

Numerical Solution of Nonlinear Volterra-Fredholm Integral Equations Using Haar Wavelet Collocation Method

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Abstract: In this paper, we present a numerical solution of nonlinear Volterra-Fredholm integral equations using Haar wavelet collocation method. Properties of Haar wavelet and its operational matrices are utilized to convert the integral equation into a system of algebraic equations, solving these equations using MATLAB to compute the Haar coefficients. The numerical results are compared with exact and existing method through error analysis, which shows the efficiency of the technique.

1. Introduction

Integral equations have motivated a large amount of research work in recent years. Integral equations find its applications in various fields of mathematics, science and technology has been studied extensively both at the theoretical and practical level. In particular, integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc. In most of the cases, it is difficult to solve them, especially analytically [1]. Analytical solutions of integral equations, however, either does not exist or are difficult to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations. Therefore variety of analytical and numerical methods have been used to handle integral equations were introduced, among which optimal homotopy asymptotic method [1], series solution method [2], Adomian decomposition method [3], Discrete Adomian Decomposition Method [4], the variational iteration method [5], Homotopy Perturbation Method [6], etc. Nonlinearity is one of the interesting topics among the physicists, mathematicians, engineers, etc. Since most physical systems are inherently nonlinear in nature. The mathematical modelling of many scientific real world problems occurs nonlinearly. Recently, many authors have solved nonlinear Volterra-Fredholm integral equations from various methods such as Linearization method [7], Hybrid Functions [8], hybrid of block-pulse functions and Taylor series [9], Bessel functions [10], new basis functions [11], radial basis functions [12], Triangular functions (TF) method [13].

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [14, 15]. Bujurke et al. [16] have proposed Fast wavelet multigrid method, Shiralashetti et al. [17, 18] have introduced new wavelet based full-approximation scheme and wavelet based decoupled method. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations. The solutions are often quite complicated and the advantages of the wavelet method get lost. Therefore any kind of simplification is welcome. One possibility for it is to make use of the Haar wavelets, which are mathematically the simplest wavelets. In the previous work, system analysis via Haar wavelets was led by Chen and Hsiao [19], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the applications for the Haar analysis into the dynamic systems. Recently, Haar wavelet method is applied for different type of problems. Namely, Bujurke et al. [20-22] used the single term Haar wavelet series for the numerical

solution of stiff systems from nonlinear dynamics, nonlinear oscillator equations and Sturm-Liouville problems. Siraj-ul-Islam et al. [23] proposed for the numerical solution of second order boundary value problems. Shiralashetti et al. [24-27] applied for the numerical solution of Klein-Gordan equations, multi-term fractional differential equations, singular initial value problems and Riccati and Fractional Riccati Differential Equations. Shiralashetti et al. [28] have introduced the adaptive gird Haar wavelet collocation method for the numerical solution of parabolic partial differential equations. Also, Haar wavelet method is applied for different kind of integral equations, which among Lepik et al. [29-32] presented the solution for differential and integral equations. Babolian et al. [33] and Shiralashetti et al. [34] applied for solving nonlinear Fredholm integral equations. Aziz et al. [35] have introduced a new algorithm for the numerical solution of nonlinear Fredholm and Volterra integral equations. In the present work, we applied the Haar wavelet collocation method for the numerical solution of nonlinear Volterra-Fredholm integral equations. The article is organized as follows: In Section 2, properties of Haar wavelets and its operational matrix is given. In Section 3, the method of solution is discussed. In section 4, we report our numerical results and demonstrated the accuracy of the proposed scheme. Lastly, the conclusion of the proposed work is given in section 5.

2. Properties of Haar Wavelets

2.1. Haar Wavelets

The scaling function $h_1(x)$ for the family of the Haar wavelet is defined as

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

The Haar Wavelet family for $x \in [0, 1)$ is defined as,

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta), \\ -1 & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere,} \end{cases} \quad (2.2)$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$,

where $m = 2^l$, $l = 0, 1, \dots, J$, J is the level of resolution; and $k = 0, 1, \dots, m-1$ is the translation parameter. Maximum level of resolution is J . The index i in (2.2) is calculated using $i = m + k + 1$. In case of minimal values $m = 1$, $k = 0$ then $i = 2$. The maximal value of i is $N = 2^{J+1}$.

