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On Enumeration of some Non-Isomorphic Dendroids

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Abstract. A dendroid is a connected semigraph without a strong cycle. In this paper, we obtain the various results on the enumeration of the non-isomorphic dendroids containing two edges and the dendroids with three edges.

Introduction

Graph theory [1] is a study of some 2-tuple (u, v) of distinct elements belonging to a set of vertices V, with a condition that for each such 2-tuple, $(u, v) = (v, u)$. Semigraph theory [2] is a generalization of graph theory and is a study of n-tuples of distinct elements belonging to a set V of vertices. Recently, many interesting generalizations of the concepts of graph theory have been obtained in semigraph.[3, 4, 5, 6].

It is a well-known fact that the enumeration of graphs is one of the most important areas of graph theory and is applied in chemistry, physics, biology, information theory and so on [7, 8, 9]. Semigraphs applications are nicely explored in [10, 11, 12]. Due to the number of varieties at each and every step of semigraph concepts, the problem of generalizing the graph-enumeration for all the semigraphs appears to be np-complete. Therefore, some graph theorist have obtained the enumeration of some special categories of semigraphs. The enumeration of some edge complete semigraphs have been studied by K. Kayathri and S. Pethanachi Selvam [13, 14] and the enumeration of labeled semigraphs containing non-adjacent s-edges is obtained by authors of the present paper [15].

Arthur Cayley is one of the pioneers of graph theory, known for his work on counting trees. His Cayley's Formula [7] gives the number of labeled trees on n-vertices. The concept of tree is generalized as a dendroid in a semigraphs. The structure of many chemical compounds of more than two elements can be represented as dendroids which otherwise not possible in case of trees. The aim of present work is to obtain the enumerations of such non-isomorphic dendroids containing two edges and the dendroids containing three edges. We expect to see many applications of this in the future.

Preliminaries

Following are the definitions related to the semigraphs. For more terminologies [2, 16, 17] may be referred.

Definition 1. [2] A *semigraph G* is an ordered pair (V, X) where V is a non-empty set, whose elements are called vertices of G and the set X is the set of n–tuples, called the edges of G, of distinct vertices, for various $n \geqslant 2$, with the following conditions :

SG1 Any two edges have at most one vertex in common.

SG2 Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal if and only if

(i) $m = n$ and

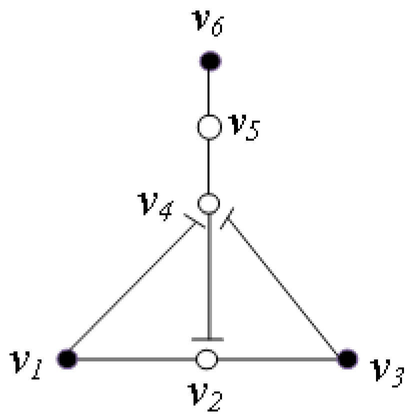

Fig. 1: Semigraph G

(ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $i = 1, 2, 3, \ldots n$.

Thus, the edge $(u_1, u_2, \ldots, u_n)$ is the same as the edge $(u_n, u_{n-1}, \ldots, u_1)$.

Let $G = (V, X)$ be a semigraph and $E = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is an edge of G. Then the vertices v_1 and v_n are called the *end vertices* of E, represented by thick dots, the vertices $v_2, \ldots, v_{n-1}$ are called the *middle vertices or m–vertices* of E, represented by small hollow circles. A vertex v in G which appears as an end vertex of one edge and the middle vertex of the other edge is known as the *middle–cum–end vertex or (m, e) vertex*, represented by a small tangent to the hollow circle of middle vertex.

Example 1. Let $G = (V, X)$ be a semigraph (Figure 1), where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $X = \{(v_1, v_2, v_3), (v_3, v_4), (v_2, v_4, v_5, v_6), (v_1, v_4)\}$. In G, v_1, v_3, v_6 are end vertices, v_5 is a middle vertex and v_2, v_4 are middle–cum–end vertices.

Definition 2. [2] Two vertices in a semigraph G are said to be *adjacent* if they belong to the same edge and are consecutively adjacent if in addition they are consecutive in order as well.

Definition 3. [2] Any two edges in a semigraph are said to be *adjacent* if they have a vertex in common.

Definition 4. [2] *Cardinality* of an edge in a semigraph is said to be k if the edge contains k number of vertices.

Definition 5. [2] An edge in a semigraph G is said to be an *s-edge* if its cardinality $k \geqslant 3$.

Definition 6. [2] *Subedge* of an edge $E = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ where $1 \leqslant i_1 < i_2 < \ldots \leqslant i_k < n$ or $1 \leqslant i_k < i_{k-1} < \ldots \leqslant i_1 < n$ and a *partial edge* of E is a $(j - i + 1)$- tuple $E\,(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$ where $1 \leqslant i \leqslant n$.

