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A Study of the Zarisky Topology through Functors

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Abstract: The aim of this paper is to study the **Zarisky topology** through a functor. In this paper we construct
i) **Zarisky topology** in K^n and a functor associated with this topology ;
ii) **Spec**(R) is a **Zarisky topology** in a ring R, where **Spec**(R) denotes the set of all prime ideals of R and construct a functor '**Spec'** associated with **Zarisky** topology; and finally
iii) we study the functor '**Spec'** associated with **Zarisky** topology

1. Introduction

The concept of affine algebraic sets plays an important role to construct Zarisky topology. In this section we give some basic definitions.

Definition 1.1 Let $K[x_1,x_2,x_3,\ldots,x_n]$ denotes the polynomial ring over an arbitrary field K in n-variables and $f_1,f_2,f_3,\ldots\ldots,f_m \in K[x_1,x_2,x_3,\ldots,x_n]$. The function $V:S \rightarrow K^n$ defined by $V(S) = \{(x)\in K^n : f(x) = 0, \forall f\in S\}$, where $S = \{f_1,f_2,f_3,\ldots\ldots,f_m\}\subset K[x_1,x_2,x_3,\ldots,x_n]$ and $(x) = (x_1,x_2,x_3,\ldots,x_n)\in K^n$ is called the algebraic set i.e., the algebraic set V(S) is the set of solutions in K^n of the system of equations:

$$f_1(x_1,x_2,x_3,\ldots,x_n) = 0,\ f_2(x_1,x_2,x_3,\ldots,x_n) = 0,\ f_3(x_1,x_2,x_3,\ldots,x_n) = 0,\ldots,\ f_m(x_1,x_2,x_3,\ldots,x_n) =0.$$

Definition 1.2 A subset A in K^n is called an affine algebraic set if A = V(S) for some $S\subset K[x]$. Thus any algebraic set A is defined by a finite set of polynomials in K[x].

Definition 1.3 An algebraic set A in K^n is called irreducible or an affine variety iff $A \neq B\cup C$, where B and C are algebraic sets in K^n and $A \neq B$, $A \neq C$.

Definition 1.4 Let k be a subfield of K. If A is an affine algebraic set in K^n admits a set of generators in $k[x_1,x_2,x_3,\ldots,x_n]\subset K[x_1,x_2,x_3,\ldots,x_n]$, then A is called an affine (K,k) algebraic set and k is called the field of definition of A. Thus an affine (K,k) algebraic set A is a subset in K^n consisting of all common zeros of a subset of polynomials in $k[x_1,x_2,x_3,\ldots,x_n]$.
If k = K, we call A is an absolute affine algebraic set in K^n.

Definition 1.5 A category ***C*** consists of
a) a class of objects X,Y,…,denoted by Ob(***C***);
b) for each ordered pair of objects X,Y a set of morphisms with domain X and range Y denoted by ***C***(X,Y);
c) for each ordered triple of objects X,Y and Z and a pair of morphisms; $f : X\rightarrow Y$ and $g : Y\rightarrow Z$, their composite is denoted by $gf : X\rightarrow Z$, satisfying the two axioms:
i) associativity ii) identity

Definition 1.6 Let ***C*** and ***D*** be two categories. A contravariant functor T from ***C*** to ***D*** consists of
a) an object function which assigns to every object X of ***C*** and object T(X) of ***D***; and
b) a morphism function which assigns to every morphism $f : X \rightarrow Y$ in ***C***, a morphism $T(f) : T(Y) \rightarrow T(X)$ in ***D*** such that
i) $T(IX) = IT(X)$,
ii) $T(g f) = T(f) T(g)$, for $g : Y\rightarrow Z$ in ***C***

Definition 1.7 Let **C** and **D** be categories. Suppose T_1 and T_2 are both contravariant functors from **C** to **D**. A natural transformation Φ from T_1 to T_2 is a function from the objects of **C** to the morphisms of **D** such that for every morphism $f : X\rightarrow Y$ in **C** the following condition holds:
$\Phi(X) T_1(f) = T_2(f) \Phi(Y)$

