

## Extrapolation Problem for Continuous Time Periodically Correlated Isotropic Random Fields

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**Abstract.** The problem of optimal linear estimation of functionals depending on the unknown values of a random field  $\zeta(t, x)$ , which is mean-square continuous periodically correlated with respect to time argument  $t \in \mathbb{R}$  and isotropic on the unit sphere  $S_n$  with respect to spatial argument  $x \in S_n$ . Estimates are based on observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x) : t < 0, x \in S_n$ , where  $\theta(t, x)$  is an uncorrelated with  $\zeta(t, x)$  random field, which is mean-square continuous periodically correlated with respect to time argument  $t \in \mathbb{R}$  and isotropic on the sphere  $S_n$  with respect to spatial argument  $x \in S_n$ . Formulas for calculating the mean square errors and the spectral characteristics of the optimal linear estimate of functionals are derived in the case of spectral certainty where the spectral densities of the fields are exactly known. Formulas that determine the least favourable spectral densities and the minimax (robust) spectral characteristics are proposed in the case where the spectral densities are not exactly known while a class of admissible spectral densities is given.

### Introduction

Cosmological Principle (first coined by Einstein): the Universe is, in the large, homogeneous and isotropic (J. G. Bartlett [3]). Last decades indicate growing interest to the spatio-temporal data measured on the surface of a sphere. These data includes cosmic microwave background (CMB) anisotropies (J. G. Bartlett [3], W. Hu and S. Dodelson [17], N. Kogo and N. Komatsu [25], T. Okamoto and W. Hu [40], P. Adshead and W. Hu [1]), medical imaging (R. Kakarala [21]), global and land-based temperature data (P. D. Jones [19], T. Subba Rao and G. Terdik [44]), gravitational and geomagnetic data, climate model (G. R. North and R. F. Cahalan [39]). Some basic results and references on the theory of isotropic random fields on a sphere can be found in the books by M. I. Yadrenko [49] and A. M. Yaglom [50, 51]. For more recent applications and results see new books by C. Gaetan and X. Guyon [10], N. Cressie and C. K. Wikle [4], D. Marinucci and G. Peccati [28] and several papers covering a number of problems in general for spatial temporal isotropic observations (T. Subba Rao and G. Terdik[45], G. Terdik [46]).

Periodically correlated processes and fields are not homogeneous but have numerous properties similar to properties of stationary processes and homogeneous fields. They describe appropriate models of numerous physical and man-made processes. A comprehensive list of the existing references up to the year 2005 on periodically correlated processes and their applications was proposed by E. Serpedin, F. Panduru, I. Sari and G. B. Giannakis[43]. See also reviews by J. Antoni [2] and A. Napolitano [38]. For more details see a survey paper by W. A. Gardner [12] and book by H. L. Hurd and A. Miamee [18]. Note, that in the literature periodically correlated processes are named in multiple different ways such as cyclostationary, periodically nonstationary or cyclic correlated processes.

The mean square optimal estimation problems for periodically correlated with respect to time isotropic on a sphere random fields are natural generalization of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and homogeneous random fields. Effective methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and random fields were developed under the condition of certainty where spectral densities of processes and fields are known exactly (see, for example, selected works of A. N. Kolmogorov [26], survey article by T. Kailath [20], books by Yu. A. Rozanov[42], N. Wiener [48], A. M. Yaglom [50, 51], M. I. Yadrenko [49], articles by M. P. Moklyachuk and M. I. Yadrenko [35] - [36]).

The classical approach to the problems of interpolation, extrapolation and filtering of stochastic processes and random fields is based on the assumption that the spectral densities of processes and fields are known. In practice, however, complete information about the spectral density is impossible in most cases. To overcome this complication one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then applies the classical estimation method provided that the estimated or selected density is the true one. This procedure can result in a significant increasing of the value of error as K. S. Vastola and H. V. Poor [47] have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error of estimates. Such problems arise when considering problems of automatic control theory, coding and signal processing in radar and sonar, pattern recognition problems of speech signals and images. A comprehensive survey of results up to the year 1985 in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor [24]. J. Franke [8], J. Franke and H. V. Poor [9] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities. The paper by Ulf Grenander [16] should be marked as the first one where the minimax approach to extrapolation problem for the functionals from stationary processes was developed. For more details see, for example, books by M. Moklyachuk [30], M. Moklyachuk and I. Golichenko [33], M. Luz and M. Moklyachuk [27], M. Moklyachuk and O. Masyutka [34]. In papers by I. I. Dubovets'ka, O. Yu. Masyutka and M.P. Moklyachuk [5], [6], I. I. Golichenko, O. Yu. Masyutka, and M. P. Moklyachuk [14], [15] results of investigation of minimax-robust estimation problems for periodically correlated isotropic random fields are proposed.

In this article we deal with the problem of mean square optimal linear estimation of the functional

$$A\zeta = \int_0^{\infty} \int_{S_n} a(t, x) \zeta(t, x) m_n(dx) dt$$

which depends on unknown values of a periodically correlated (cyclostationary with period  $T$ ) with respect to time isotropic on the unit sphere  $S_n$  in Euclidean space  $\mathbb{E}^n$  random field  $\zeta(t, x)$ ,  $t \geq 0$ ,  $x \in S_n$ . Estimates are based on observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$ , where  $\theta(t, x)$  is an uncorrelated with  $\zeta(t, x)$  periodically correlated with respect to time isotropic on the sphere  $S_n$  random field. Formulas are derived for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimate of the functional  $A\zeta$  in the case of spectral certainty, where spectral densities of the fields are known. Formulas are proposed that determine the least favourable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional  $A\zeta$  for concrete classes of spectral densities under the condition that spectral densities are not known exactly while classes  $D = D_F \times D_G$  of admissible spectral densities are given.

**Spectral properties of periodically correlated isotropic on a sphere random fields**

Let  $S_n$  be a unit sphere in the  $n$ -dimensional Euclidean space  $\mathbb{E}^n$ , let  $m_n(dx)$  be the Lebesgue measure on  $S_n$ , and let

$$S_m^l(x), l = 1, \dots, h(m, n); m = 0, 1, \dots$$

be the orthonormal spherical harmonics of degree  $m$ , where  $h(m, n)$  is the number of orthonormal spherical harmonics (see books by A. Erdelyi et al. [7] and C. Müller [37] for more details).

A mean-square continuous random field  $\zeta(t, x), t \in \mathbb{R}, x \in S_n, \zeta(t, x) \in H = L_2(\Omega, \mathcal{F}, \mathbb{P})$ , where  $L_2(\Omega, \mathcal{F}, \mathbb{P})$  denotes the Hilbert space of random variables  $\zeta$  with zero first moment,  $\mathbb{E}\zeta = 0$ , finite second moment,  $\mathbb{E}|\zeta|^2 < \infty$ , and the inner product  $\langle \zeta, \eta \rangle = \mathbb{E}\zeta\bar{\eta}$ , is called periodically correlated (cyclostationary with period  $T$ ) with respect to time isotropic on the sphere  $S_n$  if for all  $t, s \in \mathbb{R}$  and  $x, y \in S_n$  the following property holds true

$$\mathbb{E} \left( \zeta(t + T, x) \overline{\zeta(s + T, y)} \right) = B(t, s, \cos \vartheta),$$

where  $\cos \vartheta = (x, y)$ ,  $\vartheta$  is the angular distance between points  $x, y \in S_n$ .

The correlation function  $B(t, s, \cos \vartheta)$  of the mean-square continuous random field  $\zeta(t, x)$  is continuous. It can be represented in the form of the series

$$B(t, s, \cos \vartheta) = \frac{1}{\omega_n} \sum_{m=0}^{\infty} h(m, n) \frac{C_m^{(n-2)/2}(\cos \vartheta)}{C_m^{(n-2)/2}(1)} B_m^\zeta(t, s),$$

where  $\omega_n = (2\pi)^{n/2}\Gamma(n/2)$ ,  $C_m^l(z)$  are the Gegenbauer polynomials (see book by M. I. Yadrenko [49]).

It follows from the Karhunen theorem that the random field  $\zeta(t, x)$  itself can be represented in the form of the mean square convergent series (see K. Karhunen [23], I. I. Gikhman and A. V. Skorokhod [11])

$$\zeta(t, x) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(x) \zeta_m^l(t), \tag{1}$$

where

$$\zeta_m^l(t) = \int_{S_n} \zeta(t, x) S_m^l(x) m_n(dx).$$

In this representation

$$\zeta_m^l(t), l = 1, \dots, h(m, n); t \in \mathbb{R}, m = 0, 1, \dots$$

are mutually uncorrelated periodically correlated with period  $T$  stochastic processes with the correlation functions  $B_m^\zeta(t, s)$ :

$$\mathbb{E} \left( \zeta_m^l(t + T) \overline{\zeta_u^v(s + T)} \right) = \delta_m^u \delta_l^v B_m^\zeta(t, s),$$

$$l, v = 1, \dots, h(m, n); m, u = 0, 1, \dots; t, s \in \mathbb{R},$$

where  $\delta_l^v$  are the Kroneker delta-functions.