Definition 7. [2] An *fs-edge* in a semigraph G is an edge or a subedge and the *fp-edge* is an edge or a partial edge of G.

Definition 8. [2] A *$v_0 - v_n$ path* in a semigraph G is a sequence of distinct vertices $v_0, v_1, \ldots v_n$, such that (v_i, v_{i+1}) for $i = 0$ to $n - 1$ is an fs-edge of cardinality two. A $v_0 - v_n$ path is an *s-path* (or a *strong path*) if all its fs-edges are fp-edges. Otherwise, it is a *w-path* (or a *weak path*).

Definition 9. [2] A *strong cycle* is a closed s-path and a *weak cycle* is a closed w-path.

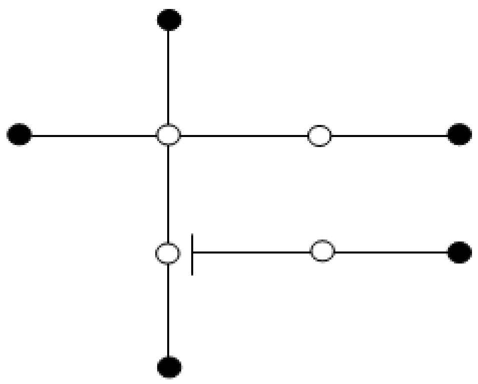

Fig. 2: Dendroid T

Definition 10. [2] A *dendroid* is a connected semigraph without strong cycles. In Figure 2 , semigraph T is a dendroid.

Definition 11. [2] Two semigraphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be *isomorphic* if there exist a bijection f from V_1 to V_2 such that for an edge $E = (v_1, v_2, \ldots, v_n)$ in G_1 there is an edge $(f(v_1), f(v_2), \ldots, f(v_n))$ in G_2.

Main Results

Here, we establish various results on the enumeration of dendroids with two edges and the dendroids with three edges. The Theorem 12 is also proved in [15], here we give an alternate proof to develop the proofs of remaining Theorems.

Theorem 12. *The number of non-isomorphic dendroids on n-vertices with exactly two edges of cardinalities $k_1 \geqslant 2$ and $k_2 \geqslant 2$ with $3 \leqslant k_1 + k_2 - 1 \leqslant n$ is*

(i) $\lceil \frac{k_1}{2} \rceil \lceil \frac{k_2}{2} \rceil$ *, if* $k_1 \neq k_2$

(ii) $\frac{1}{2} \lceil \frac{k}{2} \rceil \left(\lceil \frac{k}{2} \rceil + 1 \right)$ *, if* $k_1 = k_2 = k$.

Proof. Let G be a dendroid on n-vertices containing two edges $E_1 = (u_1, u_2, \ldots, u_{k_1-1}, u_{k_1})$ and $E_2 = (v_1, v_2, \ldots, v_{k_2-1}, v_{k_2})$ of cardinalities $k_1 \geqslant 2$ and $k_2 \geqslant 2$ with $3 \leqslant k_1 + k_2 - 1 \leqslant n$.

Let v be the common vertex between edges E_1 and E_2 and (i, j) represents the location of vertex v as i^{th} position in E_1 and j^{th} position in the edge E_2 . As an edge in a semigraph is symmetric, that is $(u_1, u_2, \ldots, u_{k-1}, u_k) = (u_k, u_{k-1}, \ldots, u_2, u_1)$, the dendroids corresponding to the position (i, j) for $i = 1, 2, 3, \ldots, k_1, j = 1, 2, 3, \ldots, k_2$ of v is isomorphic to the dendroids corresponding the position $(k_1 - i + 1, k_2 - j + 1)$ for $i = 1, 2, 3, \ldots, k_1, j = 1, 2, 3, \ldots, k_2$ of v. Henceforth, for this theorem as well as for the Theorem 13 and Theorem 14, we consider $i = 1, 2, 3, \ldots, \lceil \frac{k_1}{2} \rceil, j = 1, 2, 3, \ldots, \lceil \frac{k_2}{2} \rceil$.

Now, we consider the following two cases based on the cardinalities k_1 and k_2 of the edges E_1 and E_2.

Case (i) Let $k_1 \neq k_2$.

As $k_1 \neq k_2$, the non-isomorphic dendroids of two edges E_1 and E_2 are obtained by selecting the i^{th} position in E_1 by $\lceil \frac{k_1}{2} \rceil$ ways and the j^{th} position in the edge E_2 by $\lceil \frac{k_2}{2} \rceil$ ways.

Hence, the number of non-isomorphic dendroids is

$=\lceil \frac{k_1}{2} \rceil \lceil \frac{k_2}{2} \rceil$.