2. Elementary properties

In this section we give some basic results which are essential in the sequel.
Lemma 2.1 A subset U of K^n is an open set iff K^n - U is an affine k – algebraic set. Thus an affine k-algebraic sets in K^n are the closed sets in K^n.
Proof: Using [4] and [5], it follows.
Lemma 2.2 Any subset A of K^n is an affine k-variety, then a subset A is closed in K^n iff it is an affine k-algebraic set.
Proof: Using **Definition 1.2**, **Definition 1.4** and **Lemma 2.1**, it follows.
Lemma 2.3 Any finite subset of K^n is an algebraic set.
Proof: Using [4, art.5.5], it follows.
Lemma 2.4 An algebraic set A in K^n is affine variety iff I(A) is a prime ideal.
Proof: Using [4, art.5.5], it follows.
Lemma 2.5 Let **C** be any category and **T** be a contravariant functor from **C** to **S** (category of sets and functions). Then for any object C in **C**, there is an equivalence
Ω: $(\mathbf{h}^C,\mathbf{T}) \to \mathbf{T}(C)$, where $(\mathbf{h}^C,\mathbf{T})$ is the class of natural transformations from the set valued functor $\mathbf{h}^C$ to the set valued functor **T** such that Ω is natural in **C** and **T**.
Proof: From [4, appendix B], it follows.
Lemma 2.6 For each subset P of a ring R, let X = V(P) denotes the set of all prime ideals of R which contains P, then

i) V(0) = X and V(1) = $\varnothing$,

ii) If $(P_i)_{i\in I}$ is any family of subsets of prime ideals of R, then

$$V\left(\bigcup_{i\in I} P_i\right) = \bigcap_{i\in I} V(P_i)$$

iii) $V(P_1)\cup V(P_2) = V(P_1\cap P_2) = V(P_1 P_2)$.

Proof: Using [4, art.5.5] , it follows.
This shows that the set V(P) satisfies all the axioms for closed sets in a topological space.
Lemma 2.7 (Yoneda's Lemma) Let **C** be any category and **S** be the category of sets and functions. Let T : **C**$\to$ **S** be a contravariant functor. Then for any C in **C**, there is an equivalence
Ψ: $(h^C, T) \to T(C)$, where (h^C, T) is the class of natural transformations from the set valued functor h^C to the set valued functor T such that Ψ is natural in C and T.

3. Functor associated with Zarisky topology

In this section we construct and investigate the **Zarisky topology**. To do this we prove the following:
Theorem 3.1 All affine k-algebraic sets in Knare the closed sets in K^n. Then this sets form a topology in K^n. This topology is called the **Zarisky topology** in K^n.
Proof: Since empty set and whole set are closed sets; the intersection of any family of closed sets is a closed set; and the union of two closed set is a closed set. Using **Lemma 2.2,** it follows.
Let R be a ring. Define prime spectrum **Spec**(R) of R by X = **Spec**(R) = {P : P is prime ideal of R}.
Theorem 3.2 Let **Spec**(R) denotes the set of all prime ideals of a ring R. All subsets in the set of all prime ideals of a ring R, form a topology **Spec**(R), is called **Zarisky topology**.
Proof: Using **Lemma 2.2**, **Lemma 2.5, Lemma 2.7 & Theorem 3.1**, it follows.
Theorem 3.3 Let **R** be the category of rings and ring homomorphisms and **T** denote the category of sets and functions. Then **Spec:R$\to$T** is a contravariant functor
Proof: Let R and T are rings in **R** and f: R$\to$T is a ring homomorphism in **R**. Let X = **Spec(**R**)** and Y= **Spec(**T**).** Define f^*: Y$\to$X by $f^*(Q) = f^{-1}(Q)$, $\forall$ Q$\in$Y, then $f^{-1}(Q)$ is a prime ideal of R and hence $f^{-1}(Q)\in X$.
Let f : R$\to$T and g : T$\to$M be a ring homomorphisms in **R**. Then gf : R$\to$M is also a ring homomorphism.
Therefore **Spec**(gf) : **Spec**(M) $\to$ **Spec**(R) by