Consider two mutually uncorrelated periodically correlated with period  $T$  random fields  $\zeta(t, x)$  and  $\theta(t, x)$ . We construct the following sequences of stochastic functions

$$\{\zeta_m^l(j, u) = \zeta_m^l(u + jT), u \in [0, T), j \in \mathbb{Z}\}, \tag{2}$$

$$\{\theta_m^l(j, u) = \theta_m^l(u + jT), u \in [0, T), j \in \mathbb{Z}\} \tag{3}$$

which correspond to the random fields  $\zeta(t, x)$  and  $\theta(t, x)$ . Sequences (2) and (3) form  $L_2([0, T]; H)$ -valued stationary sequences  $\{\zeta_m^l(j, u), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j, u), j \in \mathbb{Z}\}$ , respectively, with the correlation functions

$$R_m^\zeta(k, j) = \int_0^T \mathbb{E}[\zeta_m^l(u + kT) \overline{\zeta_m^l(u + jT)}] du = \int_0^T B_m^\zeta(u + (k - j)T, u) du = R_m^\zeta(k - j),$$

$$R_m^\theta(k, j) = \int_0^T \mathbb{E}[\theta_m^l(u + kT) \overline{\theta_m^l(u + jT)}] du = \int_0^T B_m^\theta(u + (k - j)T, u) du = R_m^\theta(k - j).$$

To describe properties of the stationary sequences  $\{\zeta_m^l(j, u), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j, u), j \in \mathbb{Z}\}$  we define in the space  $L_2([0, T]; \mathbb{R})$  the following orthonormal basis

$$\left\{ \tilde{e}_k = \frac{1}{\sqrt{T}} e^{2\pi i \{(-1)^k \lfloor \frac{k}{2} \rfloor\} u/T}, k = 1, 2, \dots \right\}, \quad \langle \tilde{e}_j, \tilde{e}_k \rangle = \delta_k^j.$$

Making use of the introduced basis the stationary sequences  $\{\zeta_m^l(j, u), j \in \mathbb{Z}\}$ ,  $\{\theta_m^l(j, u), j \in \mathbb{Z}\}$  can be represented in the following forms

$$\zeta_m^l(j, u) = \sum_{k=1}^{\infty} \zeta_{mk}^l(j) \tilde{e}_k, \quad (4)$$

$$\zeta_{mk}^l(j) = \langle \zeta_m^l(j), \tilde{e}_k \rangle = \frac{1}{\sqrt{T}} \int_0^T \zeta_m^l(j, v) e^{-2\pi i \{(-1)^k \lfloor \frac{k}{2} \rfloor\} v/T} dv,$$

$$\theta_m^l(j, u) = \sum_{k=1}^{\infty} \theta_{mk}^l(j) \tilde{e}_k, \quad (5)$$

$$\theta_{mk}^l(j) = \langle \theta_m^l(j), \tilde{e}_k \rangle = \frac{1}{\sqrt{T}} \int_0^T \theta_m^l(j, v) e^{-2\pi i \{(-1)^k \lfloor \frac{k}{2} \rfloor\} v/T} dv.$$

Components  $\{\zeta_{mk}^l(j), k = 1, 2, \dots; j \in \mathbb{Z}\}$  and  $\{\theta_{mk}^l(j), k = 1, 2, \dots; j \in \mathbb{Z}\}$  of the constructed vector-valued stationary sequences  $\{\zeta_m^l(j) = (\zeta_{mk}^l(j), k = 1, 2, \dots)^\top, j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j) = (\theta_{mk}^l(j), k = 1, 2, \dots)^\top, j \in \mathbb{Z}\}$  have the following properties [22], [29]

$$\mathbb{E} \zeta_{mk}^l(j) = 0, \quad \|\zeta_m^l(j)\|_H^2 = \sum_{k=1}^{\infty} \mathbb{E} |\zeta_{mk}^l(j)|^2 = R_m^\zeta(0),$$

$$\mathbb{E} \zeta_{mk}^l(j_1) \overline{\zeta_{mn}^l(j_2)} = \langle K_m^\zeta(j_1 - j_2) e_k, e_n \rangle,$$

$$\mathbb{E} \theta_{mk}^l(j) = 0, \quad \|\theta_m^l(j)\|_H^2 = \sum_{k=1}^{\infty} \mathbb{E} |\theta_{mk}^l(j)|^2 = R_m^\theta(0),$$

$$\mathbb{E} \theta_{mk}^l(j_1) \overline{\theta_{mn}^l(j_2)} = \langle K_m^\theta(j_1 - j_2) e_k, e_n \rangle,$$

where  $\{e_k, k = 1, 2, \dots\}$  is a basis in the space  $\ell_2$ . The correlation functions  $K_m^\zeta(j)$  and  $K_m^\theta(j)$  of the stationary sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j), j \in \mathbb{Z}\}$  are correlation operator functions in  $\ell_2$ .

The vector-valued stationary sequences  $\{\zeta_m^l(j) = (\zeta_{mk}^l(j), k = 1, 2, \dots)^\top, j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j) = (\theta_{mk}^l(j), k = 1, 2, \dots)^\top, j \in \mathbb{Z}\}$  have the spectral density functions

$$F_m(\lambda) = \{f_m^{kn}(\lambda)\}_{k,n=1}^{\infty}, \quad G_m(\lambda) = \{g_m^{kn}(\lambda)\}_{k,n=1}^{\infty},$$

which are operator-valued functions of variable  $\lambda \in [-\pi, \pi)$  in the space  $\ell_2$  if their correlation functions  $K_m^\zeta(j)$  and  $K_m^\theta(j)$  can be represented in the form

$$\langle K_m^\zeta(j)e_k, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} \langle F_m(\lambda)e_k, e_n \rangle d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} f_m^{kn}(\lambda) d\lambda,$$

$$\langle K_m^\theta(j)e_k, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} \langle G_m(\lambda)e_k, e_n \rangle d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} g_m^{kn}(\lambda) d\lambda,$$

For almost all  $\lambda \in [-\pi, \pi)$  the spectral densities  $F_m(\lambda)$  and  $G_m(\lambda)$  are kernel operators with integrable kernel norm

$$\sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle F_m(\lambda)e_k, e_k \rangle d\lambda = \sum_{k=1}^{\infty} \langle K_m^\zeta(0)e_k, e_k \rangle = \|\zeta_m^l(j)\|_H^2 = R_m^\zeta(0),$$

$$\sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle G_m(\lambda)e_k, e_k \rangle d\lambda = \sum_{k=1}^{\infty} \langle K_m^\theta(0)e_k, e_k \rangle = \|\theta_m^l(j)\|_H^2 = R_m^\theta(0).$$

### Hilbert space projection method of extrapolation

Consider the problem of the mean square optimal linear estimation of the functional

$$A\zeta = \int_0^\infty \int_{S_n} a(t, x)\zeta(t, x) m_n(dx) dt$$

which depends on unknown values of a periodically correlated with period  $T$  with respect to time isotropic on the unit sphere  $S_n$  in Euclidean space  $\mathbb{E}^n$  random field  $\zeta(t, x)$ ,  $t \geq 0$ ,  $x \in S_n$ . Estimates are based on observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$ , where  $\theta(t, x)$  is an uncorrelated with  $\zeta(t, x)$  periodically correlated with respect to time isotropic on the sphere  $S_n$  random field.

It follows from representation (1) that the functional  $A\zeta$  can be represented in the form

$$A\zeta = \int_0^\infty \int_{S_n} a(t, x)\zeta(t, x) m_n(dx) dt = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_0^\infty a_m^l(t)\zeta_m^l(t) dt$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \int_0^T a_m^l(j, u)\zeta_m^l(j, u) du,$$

$$a_m^l(t) = \int_{S_n} a(t, x) S_m^l(x) m_n(dx),$$

$$a_m^l(j, u) = a_m^l(u + jT), u \in [0, T),$$

$$\zeta_m^l(j, u) = \zeta_m^l(u + jT), u \in [0, T).$$

Taking into account decomposition (4) of stationary sequence  $\{\zeta_m^l(j, u), j \in \mathbb{Z}\}$ , the functional  $A\zeta$  can be represented in the form

$$\begin{aligned} A\zeta &= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} a_{mk}^l(j) \zeta_{mk}^l(j) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} (\mathbf{a}_m^l(j))^\top \boldsymbol{\zeta}_m^l(j), \\ \boldsymbol{\zeta}_m^l(j) &= (\zeta_{mk}^l(j), k = 1, 2, \dots)^\top, \\ \mathbf{a}_m^l(j) &= (a_{mk}^l(j), k = 1, 2, \dots)^\top = \\ &= (a_{m1}^l(j), a_{m3}^l(j), a_{m2}^l(j), \dots, a_{m(2k+1)}^l(j), a_{m(2k)}^l(j), \dots)^\top, \\ a_{mk}^l(j) &= \langle a_m^l(j), \tilde{e}_k \rangle = \frac{1}{\sqrt{T}} \int_0^T a_m^l(j, v) e^{-2\pi i \{(-1)^k [\frac{k}{2}]\} v/T} dv. \end{aligned}$$

We will assume that coefficients  $\{\mathbf{a}_m^l(j), j = 0, 1, \dots\}$  which form this representation satisfy the following conditions

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \|\mathbf{a}_m^l(j)\| < \infty, \quad \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} (j+1) \|\mathbf{a}_m^l(j)\|^2 < \infty, \quad (6) \\ \|\mathbf{a}_m^l(j)\|^2 &= \sum_{k=1}^{\infty} |a_{mk}^l(j)|^2. \end{aligned}$$

Under these condition the functional  $A\zeta$  has finite second moment and operators defined below with the help of the coefficients  $\{\mathbf{a}_m^l(j), j = 0, 1, \dots\}$  are compact.