Case (ii) Let $k_1 = k_2 = k$.

Here, we consider the following different cases based on (i, j):

a) For $i, j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil, i \neq j$.

Let the common vertex v be located at (i, j), for $i, j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil, i \neq j$ in the edges E_1, E_2. Clearly, the i^{th} position in E_1 can be selected in $\left\lceil \frac{k}{2} \right\rceil$ ways and for each i, the j^{th} position in the edge E_2 edge can be selected in $\left(\left\lceil \frac{k}{2} \right\rceil - 1\right)$ ways and hence, the number of dendroids is $\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1\right)$.

As $k_1 = k_2$, the dendroids corresponding to (i, j) for $i, j = 1, 2, 3, \ldots \left\lceil \frac{k}{2} \right\rceil, i \neq j$ are isomorphic to the dendroids corresponding to the vertex v located at (j, i) for $i, j = 1, 2, 3, \ldots \left\lceil \frac{k}{2} \right\rceil$.

Hence, the number of non-isomorphic dendroids is

$=\frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1\right)$.

b) For $i = j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil$.

Let the common vertex v be located at (i, j), for $i = j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil$. As $k_1 = k_2$, for non-isomorphic dendroids, the $(i, j)^{th}$ position in E_1 and E_2 can be selected in $\left\lceil \frac{k}{2} \right\rceil$ ways.

Hence, the number of non-isomorphic dendroids is

$=\left\lceil \frac{k}{2} \right\rceil$.

Combining both the results, the number of non-isomorphic dendroids is

$$= \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1\right) + \left\lceil \frac{k}{2} \right\rceil$$
$$= \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil + 1\right).$$

Theorem 13. *The number of non-isomorphic dendroids on n-vertices with exactly three edges E_1, E_2, E_3 of cardinalities $k_1 \geqslant 2$, $k_2 \geqslant 2$ and $k_3 \geqslant 2$ $(k_1 \neq k_2 \neq k_3)$ with $7 \leqslant k_1 + k_2 + k_3 - 2 \leqslant n$ is*

(i) $(k_1 + k_2 - 1) \left\lceil \frac{k_1}{2} \right\rceil \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil$ *if both k_1 and k_2 are even,*

(ii) $\left\{(k_1 + k_2) \left\lceil \frac{k_1}{2} \right\rceil - k_1\right\} \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil$, *if k_1 is odd and k_2 is even,*

(iii) $\left\{(k_1 + k_2 + 1) \left\lceil \frac{k_1}{2} \right\rceil \left\lceil \frac{k_2}{2} \right\rceil - k_1 \left\lceil \frac{k_2}{2} \right\rceil - k_2 \left\lceil \frac{k_1}{2} \right\rceil\right\} \left\lceil \frac{k_3}{2} \right\rceil$, *if both k_1 and k_2 are odd.*

Proof. Let G be a dendroid on n-vertices containing three edges $E_1 = (u_1, u_2, \ldots, u_{k_1-1}, u_{k_1})$ and $E_2 = (v_1, v_2, \ldots, v_{k_2-1}, v_{k_2})$ and $E_3 = (w_1, w_2, \ldots, w_{k_3-1}, w_{k_3})$ of cardinalities $k_1 \geqslant 2$, $k_2 \geqslant 2$ and $k_3 \geqslant 2$ $(k_1 \neq k_2 \neq k_3)$ with $7 \leqslant k_1 + k_2 + k_3 - 2 \leqslant n$.

Let v be the common vertex between the edges E_1,E_2 and (i, j) represents the location of a vertex v as i^{th} position in E_1 and j^{th} position in the edge E_2. As dendroid with exactly three edges, can be obtained by adding a third edge to the dendroid of two edges, the number of dendroids on three edges will depend on the number of dendroids on two edges and also the position (i, j) of v in both edges. Therefore, we consider the following different cases:

Case (i) Let both k_1 and k_2 are even.

In this case, from Theorem 12, the number of dendroids on two edges E_1, E_2 of cardinality k_1 and k_2 for $k_1 \neq k_2$ is $\left\lceil \frac{k_1}{2} \right\rceil \left\lceil \frac{k_2}{2} \right\rceil$.

For each of these dendroids, as k_1 and k_2 $(k_1 \neq k_2)$ are even, the third edge can be added by choosing one vertex from the $(k_1 + k_2 - 1)$ vertices of E_1, E_2 and one vertex from $\left\lceil \frac{k_3}{2} \right\rceil$ vertices of the edge E_3, for v at any (i, j) position, $i = 1, 2, 3, \ldots, \left\lceil \frac{k_1}{2} \right\rceil$, $j = 1, 2, 3, \ldots, \left\lceil \frac{k_2}{2} \right\rceil$.

