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Novel Three Order Methods for Solving a System of Nonlinear Equations

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Abstract: In this paper, we suggest and study Simpson's formula, and Newton's two, three and four Cosed formulas iterative methods for solving the system of nonlinear equations by using Predictor-Corrector of Newton method. We present four new algorithms for solving the system of nonlinear equations (SNLE). We prove that these new algorithms have convergence. Several numerical examples are given to illustrate the efficiency and performance of the new iterative methods. These new algorithms may be viewed as an extensions and generalizations of the existing methods for solving the system of nonlinear equations.

1. Introduction

Recently, several iterative methods have been used to solve nonlinear equations and the system of nonlinear equations (SNLE) [1-6]. Ali et. al. [7] have developed a new predictor – corrector method for solving nonlinear equation by using the weight combination of the midpoint, and Trapezoidal quadrature formulas. Wang [8], using a third order family of Newton-Like iteration method for solving nonlinear equations; Noor et.al. [9] have considered an alternative decomposition technique which does not involve the derivative of the domain polynomial. Chun [10] has presented a new iterative method to solving nonlinear equations by using A domain decomposition. Yong et.al. [11] developed a new scheme for the construction of iterative methods for the solution of nonlinear equations and giving a new class of methods from any iterative method. Furthermore, several iterative methods have been developed for solving the system of nonlinear equations (SNLE) by using various techniques such as Newton's method, Revised Adomian decomposition method, homotopy perturbation method, Householder iterative method [12-18]. Darvishi et.al. [19-20] have constructed two new methods. The two methods are third order Newton type method and super cubic iterative method for solving (SNLE), H, He [21] and Golbabai et. al. [22] have applied the homotopy perturbation method to build a new family of Newton-like iterative methods for solving (SNLE). Mustafa [23] has considered a new decomposition method for solving (SNLE). Jank et.al. [24] developed two families of third-order methods for solving (SNLE). Finally, Hueso [25] presented a family of predictor–corrector methods free from second derivative for solving (SNLE).

In this paper, we aim to generalize and apply Simpson's formula , and Newton's two, three and four Cosed formulas to give four new algorithms for solving (SNLE). Finally, some numerical examples have been fulfilled with Maple software to demonstrate our method and comparison of our results with those derived from previous methods. All test problems reveal accuracy and fast convergence of our method.

2. System of Nonlinear Equations(SNLE)

The general form of a system of non-linear equations is

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_n) &= 0, \\ f_2(x_1, x_2, \cdots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \cdots, x_n) &= 0, \end{aligned} \tag{1}$$

where each function i f can be thought of as mapping a vector x = ($x_1, x_2,\ldots, x_n$) of the ndimensional space R^n, into the real line R. The system can alternatively be represented by defining a functional F, mapping R^n into R^n by :

$$F(x_1, x_2,\ldots, x_n) = f_1(x_1, x_2,\ldots, x_n),\ldots, f_n(x_1, x_2,\ldots, x_n)^T$$

Using vector notation to represent the variables $x_1, x_2,\ldots, x_n$ a system (1) can be written as the form:

$$F(\mathrm{x}) = 0 \tag{2}$$

The functions $f_1, f_2, \ldots, f_n$ are called the coordinate functions of F [26].

3. Iterative methods

Suppose that X be the simple zero of sufficiently differentiable functions and consider the numerical solution of the system of equations $F(\mathrm{x}) = 0$, where $F{:}D \subseteq R^n \rightarrow {}^n$ is a smooth mapping that has continuous second order partial derivatives on a convex open set D, and that has a locally unique root x in D, $F(\mathrm{x}) = (f_1(\mathrm{x}), f_2(\mathrm{x}),\ldots, f_n(\mathrm{x}))^T$, $\mathrm{x} = (x_1, x_2,\ldots, x_n)^T$ and $f_i : R^n$ is a nonlinear function, then,

$$F(\mathrm{x}) = F(\mathrm{x}_i) + \int_{x_i}^{x} F'(t)\,dt\,, \tag{3}$$

If we approximate the integral in equation (3) by using Simpson's methods formula for solving a definite integral $\int_{x_i}^{x} F'(t)\,dt$, are as follows [2], then

$$\int_{\mathrm{x}_i}^{\mathrm{x}} F'(t)dt = [F'(\mathrm{x}_i) + 4F'(\frac{\mathrm{x}_i + \mathrm{x}}{2}) + F'(\mathrm{x})]\frac{(\mathrm{x} - \mathrm{x}_i)}{3} \tag{4}$$