Let us define the collocation points $x_j = \frac{j-0.5}{N}$, $j = 1, 2, \dots, N$, Haar coefficient matrix $H(i, j) = h_i(x_j)$ which has the dimension $N \times N$.

For instance, $J = 3 \Rightarrow N = 16$, then we have

$$H(16,16) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Any function $f(x)$ which is square integrable in the interval $(0, 1)$ can be expressed as an infinite sum of Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \quad (2.3)$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval. Given a function $f(x) \in L^2(R)$ a multi-resolution analysis (MRA) of $L^2(R)$ produces a sequence of subspaces V_j, V_{j+1}, \dots such that the projections of $f(x)$ onto these spaces give finer and finer approximations of the function $f(x)$ as $j \rightarrow \infty$.

2.2. Operational Matrix of Haar Wavelet

The operational matrix P which is an N square matrix is defined by

$$P_{1,i}(x) = \int_0^x h_i(t) dt \quad (2.4)$$

often, we need the integrals

$$P_{r,i}(x) = \underbrace{\int_0^x \int_0^x \dots \int_0^x}_{r\text{-times}} h_i(t) dt^r = \frac{1}{(r-1)!} \int_A^x (x-t)^{r-1} h_i(t) dt \quad (2.5)$$

$r = 1, 2, \dots, n$ and $i = 1, 2, \dots, N$.

For $r = 1$, corresponds to the function $P_{1,i}(x)$, with the help of (2.2) these integrals can be calculated analytically; we get

$$P_{1,i}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta) \\ \gamma - x & \text{for } x \in [\beta, \gamma) \\ 0 & \text{Otherwise} \end{cases} \quad (2.6)$$

3. Method of Solution

In this section, we present a Haar wavelet collocation method (HWCM) based on Leibnitz rule for the numerical solution of nonlinear Volterra-Fredholm integral equation of the form,

$$u(x) = f(x) + \int_0^x K_1(x, t, u(t)) dt + \int_0^1 K_2(x, t, u(t)) dt, \quad (3.1)$$

where $K_1(x, t, u(t))$ and $K_2(x, t, u(t))$ is a nonlinear function defined on $[0, 1] \times [0, 1]$ are the known function is called the kernel of the integral equation and $f(x)$ is also a known function, while the unknown function $u(x)$ represents the solution of the integral equation. Basic principle is that for conversion of the integral equation into equivalent differential equation with initial conditions. The conversion is achieved by the well-known Leibnitz rule [36].

Numerical computational Procedure as follows,

Step 1: Differentiating (3.1) twice w.r.to x , using Leibnitz rule, we get differential equations with subject to initial conditions $u(0) = \beta$, $u'(0) = \gamma$.

Step 2: Applying Haar wavelet collocation method,

Let us assume that,

$$u''(x) = \sum_{i=1}^N a_i h_i(x) \quad (3.2)$$

Step 3: By integrating (3.2) twice and substituting the initial conditions, we get,

$$u'(x) = \gamma + \sum_{i=1}^N a_i p_{1,i}(x) \quad (3.3)$$

$$u(x) = \beta + \gamma x + \sum_{i=1}^N a_i p_{2,i}(x) \quad (3.4)$$

Step 4: Substituting (3.2) - (3.4) in the differential equation, which reduces to the nonlinear system of N equations with N unknowns and then the Newton's method is used to obtain the Haar coefficients a_i , $i = 1, 2, \dots, N$. Substituting Haar coefficients in (3.4) to obtain the required approximate solution of equation (3.1).