Denote by  $L_2(F)$  the Hilbert space of complex vector functions

$$h(\lambda) = \{h_m^l(\lambda) : m = 0, 1, \dots; l = 1, 2, \dots, h(m, n)\}, \quad h_m^l(\lambda) = (h_{mk}^l(\lambda), k = 1, 2, \dots)^\top,$$

that satisfy condition

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} (h_m^l(\lambda))^\top F_m(\lambda) \overline{h_m^l(\lambda)} d\lambda < \infty.$$

We denote by  $L_2^-(F)$  the subspace of  $L_2(F)$  generated by the functions

$$e^{ij\lambda} \delta_k, \quad \delta_k = \{\delta_k^n\}_{n=1}^{\infty}, \quad k = 1, 2, \dots, j < 0,$$

where  $\delta_k^n$  are the Kronecker delta functions:  $\delta_k^k = 1, \delta_k^n = 0, k \neq n$ .

Every linear estimate  $\hat{A}\zeta$  of the functional  $A\zeta$  which is based on observations of the sequence  $\{\zeta_m^l(j) + \boldsymbol{\theta}_m^l(j), j \in \mathbb{Z}\}$  at points  $j < 0$  is defined by the spectral characteristic  $h(\lambda) \in L_2^-(F + G)$  and is of the form

$$\hat{A}\zeta = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} (h_m^l(\lambda))^\top (Z_m^{l\zeta}(d\lambda) + Z_m^{l\theta}(d\lambda)), \quad (7)$$

where  $Z_m^{l\zeta}(\Delta) = \{Z_{mk}^{l\zeta}(\Delta)\}_{k=1}^{\infty}$  and  $Z_m^{l\theta}(\Delta) = \{Z_{mk}^{l\theta}(\Delta)\}_{k=1}^{\infty}$  are orthogonal stochastic measures of the sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  and  $\{\boldsymbol{\theta}_m^l(j), j \in \mathbb{Z}\}$ .

Suppose that spectral densities of stationary sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}, \{\boldsymbol{\theta}_m^l(j), j \in \mathbb{Z}\}$  satisfy the following minimality condition (for more details see [42], Chapter 1, Theorem 10.2.)

$$\int_{-\pi}^{\pi} \text{Tr} [(F_m(\lambda) + G_m(\lambda))^{-1}] d\lambda < \infty. \quad (8)$$

The mean square error  $\Delta(h; F, G)$  of the linear estimate  $\hat{A}\zeta$  with the spectral characteristic  $h_m^l(\lambda) = \sum_{j=1}^{\infty} h_m^l(j)e^{-ij\lambda}$  can be represented in the form

$$\Delta(h; F, G) = E|A\zeta - \hat{A}\zeta|^2$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( (A_m^l(\lambda) - h_m^l(\lambda))^{\top} F_m(\lambda) \overline{(A_m^l(\lambda) - h_m^l(\lambda))} + (h_m^l(\lambda))^{\top} G_m(\lambda) \overline{h_m^l(\lambda)} \right) d\lambda,$$

$$A_m^l(\lambda) = \sum_{j=0}^{\infty} a_m^l(j)e^{ij\lambda}.$$

The spectral characteristic  $h(F, G)$  of the optimal linear estimate  $\hat{A}\zeta$  of the functional  $A\zeta$  minimizes the value of the mean square error

$$\Delta(F, G) = \Delta(h(F, G); F, G)$$

$$= \min_{h \in L_2^-(F+G)} \Delta(h; F, G) = \min_{\hat{A}\zeta} E|A\zeta - \hat{A}\zeta|^2. \tag{9}$$

With the help of the Hilbert space projection method proposed by A. N. Kolmogorov (see, for example, book [26], p.228-280) we can find formulas for calculation the mean square error  $\Delta(F, G) = \Delta(h(F, G); F, G)$  and the spectral characteristic  $h(F, G)$ , which is a solution of the optimization problem (9), of the optimal linear estimate  $\hat{A}\zeta$  of the functional  $A\zeta$ . Following the method we find the optimal linear estimate  $\hat{A}\zeta$  as projection of  $A\zeta$  on the closed linear subspace  $H^-(\zeta + \theta)$  generated by values of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$ , in the space  $H = L_2(\Omega, \mathcal{F}, \mathbb{P})$ .

This projection is determined by conditions:

- 1)  $\hat{A}\zeta \in H^-(\zeta + \theta)$ ;
- 2)  $A\zeta - \hat{A}\zeta \perp H^-(\zeta + \theta)$ .

The second condition is satisfied if for all  $m = 0, 1, \dots; l = 1, 2, \dots, h(m, n)$  and  $j = -1, -2, \dots$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [(A_m^l(\lambda) - h_m^l(\lambda))^{\top} F_m(\lambda) - (h_m^l(\lambda))^{\top} G_m(\lambda)] e^{-ij\lambda} d\lambda = 0.$$

These relations mean that for all  $m = 0, 1, \dots; l = 1, 2, \dots, h(m, n)$

$$(A_m^l(\lambda))^{\top} F_m(\lambda) - (h_m^l(\lambda))^{\top} (F_m(\lambda) + G_m(\lambda)) = C_m^l(\lambda),$$

$$C_m^l(\lambda) = \sum_{j=0}^{\infty} c_m^l(j)e^{ij\lambda},$$

where  $\{c_m^l(j), j = 0, 1, \dots\}$  are unknown coefficients. It follows from the indicated relations that the spectral characteristic  $h(F, G)$  of the optimal linear estimate  $\hat{A}\zeta$  is of the form

$$(h_m^l(F, G))^{\top} = [(A_m^l(\lambda))^{\top} F_m(\lambda) - (C_m^l(\lambda))^{\top}] (F_m(\lambda) + G_m(\lambda))^{-1}$$

$$= (A_m^l(\lambda))^{\top} - [(A_m^l(\lambda))^{\top} G_m(\lambda) + (C_m^l(\lambda))^{\top}] (F_m(\lambda) + G_m(\lambda))^{-1}. \tag{10}$$

The first condition,  $\hat{A}\zeta \in H^-(\zeta + \theta)$ , is satisfied if for all  $m = 0, 1, \dots; l = 1, 2, \dots, h(m, n)$  and  $s = 0, 1, 2, \dots$

$$\sum_{j=0}^{\infty} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} [F_m(\lambda)(F_m(\lambda) + G_m(\lambda))^{-1}]^{\top} e^{i(j-s)\lambda} d\lambda \right] a_m^l(j)$$

$$= \sum_{j=0}^{\infty} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} [(F_m(\lambda) + G_m(\lambda))^{-1}]^{\top} e^{i(j-s)\lambda} d\lambda \right] c_m^l(j). \quad (11)$$

To write these relations in more convenient form we introduce operators  $\mathbf{B}_m$ ,  $\mathbf{D}_m$ ,  $\mathbf{R}_m$  determined by matrices

$$\mathbf{B}_m = \{B_m(j, l)\}_{j,l=0}^{\infty}, \quad \mathbf{D}_m = \{D_m(j, l)\}_{j,l=0}^{\infty}, \quad \mathbf{R}_m = \{R_m(j, l)\}_{j,l=0}^{\infty}$$

composed with the help of the Fourier coefficients

$$B_m(j, l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [(F_m(\lambda) + G_m(\lambda))^{-1}]^{\top} e^{i(j-l)\lambda} d\lambda,$$

$$D_m(j, l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [F_m(\lambda)(F_m(\lambda) + G_m(\lambda))^{-1}]^{\top} e^{i(j-l)\lambda} d\lambda,$$

$$R_m(j, l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [F_m(\lambda)(F_m(\lambda) + G_m(\lambda))^{-1}G_m(\lambda)]^{\top} e^{i(j-l)\lambda} d\lambda.$$

and vectors

$$\mathbf{a}_m^l = (a_m^l(j), j = 0, 1, 2, \dots)^{\top}, \quad \mathbf{c}_m^l = (c_m^l(j), j = 0, 1, 2, \dots)^{\top}.$$