$\mathbf{Spec}(gf)(Q) = (gf)^{-1}(Q), \forall\ Q \in \mathbf{Spec}(M)$
$= (f^{-1}g^{-1})(Q)$
$= \mathbf{Spec}(f)\ ((\mathbf{Spec}(g)(Q))$
$\Rightarrow \mathbf{Spec}(gf)(Q) = (\mathbf{Spec}(f)\ \mathbf{Spec}(g))(Q), \forall\ Q \in \mathbf{Spec}(M).$
$\Rightarrow \mathbf{Spec}(gf) = \mathbf{Spec}(f)\ \mathbf{Spec}(g)$
Also, $\mathbf{Spec}(I_R) = I_{\mathbf{Spec}(R)}$, where $I_R : R \rightarrow R$ in **R**.
Using **Lemma 2.4**, we say that the set of all subvarieties of A has a bijective correspondence with the set of all prime ideals of the ring **R**[A], where **R**[A] denotes the set of all polynomial functions on A.
Let $h^{\mathbf{R}[A]}(\mathbf{R}) = \mathrm{Hom}(\mathbf{R}, \mathbf{R}[A])$.
We define for each f: $R \rightarrow T$ in **R**, $h^{\mathbf{R}[A]}(f) = \mathrm{Hom}(T, \mathbf{R}[A]) \rightarrow \mathrm{Hom}(R, R[A])$ by

$$h^{\mathbf{R}[A]}(f)(\alpha) = \alpha \text{ o } f,\ \forall\ \alpha \in \mathrm{Hom}(T, \mathbf{R}[A]).$$

Theorem 3.4 $h^{R[A]}$: $\mathbf{R} \rightarrow \mathbf{T}$ is a contravariant functor.
Proof: From [5], it follows
Thus we have two contravariant functors **Spec** and $\mathbf{h}^{\mathbf{R}[A]}$ from the category **R** to the category **T**.
Now we have the following **Theorems**:
Theorem 3.5 For each algebraic set A in K^n, there is an equivalence Ω **:** (**Spec,** $\mathbf{h}^{\mathbf{R}[A]}$) $\rightarrow$ **Spec**(R[A]), where (**Spec,** $\mathbf{h}^{\mathbf{R}[A]}$) is the set of all natural transformations from the contravariant functor '**Spec**' to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$**.**
Proof: Using Yoneda's Lemma, it follows.
Corollary 3.6 For each set A in **Zarisky topology** in K^n**,** there is an equivalence from the set of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$ to the **Zarisky topology** '**Spec(**R[A])'
Proof: Using **Lemma 2.4**, **Lemma 2.8**, **Theorem 3.1, Theorem 3.2**,**Theorem 3.3** and **Theorem 3.4**, it follows.
Theorem 3.7 For an affine k-algebraic set A in K^n, the set of all k-subvarieties in K^n is equipotent to the set (**Spec,** $\mathbf{h}^{\mathbf{R}[A]}$) of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$**.**
Proof: From [5], it follows.
Corollary 3.8 For each set A of the **Zarisky topology** in K^n is equipotent to the set of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$**.**
Proposition 3.9 The set (**Spec**, $\mathbf{h}^{\mathbf{R}[A]}$) of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{R}[A]}$ froms a **Zarisky topology** in K^n.
Proof: Using **Theorem 3.1** and **Theorem 3.7** and **Corollary 3.8**, it follows.
Proposition 3.10
i) The **Zarisky topology** in K^n is not Hausdorff;
ii) The **Zarisky topology** in K^n may not be T1 unless K= k
Proof:
i) Let A be a k-variety in K^n. Then for any two nonempty open sets U_1 and U_2 in A such that $U_1 \cap U_2 \neq \varnothing$, for if $U_1 \cap U_2 = \varnothing$, then $A = (A- U_1) \cup (A- U_2) \cup (U_1 \cap U_2)$ i.e.,
$A = (A-U_1) \cup (A-U_2)$ and hence A is the union of two affine k-algebraic sets $A- U_1$ and $A-U_2$ which are different from A i.e., A is not a k-variety in K^n. Therefore, if x and y are distinct points of A it is not possible to find disjoint neighbourhoods U_1 and U_2 of x and y respectively. This implies that A is not a Hausdorff space. Therefore K^n cannot be a Hausdorff space, because every subspace of a Hausdorff space is a Hausdorff space.
ii) If k = K, then any point of K^n is an algebraic set and hence closed in the Zarisky topology. If $k \neq K$ and if $(x_1,x_2,x_3,\ldots,x_n) \in K^n$ is not a zero of any polynomial in $k[x_1,x_2,x_3,\ldots,x_n]$, then the point $(x_1,x_2,x_3,\ldots,x_n)$ is not a k-algebraic set and hence not closed.
Corollary 3.11 Any open subset in the **Zarisky topology** in K^n is a connected dense subset.
Proof: Using Proposition 3.10, it follows.
Corollary 3.12 Any open subset in the **Zarisky topology** in K^n is connected.

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