mathbf{T} .

Now we have the following **Theorems**:

Theorem 3.5 For each algebraic set A in K^n , there is an equivalence $\Omega : (\text{Spec}, \mathbf{h}^{\mathbf{R}[A]}) \rightarrow \text{Spec}(\mathbf{R}[A])$, where $(\text{Spec}, \mathbf{h}^{\mathbf{R}[A]})$ is the set of all natural transformations from the contravariant functor 'Spec' to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$.

Proof: Using Yoneda's Lemma, it follows.

Corollary 3.6 For each set A in **Zarisky topology** in K^n , there is an equivalence from the set of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$ to the **Zarisky topology** 'Spec($\mathbf{R}[A]$)'

Proof: Using **Lemma 2.4**, **Lemma 2.8**, **Theorem 3.1**, **Theorem 3.2**, **Theorem 3.3** and **Theorem 3.4**, it follows.

Theorem 3.7 For an affine k -algebraic set A in K^n , the set of all k -subvarieties in K^n is equipotent to the set $(\text{Spec}, \mathbf{h}^{\mathbf{R}[A]})$ of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$.

Proof: From [5], it follows.

Corollary 3.8 For each set A of the **Zarisky topology** in K^n is equipotent to the set of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$.

Proposition 3.9 The set $(\text{Spec}, \mathbf{h}^{\mathbf{R}[A]})$ of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$ forms a **Zarisky topology** in K^n .

Proof: Using **Theorem 3.1** and **Theorem 3.7** and **Corollary 3.8**, it follows.

Proposition 3.10

- i) The **Zarisky topology** in K^n is not Hausdorff;
- ii) The **Zarisky topology** in K^n may not be T1 unless $K = k$

Proof:

i) Let A be a k -variety in K^n . Then for any two nonempty open sets U_1 and U_2 in A such that $U_1 \cap U_2 \neq \emptyset$, for if $U_1 \cap U_2 = \emptyset$, then $A = (A - U_1) \cup (A - U_2) \cup (U_1 \cap U_2)$ i.e.,

$A = (A - U_1) \cup (A - U_2)$ and hence A is the union of two affine k -algebraic sets $A - U_1$ and $A - U_2$ which are different from A i.e., A is not a k -variety in K^n . Therefore, if x and y are distinct points of A it is not possible to find disjoint neighbourhoods U_1 and U_2 of x and y respectively. This implies that A is not a Hausdorff space. Therefore K^n cannot be a Hausdorff space, because every subspace of a Hausdorff space is a Hausdorff space.

ii) If $k = K$, then any point of K^n is an algebraic set and hence closed in the Zarisky topology. If $k \neq K$ and if $(x_1, x_2, x_3, \dots, x_n) \in K^n$ is not a zero of any polynomial in $k[x_1, x_2, x_3, \dots, x_n]$, then the point $(x_1, x_2, x_3, \dots, x_n)$ is not a k -algebraic set and hence not closed.

Corollary 3.11 Any open subset in the **Zarisky topology** in K^n is a connected dense subset.

Proof: Using Proposition 3.10, it follows.

Corollary 3.12 Any open subset in the **Zarisky topology** in K^n is connected.

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