Taking into consideration the introduced operators and vectors we can represent equation (11) in the form

$$\mathbf{B}_m \mathbf{c}_m^l = \mathbf{D}_m \mathbf{a}_m^l, \quad m = 0, 1, \dots; l = 1, 2, \dots, h(m, n)$$

This means that the unknown coefficients  $\mathbf{c}_m^l = (c_m^l(j), j = 0, 1, 2, \dots)^{\top}$  are determined by the equation

$$\mathbf{c}_m^l = \mathbf{B}_m^{-1} \mathbf{D}_m \mathbf{a}_m^l.$$

It follows from the derived relations that the value of the mean square error  $\Delta(F, G)$  of the optimal linear estimate  $\hat{A}\zeta$  of the functional  $A\zeta$  can be calculated by formula

$$\begin{aligned} \Delta(F, G) &= \Delta(h(F, G); F, G) \\ &= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [(A_m^l(\lambda))^{\top} G_m(\lambda) + (C_m^l(\lambda))^{\top}] (F_m(\lambda) + G_m(\lambda))^{-1} \right. \\ &\quad \times F_m(\lambda)(F_m(\lambda) + G_m(\lambda))^{-1} [(A_m^l(\lambda))^{\top} G_m(\lambda) + (C_m^l(\lambda))^{\top}]^* d\lambda \\ &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} [(A_m^l(\lambda))^{\top} F_m(\lambda) - (C_m^l(\lambda))^{\top}] (F_m(\lambda) + G_m(\lambda))^{-1} \\ &\quad \left. \times G_m(\lambda)(F_m(\lambda) + G_m(\lambda))^{-1} [(A_m^l(\lambda))^{\top} F_m(\lambda) - (C_m^l(\lambda))^{\top}]^* d \right\} \\ &= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} (\langle \mathbf{a}_m^l, \mathbf{R}_m \mathbf{a}_m^l \rangle + \langle \mathbf{c}_m^l, \mathbf{B}_m \mathbf{c}_m^l \rangle). \end{aligned} \quad (12)$$

where  $\langle a, b \rangle$  is the inner product in the space  $\ell_2$ .

Let us summarize our results and present them in the form of a theorem.



**Theorem 1.** Let  $\{\zeta(t, x), t \in \mathbb{R}, x \in S_n\}$  and  $\{\theta(t, x), t \in \mathbb{R}, x \in S_n\}$  be mutually uncorrelated random fields, which are periodically correlated with period  $T$  with respect to time argument  $t \in \mathbb{R}$  and isotropic on the unit sphere  $S_n$  with respect to spatial argument  $x \in S_n$ . Let the stationary sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j), j \in \mathbb{Z}\}$  constructed with the help of relations (2), (3), respectively, have spectral densities  $F_m(\lambda)$  and  $G_m(\lambda)$  that satisfy the minimality condition (8). Let coefficients  $\{\mathbf{a}_m^l(j), j = 0, 1, \dots\}$  that determine the functional  $A\zeta$  satisfy conditions (6). Then the spectral characteristic  $h(F, G)$  and the mean square error  $\Delta(F, G)$  of the optimal estimate of the functional  $A\zeta$  from observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x), t < 0, x \in S_n$  are given by formulas (10), (12) respectively. The optimal estimate  $A\zeta$  of the functional  $A\zeta$  is calculated by the formula (7).

**Corollary 2.** Let  $\{\zeta(t, x), t \in \mathbb{R}, x \in S_n\}$  be a random field, which is periodically correlated with respect to time argument  $t \in \mathbb{R}$  and isotropic on the unit sphere  $S_n$  with respect to spatial argument  $x \in S_n$ . Let the stationary sequence  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  constructed with the help of relations (2) has spectral densities  $F_m(\lambda)$  that satisfy the minimality condition

$$\int_{-\pi}^{\pi} \text{Tr} [(F_m(\lambda))^{-1}] d\lambda < \infty. \tag{13}$$

Let coefficients  $\{\mathbf{a}_m^l(j), j = 0, 1, \dots\}$  that determine the functional  $A\zeta$  satisfy conditions (6). Then the spectral characteristic  $h(F)$  and the mean square error  $\Delta(F)$  of the optimal linear estimate of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $(t, x), t < 0, x \in S_n$  are determined by formulas

$$(h_m^l(F))^\top = (A_m^l(\lambda))^\top - (C_m^l(\lambda))^\top (F_m(\lambda))^{-1}, \tag{14}$$

$$\Delta(F) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \langle \mathbf{c}_m^l, \mathbf{a}_m^l \rangle, \tag{15}$$

where  $\mathbf{c}_m^l = \{\mathbf{c}_m^l(j)\}_{j=0}^{\infty} = \mathbf{B}_m^{-1} \mathbf{a}_m^l$  and matrices  $\mathbf{B}_m = \{B_m(j, l)\}_{j,l=0}^{\infty}$  are defined by the Fourier coefficients

$$B_m(j, l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [(F_m(\lambda))^{-1}]^\top e^{i(j-l)\lambda} d\lambda.$$

Theorem 1 and Corollary 1 show that the Fourier coefficients of some functions from spectral densities can be used to find the spectral characteristics and the mean square error of optimal linear estimates of the functionals of the random fields for problems of extrapolation based on observations without noise as well as on observations with noise.

To solve the problem of extrapolation of stationary sequences A. N. Kolmogorov (see, for example, book [26], p.272-280) proposed a method based on factorization of the spectral density. This method is suitable for solving problems of extrapolation based on observations without noise whereas Theorem 1 describes the method of solving problem of extrapolation based on observations with noise.

We apply the indicated method based on factorization of the spectral density to the problem of estimation of the functional from observations without noise.

Suppose that spectral densities of stationary sequence  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  admit the canonical factorization (G. Kallianpur and V. Mandrekar [22], M. P. Moklyachuk [29])

$$F_m(\lambda) = P_m(\lambda)(P_m(\lambda))^*, \quad P_m(\lambda) = \sum_{u=0}^{\infty} d_m(u)e^{-iu\lambda}, \tag{16}$$

where matrices  $d_m(u) = \{d_{mk}^r(u)\}_{k=1, \dots, M}^{r=1, \dots, M}$  are coefficients of the canonical representation,  $M$  is the multiplicity of  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$ .

In this case the spectral characteristic  $h(F)$  and the mean square error  $\Delta(F)$  of the optimal estimate  $\hat{A}\zeta$  are determined by formulas

$$h_m^l(F) = A_m^l(\lambda) - (Q_m(\lambda))^\top S_m^l(\lambda), \quad (17)$$

$$\Delta(F) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \|\mathbf{A}_m^l \mathbf{d}_m\|^2, \quad (18)$$

where matrices  $Q_m(\lambda)$  are defined by equations:  $Q_m(\lambda)P_m(\lambda) = I_M$ ,

$$\|\mathbf{A}_m^l \mathbf{d}_m\|^2 = \sum_{j=0}^{\infty} \|(\mathbf{A}_m^l \mathbf{d}_m)(j)\|^2, \quad (\mathbf{A}_m^l \mathbf{d}_m)(j) = \sum_{p=0}^{\infty} (d_m(p))^\top \mathbf{a}_m^l(p+j),$$

$$S_m^l(\lambda) = \sum_{j=0}^{\infty} (\mathbf{A}_m^l \mathbf{d}_m)(j) e^{ij\lambda}.$$

Let us summarize our results and present them in the form of a theorem.

**Theorem 3.** Let  $\{\zeta(t, x), t \in \mathbb{R}, x \in S_n\}$  be a random field, which is periodically correlated with period  $T$  with respect to time argument  $t \in \mathbb{R}$  and isotropic on the unit sphere  $S_n$  with respect to spatial argument  $x \in S_n$ . Let the stationary sequence  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  constructed with the help of relations (2) has spectral densities  $F_m(\lambda)$  that admit canonical factorization (16). Let coefficients  $\{\mathbf{a}_m^l(j), j = 0, 1, \dots\}$  that determine the functional  $A\zeta$  satisfy conditions (6). Then the spectral characteristic  $h(F)$  and the mean square error  $\Delta(F)$  of the optimal linear estimate of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$  are given by formulas (17), (18).

### Minimax-robust method of extrapolation

Formulas (10) - (18) for calculating the spectral characteristic and the mean square error of the optimal linear estimate of the functional  $A\zeta$  can be applied in the case where spectral densities  $F_m(\lambda)$  and  $G_m(\lambda)$  of stationary sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j), j \in \mathbb{Z}\}$  constructed by relations (2), (3), are known. If the spectral densities are not exactly known while a set of admissible densities  $D = D_F \times D_G$  is specified, then the minimax approach to estimation of functional of unknown values is reasonable. That is we find the estimate which minimizes the mean square error for all spectral densities from a given set  $D = D_F \times D_G$  simultaneously.