From (3) and (4), we have,

$$F(\mathrm{x}) = F(\mathrm{x}_i) + [F'(\mathrm{x}_i) + 4F'(\frac{\mathrm{x}_i + \mathrm{x}}{2}) + F'(\mathrm{x})]\frac{(\mathrm{x} - \mathrm{x}_i)}{3} \tag{5}$$

Since, $F(\mathrm{x}) = 0$ then

$$\mathrm{x} = \mathrm{x}_i - 6\,[F'(\mathrm{x}_i) + 4F'(\frac{\mathrm{x}_i + \mathrm{x}}{2}) + F'(\mathrm{x})]^{-1}F(\mathrm{x}_i) \tag{6}$$

which is an implicit method. To overcome this drawback, one usually uses the prediction and correction technique. With the formulation (6) and with selecting Predictor-Corrector of Newton method we will have followed a two-step iterative method for solving the system of nonlinear equation (2) as follows

Algorithm 1: For a given X_0, compute the approximate solution X_{i+1} by iterative scheme.

$$\mathrm{y}_i = \mathrm{x}_i - J^{-1}(\mathrm{x}_i)F(\mathrm{x}_i), \tag{7}$$

$$\mathrm{x}_{i+1} = \mathrm{x}_i - 6[F'(\mathrm{x}_i) + 4F'(\frac{\mathrm{x}_i + \mathrm{y}_i}{2}) + F'(\mathrm{y}_i)]^{-1}F(\mathrm{x}_i) \tag{8}$$

We can also approximate the integral in equation(3) by using another three point Newton-Cosed, formula [27], then

$$\int_{\mathrm{x}_i}^{\mathrm{x}} F'(t)dt = [2F'(\frac{3\mathrm{x}_i + \mathrm{x}}{4}) - F'(\frac{\mathrm{x}_i + \mathrm{x}}{2}) + 2F'(\frac{\mathrm{x}_i + 3\mathrm{x}}{4})]\frac{(\mathrm{x} - \mathrm{x}_i)}{3} \tag{9}$$

From (3) and (9), we have,

$$F(\mathrm{x}) = F(\mathrm{x}_i) + [2F'(\frac{3\mathrm{x}_i + \mathrm{x}}{4}) - F'(\frac{\mathrm{x}_i + \mathrm{x}}{2}) + 2F'(\frac{\mathrm{x}_i + 3\mathrm{x}}{4})]\frac{(\mathrm{x}-\mathrm{x}_i)}{3} \tag{10}$$

Since, $F(\mathrm{x}) = 0$ then

$$\mathrm{x} = \mathrm{x}_i - 3[2F'(\frac{3\mathrm{x}_i + \mathrm{x}}{4}) - F'(\frac{\mathrm{x}_i + \mathrm{x}}{2}) + 2F'(\frac{\mathrm{x}_i + 3\mathrm{x}}{4})]^{-1} F(\mathrm{x}_i) \tag{11}$$

which is an implicit method to compute the approximate solution implicitly, we use the prediction and correction technique. With the formulation (11) for solving (SNLE) (2) as follows:

Algorithm 2: For a given X_0, compute the approximate solution X_{i+1} by iterative scheme.

$$\mathrm{y}_i = \mathrm{x}_i - J^{-1}(\mathrm{x}_i) F(\mathrm{x}_i),$$

$$\mathrm{x}_{i+1} = \mathrm{x}_i - 3[2F'(\frac{3\mathrm{x}_i + \mathrm{y}_i}{4}) - F'(\frac{\mathrm{x}_i + \mathrm{y}_i}{2}) + 2F'(\frac{\mathrm{x}_i + 3\mathrm{y}_i}{4})]^{-1} F(\mathrm{x}_i) \tag{12}$$

Furthermore, if we approximate $\int_{x_i}^{x} F'(t)\,dt$ by using two point Newton-Cotes, Open formula [27], then we have

$$\int_{\mathrm{x}_i}^{\mathrm{x}} F'(t)\,dt = [F'(\frac{2\mathrm{x}_i + x}{3}) + F'(\frac{\mathrm{x}_i + 2x}{3})]\frac{(\mathrm{x} - \mathrm{x}_i)}{2} \tag{13}$$