4. Illustrative Examples

In this section, we consider the some of the examples to demonstrate the capability of the present method and error function is presented to verify the accuracy and efficiency of the following numerical results:

$$\text{Error function} = E_{\max} = \|u_e(x_i) - u_a(x_i)\|_{\max} = \sqrt{\sum_{i=1}^n (u_e(x_i) - u_a(x_i))^2}$$

where u_e and u_a are the exact and approximate solution respectively.

Example 4.1. Let us consider the Nonlinear Volterra-Fredholm Integral equation [7],

$$u(x) = \frac{1}{6}x + \frac{1}{2}x \exp(-x^2) + \int_0^x xt \exp(-u^2(t)) dt + \int_0^1 x u^2(t) dt, \quad 0 \leq x, t \leq 1 \quad (4.1)$$

with the initial conditions $u(0) = 0, u'(0) = 1$. Which has the exact solution $u(x) = x$. We applied the present technique and solved Eq. (4.1) as follows.

Successively differentiating (4.1) twice w.r.to x and using Leibnitz rule, we get differential equation,

$$u''(x) - \frac{2x^3}{\exp(x^2)} + \frac{3x}{\exp(x^2)} - x \exp(-u^2(x)) - 2x \exp(-u^2(x)) - x^2 \exp(-u^2(x))(-2u(x)u'(x)) = 0 \quad (4.2)$$

Let us assume that,

$$u''(x) = \sum_{i=1}^N a_i h_i(x) \quad (4.3)$$

integrating (4.3) twice,

$$u'(x) - u'(0) = \sum_{i=1}^N a_i p_{1,i}(x)$$

$$u'(x) = \sum_{i=1}^N a_i p_{1,i}(x) + 1 \quad (4.4)$$

$$u(x) - u(0) = \sum_{i=1}^N a_i p_{2,i}(x) + x$$

$$u(x) = \sum_{i=1}^N a_i p_{2,i}(x) + x \quad (4.5)$$

substituting (4.3) - (4.5) in (4.2), we get the system of N equations with N unknowns,

$$\sum_{i=1}^N a_i h_i(x) - \frac{2x^3}{\exp(x^2)} + \frac{3x}{\exp(x^2)} - x \exp\left(-\left(\sum_{i=1}^N a_i p_{2,i}(x) + x\right)^2\right) - \exp\left(-\left(\sum_{i=1}^N a_i p_{2,i}(x) + x\right)^2\right) \cdot 2x$$

$$- x^2 \exp\left(-\left(\sum_{i=1}^N a_i p_{2,i}(x) + x\right)^2\right) \cdot \left(-2 \cdot \left(\sum_{i=1}^N a_i p_{2,i}(x) + x\right) \cdot \left(\sum_{i=1}^N a_i p_{1,i}(x) + 1\right)\right) = 0. \quad (4.6)$$

Solving (4.6) using Newton's Method to obtain the Haar wavelet coefficients a_i for $N = 16$ i.e., [-2.33e-09 2.32e-09 1.22e-12 4.67e-09 3.17e-12 6.85e-12 -1.32e-11 9.21e-09 3.88e-12 -5.13e-12 1.77e-11 -1.00e-11 2.10e-12 4.41e-11 1.80e-11 1.61e-08]. Substituting a_i 's, in (4.6) obtain the approximate solution are given in table 1 and fig 1 shows the comparison with exact and existing method. Maximum error analysis is presented in table 2.

Example 4.2. Next, consider the Nonlinear Volterra - Fredholm Integral equation [8],

$$u(x) = \frac{-1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4} + \int_0^x (x-t)u^2(t) dt + \int_0^1 (x+t)u(t) dt, \quad 0 \leq x, t \leq 1 \quad (4.7)$$

with the initial conditions $u(0) = -2, u'(0) = 0$. Which has the exact solution $u(x) = x^2 - 2$.