**Definition 4.** For a given class of spectral densities  $D = D_F \times D_G$  the spectral densities  $F_m^0(\lambda) \in D_F$  and  $G_m^0(\lambda) \in D_G$  are called the least favorable in  $D$  for the optimal estimate of functional  $A\zeta$  if

$$\Delta(F^0, G^0) = \Delta(h(F^0, G^0); F^0, G^0) = \max_{(F,G) \in D} \Delta(h(F, G); F, G).$$

**Definition 5.** For a given class of spectral densities  $D = D_F \times D_G$  the spectral characteristic  $h^0(\lambda)$  of the optimal linear estimate of the functional  $A\zeta$  is called minimax-robust if the following relations hold true

$$h^0(\lambda) \in H_D = \bigcap_{(F,G) \in D} L_2^-(F+G),$$

$$\min_{h \in H_D} \max_{(F,G) \in D} \Delta(h; F, G) = \max_{(F,G) \in D} \Delta(h^0; F, G).$$

Taking into account the introduced definitions and relations (10) - (18) we can verify that the following lemma holds true.

**Lemma 6.** Spectral densities  $F_m^0(\lambda) \in D_F$  and  $G_m^0(\lambda) \in D_G$  which satisfy the minimality condition (8) are the least favorable in the class  $D = D_F \times D_G$  for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$  if the Fourier coefficients of the functions

$$(F_m^0(\lambda) + G_m^0(\lambda))^{-1}, F_m^0(\lambda)(F_m^0(\lambda) + G_m^0(\lambda))^{-1}, \\ F_m^0(\lambda)(F_m^0(\lambda) + G_m^0(\lambda))^{-1}G_m^0(\lambda)$$

determine matrices  $\mathbf{B}_m^0, \mathbf{D}_m^0, \mathbf{R}_m^0$  giving a solution of the constrained optimization problem

$$\max_{(F,G) \in D} \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} (\langle \mathbf{a}_m^l, \mathbf{R}_m^0 \mathbf{a}_m^l \rangle + \langle \mathbf{B}_m^{-1} \mathbf{D}_m^0 \mathbf{a}_m^l, \mathbf{D}_m^0 \mathbf{a}_m^l \rangle) = \\ = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} (\langle \mathbf{a}_m^l, \mathbf{R}_m^0 \mathbf{a}_m^l \rangle + \langle (\mathbf{B}_m^0)^{-1} \mathbf{D}_m^0 \mathbf{a}_m^l, \mathbf{D}_m^0 \mathbf{a}_m^l \rangle). \tag{19}$$

**Lemma 7.** Spectral densities  $F_m^0(\lambda) \in D_F$  which satisfy the minimality condition (13) are the least favorable in the class  $D_F$  for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $t < 0$ ,  $x \in S_n$  if the Fourier coefficients of the functions  $(F_m^0(\lambda))^{-1}$  determine matrices  $\mathbf{B}_m^0$  giving a solution of the constrained optimization problem

$$\max_{F_m \in D_F} \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \langle \mathbf{B}_m^{-1} \mathbf{a}_m^l, \mathbf{a}_m^l \rangle = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \langle (\mathbf{B}_m^0)^{-1} \mathbf{a}_m^l, \mathbf{a}_m^l \rangle. \tag{20}$$

**Lemma 8.** Spectral densities  $F_m^0(\lambda) \in D_F$  which admit the canonical factorization (16) are the least favorable in the class  $D_F$  for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $t < 0$ ,  $x \in S_n$  if the coefficients of factorizations define a solution of the constrained optimization problem

$$\sup_{F_m \in D_F} \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \|\mathbf{A}_m^l \mathbf{d}_m\|^2 = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \|\mathbf{A}_m^l \mathbf{d}_m^0\|^2, \tag{21}$$

$$F_m(\lambda) = \left( \sum_{u=0}^{\infty} d_m(u) e^{-iu\lambda} \right) \left( \sum_{u=0}^{\infty} d_m(u) e^{-iu\lambda} \right)^* \in D_F.$$

For more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics we observe that the least favorable spectral densities  $F_m^0(\lambda) \in D_F$ ,  $G_m^0(\lambda) \in D_G$  and the minimax spectral characteristic  $h^0 = h(F^0, G^0)$  form a saddle point of the function  $\Delta(h; F, G)$  on the set  $H_D \times D$ . The saddle point inequalities

$$\Delta(h^0; F, G) \leq \Delta(h^0; F^0, G^0) \leq \Delta(h; F^0, G^0),$$

$$\forall h \in H_D, \quad \forall F \in D_F, \quad \forall G \in D_G$$

hold if  $h^0 = h(F^0, G^0)$ ,  $h(F^0, G^0) \in H_D$  and  $(F^0, G^0)$  is a solution of the constrained optimization problem

$$\max_{(F,G) \in D} \Delta(h(F^0, G^0); F, G) = \Delta(h(F^0, G^0); F^0, G^0), \tag{22}$$

where the functional

$$\Delta(h(F^0, G^0); F, G) =$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} [(A_m^l(\lambda))^{\top} G_m^0(\lambda) + (C_m^{l0}(\lambda))^{\top}] (F_m^0(\lambda) + G_m^0(\lambda))^{-1} \\
&\quad \times F_m(\lambda) (F_m^0(\lambda) + G_m^0(\lambda))^{-1} \left[ G_m^0(\lambda) \overline{A_m^l(\lambda)} + \overline{C_m^{l0}(\lambda)} \right] d\lambda \\
&+ \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} [(A_m^l(\lambda))^{\top} F_m^0(\lambda) - (C_m^{l0}(\lambda))^{\top}] (F_m^0(\lambda) + G_m^0(\lambda))^{-1} \\
&\quad \times G_m(\lambda) (F_m^0(\lambda) + G_m^0(\lambda))^{-1} \left[ F_m^0(\lambda) \overline{A_m^l(\lambda)} - \overline{C_m^{l0}(\lambda)} \right] d\lambda. \tag{23}
\end{aligned}$$

The constrained optimization problem (22) is equivalent to the following unconstrained optimization problem

$$\min_{(F,G) \in D} \Delta_D(F, G) = \Delta_D(F^0, G^0), \tag{24}$$

$$\Delta_D(F, G) = -\Delta(h(F^0, G^0); F, G) + \delta((F, G)|D),$$

where  $\delta((F, G)|D)$  is the indicator function of the set  $D$ . Solution  $(F^0, G^0)$  to the optimization problem (24) is determined by the condition  $0 \in \partial\Delta_D(F^0, G^0)$  which is necessary for the point  $(F^0, G^0)$  to belong to the set of minimums of a convex functional. Here  $\partial\Delta_D(F^0, G^0)$  is a subdifferential of the convex functional  $\Delta_D(F, G)$  at point  $(F, G) = (F^0, G^0)$  (see R. T. Rockafellar [41], M. P. Moklyachuk [31]).

The form (23) of the functional  $\Delta(h(F^0, G^0); F, G)$  is convenient for application the method of Lagrange multipliers for finding solution to the problem (24). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions  $\delta((F, G)|D)$  we describe relations that determine the least favourable spectral densities in some special classes of spectral densities (see books by M. Moklyachuk [30], M. Moklyachuk and I. Golichenko [33], M. Moklyachuk and O. Masytka [34] for more details).

### The least favorable spectral densities in the class $D_0 \times D_\varepsilon$

Consider the problem of minimax estimation of the functional  $A\zeta$  depending on the unknown values of the random field  $\{\zeta(t, x), t \in \mathbb{R}, x \in S_n\}$ , which is periodically correlated with respect to the time argument  $t \in \mathbb{R}$  and isotropic on the sphere  $S_n$  with respect to spatial argument  $x \in S_n$  based on observations of the random field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x) : t < 0, x \in S_n$ , under the condition that spectral densities  $F_m(\lambda), G_m(\lambda)$  of stationary sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j), j \in \mathbb{Z}\}$  which are constructed with the help of relations (2), (3), respectively, are not known exactly while there are specified the following pairs of sets of admissible spectral densities.