From (3) and (13), we have

$$F(\mathrm{x}) = F(\mathrm{x}_i) + [F'(\frac{2\mathrm{x}_i + x}{3}) + F'(\frac{\mathrm{x}_i + 2x}{3})]\frac{(\mathrm{x} - \mathrm{x}_i)}{2} \tag{14}$$

Since, $F(\mathrm{x}) = 0$ then

$$\mathrm{x}_{i+1} = \mathrm{x}_i - 2\,[F'(\frac{2\mathrm{x}_i + x}{3}) + F'(\frac{\mathrm{x}_i + 2x}{3})]^{-1} F(\mathrm{x}_i) \tag{15}$$

to compute the approximate solution implicitly, we use the prediction and correction technique. With the formulation (15) for solving the system of nonlinear equation (2) as follows:

Algorithm 3: For a given X_0, compute the approximate solution X_{i+1} by iterative scheme

$$\mathrm{y}_i = \mathrm{x}_i - J^{-1}(\mathrm{x}_i) F(\mathrm{x}_i), \tag{16}$$

$$\mathrm{x}_{i+1} = \mathrm{x}_i - 2\,[F'(\frac{2\mathrm{x}_i + y_i}{3}) + F'(\frac{\mathrm{x}_i + 2y_i}{3})]^{-1} F(\mathrm{x}_i) \tag{17}$$

Finally, if we approximate $\int_{x_i}^{x} F'(t)\,dt$ using four points Newton-Cotes, Closed formula [27], then we have

$$\int_{\mathrm{x}_i}^{\mathrm{x}} F'(t)\,dt = [F'(\mathrm{x}_i) + 3F'(\frac{2\mathrm{x}_i + x}{3}) + 3F'(\frac{\mathrm{x}_i + 2x}{3}) + F'(\mathrm{x})]\frac{(\mathrm{x} - \mathrm{x}_i)}{8} \tag{18}$$

From (3) and (18), we have

$$F(\mathrm{x}) = F(\mathrm{x}_i) + [F'(\mathrm{x}_i) + 3F'(\frac{2\mathrm{x}_i + x}{3}) + 3F'(\frac{\mathrm{x}_i + 2x}{3}) + F'(\mathrm{x})]\frac{(\mathrm{x} - \mathrm{x}_i)}{8} \quad (19)$$

Since, $F(\mathrm{x}) = 0$ then

$$\mathrm{x}_{i+1} = \mathrm{x}_i - 8\,[F'(\mathrm{x}_i) + 3F'(\frac{2\mathrm{x}_i + x}{3}) + 3F'(\frac{\mathrm{x}_i + 2x}{3}) + F'(x)]^{-1} F(\mathrm{x}_i) \quad (20)$$

by using the prediction and correction technique. With the formulation (20) the system of nonlinear equation (2) as follows:

Algorithm 4: For a given X_0 , compute the approximate solution X_{i+1} by iterative scheme.

$$\mathrm{y}_i = \mathrm{x}_i - J^{-1}(\mathrm{x}_i)F(\mathrm{x}_i), \quad (21)$$

$$\mathrm{x}_{i+1} = \mathrm{x}_i - 8\,[F'(\mathrm{x}_i) + 3F'(\frac{2\mathrm{x}_i + y_i}{3}) + 3F'(\frac{\mathrm{x}_i + 2y_i}{3}) + F'(y_i)]^{-1} F(\mathrm{x}_i) \quad (22)$$

4. Convergence analysis

In this section, we consider the convergence of our algorithm using the Taylor's series technique.

Theorem 1: Let x^* be a sample zero of sufficient differentiable function $F :\subseteq R^n \to R^n$ for an open interval. If 0 x is sufficiently close to x^*, then the two step method defined by our algorithm (5)has convergence is at least of order 3.