Differentiating (4.7) twice w.r.t x and using Leibnitz rule, we get the differential equation,

$$u''(x) - u^2(x) + x^4 - 4x^2 + 2 = 0 \quad (4.8)$$

Let us assume that,

$$u''(x) = \sum_{i=1}^N a_i h_i(x) \quad (4.9)$$

integrating (4.9) twice,

$$u'(x) = \sum_{i=1}^N a_i p_{1,i}(x) \quad (4.10)$$

$$u(x) = \sum_{i=1}^N a_i p_{2,i}(x) - 2 \quad (4.11)$$

substituting (4.9) - (4.11) in (4.8), we get the system of N equations with N unknowns

$$\sum_{i=1}^N a_i h_i(x) - \left[\sum_{i=1}^N a_i p_{2,i}(x) - 2 \right]^2 + x^4 - 4x^2 + 2 = 0. \tag{4.12}$$

Solving (4.12) using Newton’s Method to obtain Haar wavelet coefficients a_i for $N = 16$ i.e., [2.0 8.66e-07 6.02e-09 1.29e-06 2.58e-11 1.07e-08 2.70e-07 1.04e-06 -6.08e-12 1.22e-10 8.55e-10 1.27e-08 6.99e-08 2.12e-07 4.29e-07 6.03e-07]. Substituting a_i ’s, in (4.12) and obtain the approximate solution, which is given in table 3 and fig 2 shows the comparison with the exact and existing method. Maximum error analysis is presented in table 4.

Table 1. Numerical results of example 4.1.

x	Exact	HWCM for $N = 8$	Method [1]	Error (HWCM)	Error Method [1]
0	0	0.00000000	0.00000000	5.25e-12	0.00e+00
0.1	0.1	0.10000000	0.06616916	4.65e-13	3.38e-02
0.2	0.2	0.20000000	0.13333039	4.56e-14	6.66e-02
0.3	0.3	0.30000000	0.20355479	4.91e-14	9.64e-02
0.4	0.4	0.40000000	0.27915784	1.31e-13	1.20e-01
0.5	0.5	0.50000000	0.36269547	6.73e-14	1.37e-01
0.6	0.6	0.60000000	0.45678660	8.33e-14	1.43e-01
0.7	0.7	0.70000000	0.56375872	4.21e-13	1.36e-01
0.8	0.8	0.80000000	0.68523040	1.09e-12	1.14e-01
0.9	0.9	0.90000000	0.82191762	1.14e-11	7.80e-02
1	1	1.00000000	0.97404036	7.18e-11	2.59e-02

Table 2. Maximum error analysis of example 4.1.

N	E_{\max} (HWCM)
4	1.7360e-11
8	2.4378e-11
16	3.0342e-11
32	4.3488e-11
64	5.7022e-11
128	6.8005e-11

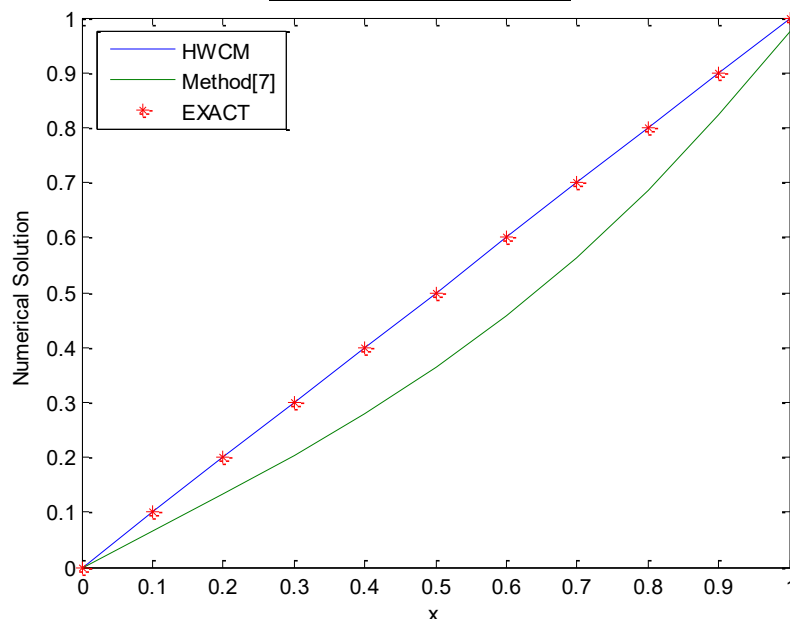


Figure 1. Comparison of HWCM with exact and existing method of example 4.1.