The first pair is

$$D_0^1 = \left\{ F(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} \text{Tr} F_m(\lambda) d\lambda = p \right\},$$

$$D_\varepsilon^1 = \left\{ G(\lambda) \mid \text{Tr} G_m(\lambda) = (1 - \varepsilon) \text{Tr} U_m(\lambda) + \varepsilon \text{Tr} V_m(\lambda), \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} \text{Tr} G_m(\lambda) d\lambda = q \right\}.$$

The second pair of sets of admissible spectral densities is

$$D_0^2 = \left\{ F(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} F_m^{kk}(\lambda) d\lambda = p_k, k = 1, 2, \dots \right\},$$

$$D_\varepsilon^2 = \left\{ G(\lambda) \mid G_m^{kk}(\lambda) = (1 - \varepsilon)U_m^{kk}(\lambda) + \varepsilon V_m^{kk}(\lambda), \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} G_m^{kk}(\lambda) d\lambda = q_k, k = 1, 2, \dots \right\}.$$

The third pair of sets of admissible spectral densities is

$$D_0^3 = \left\{ F(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} \langle B_1, F_m(\lambda) \rangle d\lambda = p \right\},$$

$$D_\varepsilon^3 = \left\{ G(\lambda) \mid \langle B_2, G_m(\lambda) \rangle = (1 - \varepsilon) \langle B_2, U_m(\lambda) \rangle + \varepsilon \langle B_2, V_m(\lambda) \rangle, \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} \langle B_2, G_m(\lambda) \rangle d\lambda = q \right\}.$$

The fourth pair of sets of admissible spectral densities is

$$D_0^4 = \left\{ F(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} F_m(\lambda) d\lambda = P \right\},$$

$$D_\varepsilon^4 = \left\{ G(\lambda) \mid G_m(\lambda) = (1 - \varepsilon)U_m(\lambda) + \varepsilon V_m(\lambda), \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} G_m(\lambda) d\lambda = Q \right\}.$$

Here  $V_m(\lambda), U_m(\lambda)$  are given matrices of spectral densities,  $p, q, p_k, q_k, k = 1, 2, \dots$  are given numbers,  $B_1, B_2, P, Q$  are given positive-definite Hermitian matrices.

From the condition  $0 \in \partial\Delta_D(F^0, G^0)$  we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair  $D_0^1 \times D_\varepsilon^1$  we get equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = \alpha_m^2 (F_m^0(\lambda) + G_m^0(\lambda))^2, \tag{25}$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = (\beta_m^2 + \gamma_m(\lambda))(F_m^0(\lambda) + G_m^0(\lambda))^2, \tag{26}$$

where

$$r_F(\lambda) = (A_m^l(\lambda))^T F_m(\lambda) - (C_m^l(\lambda))^T,$$

$$r_G(\lambda) = (A_m^l(\lambda))^T G_m(\lambda) + (C_m^l(\lambda))^T,$$

$$\gamma_m(\lambda) \leq 0 \text{ and } \gamma_m(\lambda) = 0 \text{ if } \text{Tr } G_m^0(\lambda) > (1 - \varepsilon)\text{Tr } U_m(\lambda),$$

and  $\alpha_m^2, \beta_m^2$  are unknown Lagrange multipliers.

For the second pair  $D_0^2 \times D_\varepsilon^2$  we get equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = (F_m^0(\lambda) + G_m^0(\lambda)) \{ \alpha_{mk}^2 \delta_k^n \}_{k,n=1}^\infty (F_m^0(\lambda) + G_m^0(\lambda)), \quad (27)$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = (F_m^0(\lambda) + G_m^0(\lambda)) \{ (\beta_{mk}^2 + \gamma_{mk}(\lambda)) \delta_k^n \}_{k,n=1}^\infty (F_m^0(\lambda) + G_m^0(\lambda)), \quad (28)$$

where

$$\gamma_{mk}(\lambda) \leq 0 \text{ and } \gamma_{mk}(\lambda) = 0 \text{ if } G_m^{0kk}(\lambda) > (1 - \varepsilon)U_m^{kk}(\lambda),$$

and  $\alpha_{mk}^2, \beta_{mk}^2$  are unknown Lagrange multipliers.

For the third pair  $D_0^3 \times D_\varepsilon^3$  we get equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = \alpha_m^2 (F_m^0(\lambda) + G_m^0(\lambda)) (B_1)^\top (F_m^0(\lambda) + G_m^0(\lambda)), \quad (29)$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = (\beta_m^2 + \gamma'_m(\lambda)) (F_m^0(\lambda) + G_m^0(\lambda)) (B_2)^\top (F_m^0(\lambda) + G_m^0(\lambda)), \quad (30)$$

where

$$\gamma'_m(\lambda) \leq 0 \text{ and } \gamma'_m(\lambda) = 0 \text{ if } \langle B_2, G_m^0(\lambda) \rangle > (1 - \varepsilon) \langle B_2, U_m(\lambda) \rangle,$$

and  $\alpha_m^2, \beta_m^2$  are unknown Lagrange multipliers.

For the fourth pair  $D_0^4 \times D_\varepsilon^4$  we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = (F_m^0(\lambda) + G_m^0(\lambda)) \alpha_m \cdot \alpha_m^* (F_m^0(\lambda) + G_m^0(\lambda)), \quad (31)$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = (F_m^0(\lambda) + G_m^0(\lambda)) (\beta_m \cdot \beta_m^* + \Gamma_m(\lambda)) (F_m^0(\lambda) + G_m^0(\lambda)), \quad (32)$$

where  $\Gamma_m(\lambda)$  are Hermitian matrices,

$$\Gamma_m(\lambda) \leq 0 \text{ and } \Gamma_m(\lambda) = 0 \text{ if } G_m^0(\lambda) > (1 - \varepsilon)U_m(\lambda),$$

and  $\alpha_m, \beta_m$  are vectors of unknown Lagrange multipliers.

The following theorem holds true.

**Theorem 9.** *Let the minimality condition (8) hold true. The least favorable spectral densities  $F_m^0(\lambda), G_m^0(\lambda)$  in the classes  $D_0 \times D_\varepsilon$  for the optimal estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x), t < 0, x \in S_n$  are determined by relations (25), (26) for the first pair  $D_0^1 \times D_\varepsilon^1$  of sets of admissible spectral densities, (27), (28) for the second pair  $D_0^2 \times D_\varepsilon^2$  of sets of admissible spectral densities, (29), (30) for the third pair  $D_0^3 \times D_\varepsilon^3$  of sets of admissible spectral densities, (31), (32) for the fourth pair  $D_0^4 \times D_\varepsilon^4$  of sets of admissible spectral densities, constrained optimization problem (19) and restrictions on densities from the corresponding classes  $D_0 \times D_\varepsilon$ . The minimax spectral characteristic  $h(F^0, G^0)$  of the optimal estimate  $\hat{A}\zeta$  is calculated by formula (10). The mean square error  $\Delta(F^0, G^0)$  is calculated by formula (12).*

**Corollary 10.** *Let the minimality condition (13) hold true. The least favorable spectral densities  $F_m^0(\lambda)$  in the classes  $D_0^k$ ,  $k = 1, 2, 3, 4$ , for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  for  $t < 0$ ,  $x \in S_n$  are determined by the following equations, respectively,*

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = \alpha_m^2 (F_m^0(\lambda))^2, \tag{33}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda) \{ \alpha_{mk}^2 \delta_k^n \}_{k,n=1}^\infty F_m^0(\lambda), \tag{34}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = \alpha_m^2 F_m^0(\lambda) (B_1)^\top F_m^0(\lambda), \tag{35}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda) \alpha_m \cdot \alpha_m^* F_m^0(\lambda), \tag{36}$$

*constrained optimization problem (20) and restrictions on densities from the corresponding classes  $D_0^k$ ,  $k = 1, 2, 3, 4$ . The minimax spectral characteristic  $h(F^0)$  of the optimal estimate  $\hat{A}\zeta$  is calculated by (14). The mean square error  $\Delta(F^0)$  is calculated by (15).*

**Corollary 11.** *Let the minimality condition (13) hold true. The least favorable spectral densities  $F_m^0(\lambda)$  in the classes  $D_\varepsilon^k$ ,  $k = 1, 2, 3, 4$ , for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $t < 0$ ,  $x \in S_n$  are determined by the following equations, respectively,*

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = (\alpha_m^2 + \gamma_m(\lambda)) (F_m^0(\lambda))^2, \tag{37}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda) \{ (\alpha_{mk}^2 + \gamma_{mk}(\lambda)) \delta_k^n \}_{k,n=1}^\infty F_m^0(\lambda), \tag{38}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = (\alpha_m^2 + \gamma'_m(\lambda)) F_m^0(\lambda) (B_2)^\top F_m^0(\lambda), \tag{39}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda) (\alpha_m \cdot \alpha_m^* + \Gamma_m(\lambda)) F_m^0(\lambda), \tag{40}$$

*constrained optimization problem (20) and restrictions on densities from the corresponding classes  $D_\varepsilon^k$ ,  $k = 1, 2, 3, 4$ . The minimax spectral characteristic  $h(F^0)$  of the optimal estimate  $\hat{A}\zeta$  is calculated by formula (14). The mean square error  $\Delta(F^0)$  is calculated by formula (15).*

Consider the problem of optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $t < 0$ ,  $x \in S_n$  under condition that the spectral densities  $F_m(\lambda)$  admit canonical factorization (17).