Proof. Consider to

$$\mathrm{y}_n = \mathrm{x}_n - \frac{F(\mathrm{x}_n)}{F'(\mathrm{x}_n)}, \quad (23)$$

$$\mathrm{x}_{n+1} = \mathrm{x}_n - 6[F'(\mathrm{x}_n) + 4F(\mathrm{w}_n) + F'(\mathrm{y}_n)]^{-1} F(\mathrm{x}_n) \quad (24)$$

where $\mathrm{w}_n = \frac{1}{2}(\mathrm{x}_n + \mathrm{y}_n)$, Let x^* be a sample zero of F. Since F is sufficiently differentiable, by expanding $F(\mathrm{x}_n)$ and $F'(\mathrm{x}_n)$ about x^*, we get

$$F(\mathrm{x}_n) = F(\mathrm{x}^*) + F'(\mathrm{x}^*)(\mathrm{x}_n - \mathrm{x}^*) + F^{(2)}(\mathrm{x}^*)\frac{(\mathrm{x}_n - \mathrm{x}^*)^2}{2!} + F^{(3)}(\mathrm{x}^*)\frac{(\mathrm{x}_n - \mathrm{x}^*)^3}{3!} + \cdots,$$

But $F(\mathrm{x}^*) = 0$ then

$$F(\mathrm{x}_n) = F'(\mathrm{x}^*)[E_n + C_2E_n^2 + C_3E_n^3 + O(\|E_n^4\|)], \quad (25)$$

and

$$F'(\mathrm{x}_n) = F'(\mathrm{x}^*)[I + 2C_2E_n + 3C_3E_n^2 + O(\|E_n^3\|)], \quad (26)$$

where $C_k = \frac{1}{k!}\frac{F^{(k)}(\mathrm{x}^*)}{F'(\mathrm{x}^*)}, k = 2, 3, \ldots$ and $E_n = \mathrm{x}_n - \mathrm{x}^*$.

Now from (25) and (26), we have (see [28])

$$\frac{F(\mathrm{x}_n)}{F'(\mathrm{x}_n)} = E_n - C_2E_n^2 + 2(C_2^2 - C_3)E_n^3 + O(\|E_n^4\|), \quad (27)$$

From (23) and (27), we get

$$\mathrm{y}_n = \mathrm{x}^* + C_2E_n^2 + 2(C_3 - C_2^2)E_n^3 + O(\|E_n^4\|), \quad (28)$$

From (22), we get,

$$F(\mathrm{y}_n) = F'(\mathrm{x}^*)[(\mathrm{y}_n - \mathrm{x}^*) + C_2(\mathrm{y}_n - \mathrm{x}^*)^2 + C_3(\mathrm{y}_n - \mathrm{x}^*)^3 + O(\|E_n^4\|)]$$

and

$$F'(\mathrm{y}_n) = F'(\mathrm{x}^*)[I + 2C_2(\mathrm{y}_n - \mathrm{x}^*) + 3C_3(\mathrm{y}_n - \mathrm{x}^*)^2 + O(\|E_n^3\|)]$$
$$= F'(\mathrm{x}^*)[I + 2C_2^2E_n^2 + O(\|E_n^3\|)].$$

Expanding $F'(\mathrm{w}_n)$ about x^*, we get

$$F'(\mathrm{w}_n) = F'(\mathrm{x}^*)[I + 2C_2(\mathrm{w}_n - \mathrm{x}^*) + 3C_3(\mathrm{w}_n - \mathrm{x}^*)^2 + O(\|E_n^3\|)]$$

Where $\mathrm{w}_n = \frac{1}{2}(\mathrm{x}_n + \mathrm{y}_n)$ from (22) we have

$$F'(\mathrm{w}_n) = F'(\mathrm{x}^*)[I + C_2E_n + (C_2^2 + \frac{3}{4}C_3)E_n^2 + O(\|E_n^3\|)]$$

then

$$F'(\mathrm{x}_n) + 4F'(\mathrm{w}_n) + F'(\mathrm{y}_n) = F'(\mathrm{x}^*)[6I + 6C_2E_n + (6C_2^2 + 9C_3)E_n^2 + O(\|E_n^3\|)]$$
$$= 6F'(\mathrm{x}^*)[I + C_2E_n + (C_2^2 + \frac{3}{2}C_3)E_n^2 + O(\|E_n^3\|)]$$

From (24), $E_{n+1} = \mathrm{x}_{n+1} - \mathrm{x}^*$ and $E_n = \mathrm{x}_n - \mathrm{x}^*$.