Table 3. Numerical results of example 4.2.

x	Exact	HWCM for $N = 16$	Method [10]	Error (HWCM)	Error Method [10]
0	-2.0	-2.000000	-2.000085	1.4e-9	8.5e-5
0.1	-1.99	-1.990000	-1.990104	5.4e-12	1.0e-4
0.2	-1.96	-1.960000	-1.960125	1.6e-13	1.2e-4
0.3	-1.91	-1.910000	-1.910152	9.3e-13	1.5e-4
0.4	-1.84	-1.840000	-1.840174	1.0e-11	1.7e-4
0.5	-1.75	-1.750000	-1.750195	1.5e-10	1.9e-4
0.6	-1.64	-1.640000	-1.640208	1.0e-9	2.0e-4
0.7	-1.51	-1.510000	-1.510204	4.8e-9	2.0e-4
0.8	-1.36	-1.360000	-1.360192	1.6e-8	1.9e-4
0.9	-1.19	-1.190000	-1.190149	4.7e-8	1.4e-4
1	-1.0	-1.000000	-1.000086	1.1e-7	8.6e-5

Table 4. Maximum error analysis of example 4.2.

N	E_{\max} (HWCM)
4	4.6473e-09
8	3.6298e-08
16	8.7203e-08
32	1.3162e-07
64	1.6075e-07
128	1.7743e-07

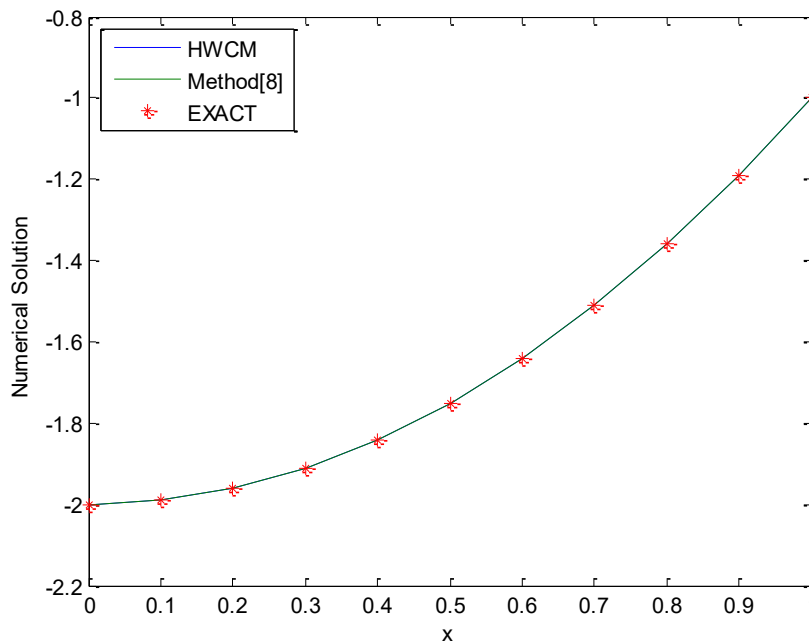


Figure 2. Comparison of HWCM with exact and existing method of example 4.2.

5. Conclusion

In the present work, Haar wavelet collocation method based on Leibnitz rule is applied to obtain the numerical solution of nonlinear Volterra-Fredholm integral equation of the second kind. The Haar wavelet function and its operational matrix were employed to solve the resultant integral equations. The numerical results are obtained by the proposed method have been compared with existing methods [7, 8]. Illustrative examples are tested with error analysis to justify the efficiency and possibility of the proposed technique.

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