From the condition  $0 \in \partial\Delta_D(F^0)$  we find the following equations which determine the least favourable spectral densities for the classes  $D_0^k$ ,  $k = 1, 2, 3, 4$ , respectively

$$\sum_{l=1}^{h(m,n)} (S_m^{l0}(\lambda))^\top \overline{(S_m^{l0}(\lambda))} = \alpha_m^2 (P_m^0(\lambda))^\top \overline{P_m^0(\lambda)}, \tag{41}$$

$$\sum_{l=1}^{h(m,n)} (S_m^{l0}(\lambda))^\top \overline{(S_m^{l0}(\lambda))} = (P_m^0(\lambda))^\top \left\{ \alpha_{mk}^2 \delta_k^n \right\}_{k,n=1}^\infty \overline{P_m^0(\lambda)}, \quad (42)$$

$$\sum_{l=1}^{h(m,n)} (S_m^{l0}(\lambda))^\top \overline{(S_m^{l0}(\lambda))} = (P_m^0(\lambda))^\top (B_1)^\top \overline{P_m^0(\lambda)}, \quad (43)$$

$$\sum_{l=1}^{h(m,n)} (S_m^{l0}(\lambda))^\top \overline{(S_m^{l0}(\lambda))} = (P_m^0(\lambda))^\top \alpha_m \cdot \alpha_m^* \overline{P_m^0(\lambda)}. \quad (44)$$

**Theorem 12.** *The least favorable spectral densities  $F_m^0(\lambda)$  in the classes  $D_0^k$ ,  $k = 1, 2, 3, 4$ , for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  for  $t < 0$ ,  $x \in S_n$  are determined by relations (41) - (44), respectively, constrained optimization problem (20) and restrictions on densities from the corresponding classes  $D_0^k$ ,  $k = 1, 2, 3, 4$ . The minimax spectral characteristic of the optimal estimate of the functional  $A\zeta$  is calculated by the formula (17). The mean square error  $\Delta(F^0)$  is calculated by formula (18).*

### The least favorable spectral densities in the class $D_V^U \times D_{1\varepsilon}$

Consider the problem of minimax estimation of the functional  $A\zeta$  depending on the unknown values of the random field  $\{\zeta(t, x), t \in \mathbb{R}, x \in S_n\}$ , which is periodically correlated with respect to the time argument  $t \in \mathbb{R}$  and isotropic on the sphere  $S_n$  with respect to spatial argument  $x \in S_n$  based on observations of the random field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x) : t < 0, x \in S_n$ , under the condition that spectral densities  $F_m(\lambda), G_m(\lambda)$  of stationary sequences  $\{\zeta_m^l(j), j \in \mathbb{Z}\}$  and  $\{\theta_m^l(j), j \in \mathbb{Z}\}$  which are constructed with the help of relations (2), (3), respectively, are not known exactly while there are specified the following pairs of sets of admissible spectral densities.

The first pair is

$$D_V^{U1} = \left\{ F(\lambda) \mid \text{Tr } V_m(\lambda) \leq \text{Tr}(F_m(\lambda)) \leq \text{Tr}(U_m(\lambda)), \right. \\ \left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} \text{Tr}(F_m(\lambda)) d\lambda = p \right\}, \\ D_{1\varepsilon}^1 = \left\{ G(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} |\text{Tr}(G_m(\lambda)) - \text{Tr}(G_m^1(\lambda))| d\lambda \leq \varepsilon \right\}.$$

The second pair of sets of admissible spectral densities is

$$D_V^{U2} = \left\{ F(\lambda) \mid V_m^{kk}(\lambda) \leq F_m^{kk}(\lambda) \leq U_m^{kk}(\lambda), \right. \\ \left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} F_m^{kk}(\lambda) d\lambda = p_k, k = 1, 2, \dots \right\}, \\ D_{1\varepsilon}^2 = \left\{ G(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} |G_m^{kk}(\lambda) - G_m^{1kk}(\lambda)| d\lambda \leq \varepsilon_k, k = 1, 2, \dots \right\}.$$



The third pair of sets of admissible spectral densities is

$$D_V^{U^3} = \left\{ F(\lambda) \mid \langle B_1, V_m(\lambda) \rangle \leq \langle B_1, F_m(\lambda) \rangle \leq \langle B_1, U_m(\lambda) \rangle, \right. \\ \left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} \langle B_1, F_m(\lambda) \rangle d\lambda = p \right\},$$

$$D_{1\varepsilon}^3 = \left\{ G(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} |\langle B_2, G_m(\lambda) \rangle - \langle B_2, G_m^1(\lambda) \rangle| d\lambda \leq \varepsilon \right\}.$$

The fourth pair of sets of admissible spectral densities is

$$D_V^{U^4} = \left\{ F(\lambda) \mid V_m(\lambda) \leq F_m(\lambda) \leq U_m(\lambda), \right. \\ \left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} F_m(\lambda) d\lambda = P \right\},$$

$$D_{1\varepsilon}^4 = \left\{ G(\lambda) \mid \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m, n) \int_{-\pi}^{\pi} |G_m^{kj}(\lambda) - G_m^{1kj}(\lambda)| d\lambda \leq \varepsilon_{kj}, k, j = 1, 2, \dots \right\}.$$

Here  $V_m(\lambda), U_m(\lambda), G_m^1(\lambda)$  are given matrices of spectral densities,  $B_1, B_2, P$  are given positive-definite Hermitian matrices,  $p, p_k, k = 1, 2, \dots; \varepsilon, \varepsilon_k, \varepsilon_{kj}, k, j = 1, 2, \dots$  are given numbers.

From the condition  $0 \in \partial\Delta_D(F^0, G^0)$  we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities  $D_V^U \times D_{1\varepsilon}$ .

For the first pair  $D_V^{U^1} \times D_{1\varepsilon}^1$  we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = (\alpha_m^2 + \gamma_{m_1}(\lambda) + \gamma_{m_2}(\lambda))(F_m^0(\lambda) + G_m^0(\lambda))^2, \tag{45}$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = \beta_m^2 \gamma_m(\lambda)(F_m^0(\lambda) + G_m^0(\lambda))^2, \tag{46}$$

where

$$r_F(\lambda) = (A_m^l(\lambda))^T F_m(\lambda) - (C_m^l(\lambda))^T, \\ r_G(\lambda) = (A_m^l(\lambda))^T G_m(\lambda) + (C_m^l(\lambda))^T, \\ \gamma_{m_1}(\lambda) \leq 0 \text{ and } \gamma_{m_1}(\lambda) = 0 \text{ if } \text{Tr}(F_m^0(\lambda)) > \text{Tr}(V_m(\lambda)), \\ \gamma_{m_2}(\lambda) \geq 0 \text{ and } \gamma_{m_2}(\lambda) = 0 \text{ if } \text{Tr}(F_m^0(\lambda)) < \text{Tr}(U_m(\lambda)), \\ \gamma_m(\lambda) = \text{sign}(\text{Tr}(G_m^0(\lambda)) - \text{Tr}(G_m^1(\lambda))) \text{ if } (\text{Tr}(G_m^0(\lambda)) - \text{Tr}(G_m^1(\lambda))) \neq 0,$$

and  $\alpha_m^2, \beta_m^2$  are unknown Lagrange multipliers.

For the second pair  $D_V^{U^2} \times D_{1\varepsilon}^2$  we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = (F_m^0(\lambda) + G_m^0(\lambda)) \\ \times \left\{ (\alpha_{mk}^2 + \gamma_{m_1k}(\lambda) + \gamma_{m_2k}(\lambda)) \delta_k^n \right\}_{k,n=1}^{\infty} (F_m^0(\lambda) + G_m^0(\lambda)), \tag{47}$$

$$\begin{aligned}
& \sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = \\
& = (F_m^0(\lambda) + G_m^0(\lambda)) \{(\beta_{mk}^2 \gamma_{mk}(\lambda)) \delta_k^n\}_{k,n=1}^{\infty} (F_m^0(\lambda) + G_m^0(\lambda)), \tag{48}
\end{aligned}$$

where

$$\begin{aligned}
& \gamma_{m_1 k}(\lambda) \leq 0 \text{ and } \gamma_{m_1 k}(\lambda) = 0 \text{ if } F_m^{0kk}(\lambda) > V_m^{kk}(\lambda), \\
& \gamma_{m_2 k}(\lambda) \geq 0 \text{ and } \gamma_{m_2 k}(\lambda) = 0 \text{ if } F_m^{0kk}(\lambda) < U_m^{kk}(\lambda), \\
& \gamma_{mk}(\lambda) = \text{sign}(G_m^{0kk} - G_m^{1kk}) \text{ if } (G_m^{0kk} - G_m^{1kk}) \neq 0,
\end{aligned}$$

and  $\alpha_{mk}^2, \beta_{mk}^2$  are unknown Lagrange multipliers.