$$E_{n+1} = E_n - 6\left[J(\mathrm{x}_n) + 4J(\mathrm{w}_n) + J(\mathrm{y}_n)\right]^{-1} F(\mathrm{x}_n)$$

$$E_{n+1} = E_n - 6\left[6F'(\mathrm{x}^*)[I + C_2E_n + (C_2^2 + \frac{3}{2}C_3)E_n^2 + O(\|E_n^3\|)]\right]^{-1} F(\mathrm{x}_n)$$

In general, according to [28]:

$$\left[I + M_2E_n + M_3E_n^2 + O_3\right]^{-1} = \left[I - M_2E_n + (M_2^2 - M_3)E_n^2 + O_3\right]$$

Set $M_2 = C_2, M_3 = C_2^2 + \frac{3}{2}C_3$

Then we will have

$$E_{n+1} = E_n - \left\{F'(\mathrm{x}^*)\left[I - C_2E_n - \frac{3}{2}C_3E_n^2 + O(\|E_n^3\|)\right]\right\} F(\mathrm{x}_n)$$

From (25)

$$E_{n+1} = E_n - [I - C_2E_n - \frac{3}{2}C_3E_n^2 + O(\|E_n^3\|)][E_n + C_2E_n^2 + C_3E_n^3 + O(\|E_n^4\|)]$$

Finally,

$$E_{n+1} = (C_2^2 + \frac{1}{2}C_3)E_n^3 + O(\|E_n^4\|)$$

which shows that Algorithm (1) is at least a third order convergent method, the required result. Since asymptotic convergence of Newton method is 2 c and from Theorem 1, we deduce that the convergence rate of our algorithm is better than the Newton's method. And the cubic convergent method is vastly superior to the linear and the quadratically convergent methods [29].

5. Numerical examples

For comparisons, we have used the third-order Hafiz and Bahgat method (HBM) [16] and Darvishi (DAM) [14] algorithms, respectively.

$$x_{n+1} = x_n - 12\left[J(x_n) + 10J(w_n) + J(y_n)\right]^{-1} F(x_n),$$

$$x_{n+1} = x_n - 2\left[F'(x_n) + F'(y_n)\right]^{-1} F(x_n),$$

where $y_n = x_n - J^{1}(x_n)F(x_n)$.

We present some examples to illustrate the efficiency of our proposed methods. Here, numerical results are performed by Maple 15 with 200 digits but only 14 digits are displayed. In Tables [1-5] we list the results obtained in Algorithm [1-4], which we called, Khirallah and Hafiz Methods (KHM1, KHM2, KHM3, KHM4), respectively and compare them with Newton–Raphson method (NM), Hafiz and Bahgat method (HBM) [16], and Darvishi (DAM). The following stopping criteria is used for computer programs:

$$\| x^{(n+1)} - x^{(n)} \| + \| F(x^{(n)}) \| < 10^{-15}$$

and the computational order of convergence (COC) can be approximated using the following formula:

$$\mathrm{COC} \approx \frac{\ln\left(\| x^{(n+1)} - x^{(n)} \| / \| x^{(n)} - x^{(n-1)} \|\right)}{\ln\left(\| x^{(n)} - x^{(n-1)} \| / \| x^{(n-1)} - x^{(n-2)} \|\right)}$$

Table 2 shows the number of iterations and the computational order of convergence (COC). $\| x^{(n+1)} - x^{(n)} \|$ and the norm of the function $F(x^{(n)})$ is also shown in Table 2 for various methods.

Table 1. Number of iterations for Example 1.

Functions & Methods	IT	COC	$\| x^{(n+1)} - x^{(n)} \|_2$	$\| F(x^{(n)}) \|_2$
F_1, x_0 =-3.				
NM	15	2.00000001842997	4.62461580585008E-28	6.52399684550115E-54
HBM	10	2.99999933808324	1.05721596802914E-38	4.81394938579303E-113
DAM	10	2.99922897924987	3.70578223790695E-18	3.36653193192208E-51
KHM1	10	2.99922897924987	3.70578223790695E-18	3.36653193192208E-51
KHM2	10	2.99999480789091	3.01303999741227E-33	1.2533863797071E-96
KHM3	10	2.99999974385679	6.65989967725116E-41	1.15336367836786E-119
KHM4	10	2.99999461902748	4.01650938788201E-33	2.96904403993932E-96
F_2, x_0 =4				
NM	20	2.00000177738031	7.04037622176027E-21	4.23796972290579E-39
HBM	14	2.99983859733573	1.59297520186683E-25	2.08149877265886E-72
DAM	14	2.99980584903215	1.86433543978608E-25	4.87234659996821E-72
KHM1	13	2.99927894804511	9.88081115604762E-21	5.42458694342177E-58
KHM2	13	2.99930488585053	7.47548441892867E-21	2.34912917724958E-58
KHM3	13	2.99991089189577	1.8889427520925E-27	3.36412572863912E-78
KHM4	13	2.99928670711787	9.09952966462796E-21	4.23687559822413E-58

5.1. Small systems of nonlinear equations

Example 1. In a case of one dimension, consider the following nonlinear functions [12], $f_1(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, with $x_0 = -3$ and $f_2(x) = e^{x^2+7x-30} - 1$ with $x_0 = 4$.

Example 2. In a case two dimension, consider the following systems of nonlinear functions [30],

$$F_3(\mathrm{x}) = \begin{cases} f_1(x,y) = x^2 - 10x + y^2 + 8 = 0 \\ f_2(x,y) = xy^2 + x - 10y + 8 = 0 \end{cases}, (x_0, y_0)=(2, 2).$$

$$F_4(\mathrm{x}) = \begin{cases} f_1(x,y) = x^4 y - xy + 2x - y - 1 = 0 \\ f_2(x,y) = ye^{-x} + x - y - e^{-1} = 0 \end{cases}, (x_0, y_0)=(0.8, 0.8).$$

Example 3. In a case three dimension, consider the following systems of nonlinear functions [30].

$$F_5(\mathrm{x}) = \begin{cases} f_1(x,y,z) = 15x + y^2 - 4z - 13 = 0 \\ f_2(x,y,z) = x^2 + 10y - e^{-z} - 11 = 0 \\ f_3(x,y,z) = y^3 - 25z + 22 = 0 \end{cases}, X_0=(10, 6, -5).$$

$$F_6(\mathrm{x}) = \begin{cases} f_1(x,y,z) = 3x - \cos(yz) - 0.5 = 0 \\ f_2(x,y,z) = x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0 \\ f_3(x,y,z) = e^{-xy} + 20z + \dfrac{10\pi - 3}{3} = 0 \end{cases}, X_0=(1.1, 1.1, 1.1).$$

Table 2. Number of iterations for Example 2.

Methods & functions	IT	COC	$\lVert \mathrm{x}^{(n+1)} - \mathrm{x}^{(n)} \rVert_2$	$\lVert F(\mathrm{x}^{(n)}) \rVert_2$
F_3,X_0=(2, 2)				
NM	9	2.00000140914961	4.37934644418127E-21	8.47815841458272E-41
HBM	6	3.00155165046022	2.89152305962132E-20	5.42470874639365E-59
DAM	6	3.00384615129491	2.35456290038799E-17	3.23994953735791E-50
KHM1	6	3.0018969782973	1.28314015297702E-19	4.8413637367635E-57
KHM2	6	3.0018969782973	1.28314015297702E-19	4.8413637367635E-57
KHM3	6	3.00144738591666	1.7256860796277E-20	1.14494920674716E-59
KHM4	6	3.0018969782973	1.28314015297702E-19	4.8413637367635E-57
F_4,X_0=(2, 2)				
NM	9	2.0003080790074	7.53061926402066E-19	2.30110082351703E-36
HBM	6	3.00233622614329	1.18980970698501E-18	1.02194895526103E-53
DAM	7	3.00021436384324	1.17783537421347E-43	1.37875704566421E-128
KHM1	6	3.00174818657116	7.02465997001949E-18	2.26756512830416E-51
KHM2	6	3.00174798833245	7.00487395265092E-18	2.24845784528931E-51
KHM3	6	3.00251772607513	6.41769417427871E-19	1.56198405876415E-54
KHM4	6	3.00174812784528	7.01879180165014E-18	2.26188701404281E-51

Table 3. Number of iterations for Example 3.