For the third pair  $D_V^{U^3} \times D_{1\varepsilon}^3$  we have equations

$$\begin{aligned}
& \sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = \\
& = (\alpha_m^2 + \gamma'_{m_1}(\lambda) + \gamma'_{m_2}(\lambda))(F_m^0(\lambda) + G_m^0(\lambda))(B_1)^\top (F_m^0(\lambda) + G_m^0(\lambda)), \tag{49}
\end{aligned}$$

$$\begin{aligned}
& \sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = \\
& = (\beta_m^2 \gamma'_m(\lambda))(F_m^0(\lambda) + G_m^0(\lambda))(B_2)^\top (F_m^0(\lambda) + G_m^0(\lambda)), \tag{50}
\end{aligned}$$

where

$$\begin{aligned}
& \gamma'_{m_1}(\lambda) \leq 0 \text{ and } \gamma'_{m_1}(\lambda) = 0 \text{ if } \langle B_1, F_m^0(\lambda) \rangle > \langle B_1, V_m(\lambda) \rangle, \\
& \gamma'_{m_2}(\lambda) \geq 0 \text{ and } \gamma'_{m_2}(\lambda) = 0 \text{ if } \langle B_1, F_m^0(\lambda) \rangle < \langle B_1, U_m(\lambda) \rangle, \\
& \gamma'_m(\lambda) = \text{sign}(\langle B_2, G_m^0(\lambda) \rangle - \langle B_2, G_m^1(\lambda) \rangle) \text{ if } \langle B_2, G_m^0(\lambda) \rangle - \langle B_2, G_m^1(\lambda) \rangle \neq 0,
\end{aligned}$$

and  $\alpha_m^2, \beta_m^2$  are unknown Lagrange multipliers.

For the fourth pair  $D_V^{U^4} \times D_\varepsilon^4$  we have equations

$$\begin{aligned}
& \sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = \\
& = (F_m^0(\lambda) + G_m^0(\lambda))(\alpha_m \cdot \alpha_m^* + \Gamma_{m_1}(\lambda) + \Gamma_{m_2}(\lambda))(F_m^0(\lambda) + G_m^0(\lambda)), \tag{51}
\end{aligned}$$

$$\begin{aligned}
& \sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = \\
& = (F_m^0(\lambda) + G_m^0(\lambda))(\beta_m \Gamma_m(\lambda) \beta_m^*)(F_m^0(\lambda) + G_m^0(\lambda)), \tag{52}
\end{aligned}$$

where  $\Gamma_{m_1}(\lambda), \Gamma_{m_2}(\lambda), \Gamma_m(\lambda)$  are Hermitian matrices,

$$\begin{aligned}
& \Gamma_{m_1}(\lambda) \leq 0 \text{ and } \Gamma_{m_1}(\lambda) = 0 \text{ if } F_m^0(\lambda) > V_m(\lambda), \\
& \Gamma_{m_2}(\lambda) \geq 0 \text{ and } \Gamma_{m_2}(\lambda) = 0 \text{ if } F_m^0(\lambda) < U_m(\lambda), \\
& \Gamma_m^{kj}(\lambda) = \text{sign}(G_m^{0kj} - G_m^{1kj}) \text{ if } (G_m^{0kj} - G_m^{1kj}) \neq 0,
\end{aligned}$$

and  $\alpha_m, \beta_m$  are vectors of unknown Lagrange multipliers.

The following theorem holds true.

**Theorem 13.** Let the minimality condition (8) hold true. The least favorable spectral densities  $F_m^0(\lambda)$ ,  $G_m^0(\lambda)$  in the classes  $D_V^U \times D_{1\epsilon}$  for the optimal estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$  are determined by relations (45), (46) for the first pair  $D_V^{U^1} \times D_{1\epsilon}^1$  of sets of admissible spectral densities, (47), (48) for the second pair  $D_V^{U^2} \times D_{1\epsilon}^2$  of sets of admissible spectral densities, (49), (50) for the third pair  $D_V^{U^3} \times D_{1\epsilon}^3$  of sets of admissible spectral densities, (51), (52) for the fourth pair  $D_V^{U^4} \times D_{1\epsilon}^4$  of sets of admissible spectral densities, constrained optimization problem (19) and restrictions on densities from the corresponding classes  $D_0 \times D_\epsilon$ . The minimax spectral characteristic  $h(F^0, G^0)$  of the optimal estimate  $\hat{A}\zeta$  is calculated by formula (10). The mean square error  $\Delta(F^0, G^0)$  is calculated by formula (12).

**Corollary 14.** Let the minimality condition (13) hold true. The least favorable spectral densities  $F_m^0(\lambda)$  in the classes  $D_V^{U^k}$ ,  $k = 1, 2, 3, 4$ , for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  for  $t < 0$ ,  $x \in S_n$  are determined by the following equations, respectively,

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = (\alpha_m^2 + \gamma_{m1}(\lambda) + \gamma_{m2}(\lambda))(F_m^0(\lambda))^2, \tag{53}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda) \{(\alpha_{mk}^2 + \gamma_{m1k}(\lambda) + \gamma_{m2k}(\lambda))\delta_k^n\}_{k,n=1}^\infty F_m^0(\lambda), \tag{54}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = (\alpha_m^2 + \gamma'_{m1}(\lambda) + \gamma'_{m2}(\lambda))F_m^0(\lambda)(B_1)^\top F_m^0(\lambda), \tag{55}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda)(\alpha_m \cdot \alpha_m^* + \Gamma_{m1}(\lambda) + \Gamma_{m2}(\lambda))F_m^0(\lambda), \tag{56}$$

constrained optimization problem (20) and restrictions on densities from the corresponding classes  $D_V^{U^k}$ ,  $k = 1, 2, 3, 4$ . The minimax spectral characteristic  $h(F^0)$  of the optimal estimate  $\hat{A}\zeta$  is calculated by (14). The mean square error  $\Delta(F^0)$  is calculated by (15).

**Corollary 15.** Let the minimality condition (13) hold true. The least favorable spectral densities  $F_m^0(\lambda)$  in the classes  $D_{1\epsilon}^k$ ,  $k = 1, 2, 3, 4$ , for the optimal linear estimation of the functional  $A\zeta$  from observations of the field  $\zeta(t, x)$  at points  $t < 0$ ,  $x \in S_n$  are determined by the following equations, respectively,

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = (\alpha_m^2 \gamma_m(\lambda))(F_m^0(\lambda))^2, \tag{57}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda) \{(\alpha_{mk}^2 \gamma_{mk}(\lambda))\delta_k^n\}_{k,n=1}^\infty F_m^0(\lambda), \tag{58}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = (\alpha_m^2 \gamma'_m(\lambda))F_m^0(\lambda)(B_2)^\top F_m^0(\lambda), \tag{59}$$

$$\sum_{l=1}^{h(m,n)} ((C_m^{l0}(\lambda))^\top)^* \cdot (C_m^{l0}(\lambda))^\top = F_m^0(\lambda)(\beta_m \Gamma_m(\lambda) \beta_m^*)F_m^0(\lambda), \tag{60}$$

constrained optimization problem (20) and restrictions on densities from the corresponding classes  $D_{1\epsilon}^k$ ,  $k = 1, 2, 3, 4$ . The minimax spectral characteristic  $h(F^0)$  of the optimal estimate  $\hat{A}\zeta$  is calculated by formula (14). The mean square error  $\Delta(F^0)$  is calculated by formula (15).

## Conclusions

In this paper we propose formulas for calculating the mean square error and the spectral characteristic of the optimal linear estimate of the functional

$$A\zeta = \int_0^{\infty} \int_{S_n} a(t, x) \zeta(t, x) m_n(dx) dt$$

depending on unknown values of a mean-square continuous periodically correlated (cyclostationary with period  $T$ ) with respect to time argument and isotropic on the unit sphere  $S_n$  in Euclidean space  $\mathbb{E}^n$  random field  $\zeta(t, x)$ ,  $t \in \mathbb{R}$ ,  $x \in S_n$ . Estimates are based on observations of the field  $\zeta(t, x) + \theta(t, x)$  at points  $(t, x)$ ,  $t < 0$ ,  $x \in S_n$  where  $\theta(t, x)$  is an uncorrelated with  $\zeta(t, x)$  mean-square continuous periodically correlated with respect to time argument and isotropic on the sphere  $S_n$  random field. The problem is investigated in the case of spectral certainty where matrices of spectral densities of random fields are known exactly and in the case of spectral uncertainty where matrices of spectral densities of random fields are not known exactly while some classes of admissible spectral density matrices are given. We derive formulas for calculation the spectral characteristic and the mean-square error of the optimal linear estimate of the functional  $A\zeta$  in the case of spectral certainty, where spectral densities  $F_m(\lambda)$ ,  $G_m(\lambda)$  of the stationary sequences that generate the random fields  $\zeta(t, x)$ ,  $\theta(t, x)$  are known exactly.

We propose a representation of the mean square error in the form of a linear functional in the  $L_1 \times L_1$  space with respect to spectral densities  $(F, G)$ , which allows us to solve the corresponding constrained optimization problem and describe the minimax (robust) estimates of the functional  $A\zeta$  for concrete classes of spectral densities under the condition that spectral densities are not known exactly while classes  $D = D_F \times D_G$  of admissible spectral densities are given.

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