Methods & functions	IT	COC	$\|\| x^{(n+1)} - x^{(n)} \|\|_2$	$\|\| F(x^{(n)}) \|\|_2$
F_5,X_0=(10,6,-5)				
NM	8	2.26089072336615	1.30827439612494E-16	5.41580264395725E-32
HBM	5	3.11050550213485	6.09915403468275E-22	2.91039586070281E-65
ADM	6	3.19324326204977	5.23998126509222E-19	6.94886127056805E-56
KHM1	5	3.23147391984341	1.81427047173709E-16	1.18314011923307E-49
KHM2	5	3.22620879334894	4.10586956837708E-18	1.11920543355639E-54
KHM3	5	3.24321465344843	8.23705630178016E-17	6.49793389946669E-50
KHM4	5	3.22914844338236	7.05132295496965E-17	6.82403435648476E-51
F_6,X_0=(1.1,1.1,1.1)				
NM	9	2.0000000032895	6.45188221156175E-20	3.36918186538804E-37
HBM	6	2.99634279195962	4.93648808913231E-19	4.87596025092796E-53
DAM	6	2.99606536543964	8.48219679389847E-19	2.4736197503045E-52
KHM1	6	2.99627938972062	5.60696523234285E-19	7.14479665255229E-53
KHM2	6	2.99626324434839	5.78960142628486E-19	7.86596995617356E-53
KHM3	6	2.99635743716414	4.79178354244989E-19	4.45961446054831E-53
KHM4	6	2.99627454598102	5.66122916703125E-19	7.35425170801941E-53

5.2. Large systems of nonlinear equations

In this subsection, we test HPM with some sparse systems with m unknown variables. In examples [4-6], we compare the NR method with the proposed HPM method focusing on iteration numbers [15].

Table 4. Number of iterations for Examples 4-6.

Methods	$F7$	$F8$	$F9$	$F7$	$F8$	$F9$	$F7$	$F8$	$F9$
$\varepsilon = 10^{-15}$	m =50			m =75			m =100		
NM	6	7	53	6	7	53	6	7	53
HBM	4	5	34	4	5	34	4	5	35
DAM	5	5	33	5	5	34	5	5	34
KHM1	4	5	34	4	5	35	4	5	35
KHM2	4	5	34	4	5	35	4	5	35
KHM3	4	5	35	4	5	35	4	5	35
KHM4	4	5	34	4	5	35	4	5	35

Example 4. Consider the following system of nonlinear equations:

$$F_7 : f_i = e^{x_i} - 1, \quad i = 1, 2, \ldots, m.$$

The exact solution of this system is $X^* = [0,0,...,0]^T$. To solve this system, we set x_0 $[0.5,0.5,...,0.5]^T$ as an initial value. Tables 4, 5 are shown the result.

Example 5. Consider the following system of nonlinear equations:

$$F_8 : f_i = x_i^2 - \cos(x_i - 1), \quad i = 1, 2, \ldots, m.$$

One of the exact solutions of this system is $X^* = [1,1,...,1]^T$. To solve this system, we set $x_0 = [2,2.,...,2]^T$ as an initial value. The results are presented in Tables 4, 5.

Example 6. Consider the following system of nonlinear equations [13]:

$$F_9 : f_i = \cos x_i - 1, \quad i = 1, 2, \ldots, m.$$

One of the exact solutions of this system is $x^* = [0,0,...,0]^T$. To solve this system, we set $x_0 = [2,2.,...,2]^T$ as an initial guess. The results are presented in Tables 4, 5.

Example 7. Consider the nonlinear boundary value problem [17]

$$y'' = -(y')^2 - y + \ln x, \ 1 \le x \le 2, \ y(1) = 0, \ y(2) = \ln 2.$$

whose exact solution is $y = \ln x$. We consider the following partition of the interval:

$$x_0 = 1, \ x_n = 2, \ x_j = x_0 + jh, \ h = \frac{1}{m}, \ j = 1, 2, \cdots, m-1$$

Let us define now

$$y_0 = y(x_0) = 0, \ y_m = \ln 2, \ y_i = f(x_i), \ i = 1, 2, \cdots, m-1.$$

If we discretize the problem by using the second order finite differences method defined by the numerical formulas

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}, \ i = 1, 2, \cdots, m-1,$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \ i = 1, 2, \cdots, m-1,$$

then, we obtain a $(m - 1) \times (m - 1)$ system of nonlinear equations F_{10}:

$$4y_2 + y_2^2 + 4y_1(h^2 - 2) - 4h^2 \ln x_2 = 0,$$

$$4(y_{i+1} + y_{i-1}) + (y_{i+1} - y_{i-1})^2 + 4y_i(h^2 - 2) - 4h^2 \ln x_{i+1} = 0, \ i = 2, \cdots, m-1,$$

$$4(\ln 2 + y_{m-2}) + (\ln 2 - y_{m-2})^2 + 4y_{m-1}(h^2 - 2) - 4h^2 \ln x_m = 0$$

We take X_0 with $y_k^{(0)} = \ln(\frac{k}{2}), \ k = 1, 2, \cdots, m-1$, as a starting point. In particular, we solve this problem for m = 50, 75 and 100. The numerical results for the above system of nonlinear equations are presented in Table 5. The number of iterations of methods HBM, KHM1, KHM2, KHM3 and KHM4 are equal but method KHM2, … KHM4 has advantage of they are free from second derivatives, over methods NM and HBM, because the cost of computing second derivatives is very high, see Table 6.

Table 5. Comparison of the computational order of convergence (COC) for Examples 4-6.

Methods	$F7$	$F8$	$F9$	$F7$	$F8$	$F9$	$F7$	$F8$	$F9$
$\varepsilon = 10^{-15}$	m =50			m =75			m =100		
NM	2.00	1.99	1.00	2.00	1.99	1.00	2.00	1.99	1.00
HBM	2.99	2.99	1.00	2.99	2.99	1.00	2.99	2.99	1.00
DAM	2.99	2.99	1.00	2.99	2.99	1.00	2.99	2.99	1.00
KHM1	2.99	2.99	1.00	2.99	2.99	1.00	2.99	2.99	1.00
KHM2	2.99	2.99	1.00	2.99	2.99	1.00	2.99	2.99	1.00
KHM3	2.99	2.99	1.00	2.99	2.99	1.00	2.99	2.99	1.00
KHM4	2.99	2.99	1.00	2.99	2.99	1.00	2.99	2.99	1.00

In Tables 1-6, we list the results obtained by modified iterations methods. As we see from this Tables, it is clear that, in most cases, the result obtained by DAM, HBM, KHM1,… KHM4 are equivalent and they very superior to that obtained NM.

6. Conclusions

In this paper, we presented four new algorithms for solving the system of nonlinear equations by using Simpson's formula , and Newton's two, three and four Cosed formulas iterative methods and used these algorithms for the first time for solving initial value problem. These methods have the same efficiency as the other third-order methods in the literature. We conclude from the numerical examples that the proposed methods have at least equal performance as compared with the other methods of the same order. Moreover, our proposed methods provid

Table 6. Number of iterations for Examples 7.

Methods	m	IT	COC	$\| x^{(n+1)} - x^{(n)} \|_2$	$\| F(x^{(n)}) \|_2$
$F10$					
NM		9	1.99590511842648	5.55653355591706E-22	3.36918186538804E-37
HBM		6	2.95997350151316	7.30961049290975E-43	1.85852506647941E-128
DAM		6	2.95997350151316	7.30961049290975E-43	1.85852506647941E-128
KHM1	50	6	2.95997350151316	7.30961049290975E-43	1.85852506647941E-128
KHM2		6	2.95997350151316	7.30961049290975E-43	1.85852506647941E-128
KHM3		6	2.95997350151316	7.30961049290975E-43	1.85852506647941E-128
KHM4		6	2.95997350151316	7.30961049290975E-43	1.85852506647941E-128
F_{10}					
NM		7	2.00919187729643	6.66382516768248E-20	4.13670932812775E-40
HBM		5	2.99888096001969	1.19842575187367E-33	2.61996885836031E-101
ADM		5	2.99888096001969	1.19842575187367E-33	2.61996885836031E-101
KHM1	100	5	2.99888096001969	1.19842575187367E-33	2.61996885836031E-101
KHM2		5	2.99888096001969	1.19842575187367E-33	2.61996885836031E-101
KHM3		5	2.99888096001969	1.19842575187367E-33	2.61996885836031E-101
KHM4		5	2.99888096001969	1.19842575187367E-33	2.61996885836031E-101

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