Hence, the number of non-isomorphic dendroids is

$= (k_1 + k_2 - 1) \left\lceil \frac{k_1}{2} \right\rceil \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil$.

Case (ii) Without loss of generality, let k_1 be odd and k_2 be even.

As k_1 is odd, the number of dendroids depend on position of the common vertex v in the edge E_1. Therefore, here the following two sub-cases will arise:

a) For $i = 1, 2, 3, \ldots, \left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right)$, $j = 1, 2, 3, \ldots, \left\lceil \frac{k_2}{2} \right\rceil$ that is v be at the non-centre position of E_1.

Let the common vertex v be located at (i, j) for $i = 1, 2, \ldots, \left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right)$, $j = 1, 2, \ldots, \left\lceil \frac{k_2}{2} \right\rceil$. Clearly, the i^{th} position in E_1 can be selected in $\left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right)$ ways and for each i, the j^{th} position in the edge E_2 can be selected in $\left\lceil \frac{k_2}{2} \right\rceil$ ways, hence the number of dendroids is $\left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right) \left\lceil \frac{k_2}{2} \right\rceil$.

For each of these dendroids, the third edge can be added by choosing one of the vertices from the $(k_1 + k_2 - 1)$ vertices of E_1, E_2 and one vertex from $\left\lceil \frac{k_3}{2} \right\rceil$ vertices of the edge E_3.

Hence, the number of non-isomorphic dendroids with v at the non-centre position is

$=\left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right) \left\lceil \frac{k_2}{2} \right\rceil (k_1 + k_2 - 1) \left\lceil \frac{k_3}{2} \right\rceil$.

b) For $i = \left\lceil \frac{k_1}{2} \right\rceil$, $j = 1, 2, 3, \ldots, \left\lceil \frac{k_2}{2} \right\rceil$ that is v be at the centre position of E_1.

Let the common vertex v be located at (i, j), for $i = \left\lceil \frac{k_1}{2} \right\rceil$, $j = 1, 2, 3, \ldots, \left\lceil \frac{k_2}{2} \right\rceil$ in the edges E_1, E_2. Clearly, the i^{th} position in E_1 can be selected in one way and the j^{th} position in the edge E_2 edge can be selected in $\left\lceil \frac{k_2}{2} \right\rceil$ ways, hence the number of dendroids is $\left\lceil \frac{k_2}{2} \right\rceil$.

If v is at the centre position of the edge E_1, then the dendroids obtained by adding a third edge by choosing the common vertices as $u_1, u_2, \ldots u_{\left(\left\lceil \frac{k_1}{2} \right\rceil\right)}$ of the edge E_1 is isomorphic to the dendroids obtained by adding a third edge by choosing the common vertex as $u_{\left(\left\lceil \frac{k_1}{2} \right\rceil\right)}, u_{\left(\left\lceil \frac{k_1}{2} \right\rceil + 1\right)}, \ldots u_{k_1}$ of the edge E_1. Hence, the non-isomorphic dendroids are obtained by adding a third edge by choosing a common vertex from either $\left\lceil \frac{k_1}{2} \right\rceil$ vertices of the edge E_1 or k_2 vertices of E_2 that is $\left(\left\lceil \frac{k_1}{2} \right\rceil + k_2 - 1\right)$ vertices of E_1, E_2 and a vertex from $\left\lceil \frac{k_3}{2} \right\rceil$ to vertices of the edge E_3.

Therefore, the number of such non-isomorphic dendroids with exactly three edges is

$=\left(\left\lceil \frac{k_1}{2} \right\rceil + k_2 - 1\right) \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil$.

Combining both the results of (a) and (b), the number of non-isomorphic dendroids with k_1 odd and k_2 is even $(k_1 \neq k_2)$ is

$$=\left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right) \left\lceil \frac{k_2}{2} \right\rceil (k_1 + k_2 - 1) \left\lceil \frac{k_3}{2} \right\rceil + \left(\left\lceil \frac{k_1}{2} \right\rceil + k_2 - 1\right) \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil$$
$$= \left\{(k_1 + k_2) \left\lceil \frac{k_1}{2} \right\rceil - k_1\right\} \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil.$$

Case (iii) Let both k_1 and k_2 be odd. $(k_1 \neq k_2)$.

As both k_1 and k_2 are odd, the number of dendroids depends on the (i, j) position of common vertex v of E_1, E_2 in both the edges E_1 and E_2 . Therefore, here the following four sub-cases will arise:

a) For $i = 1, 2, 3, \ldots, \left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right)$, $j = 1, 2, 3, \ldots, \left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right)$ that is v be at the non-centre position of both the edges E_1, E_2.

Let the common vertex v be located at (i, j), for $i = 1, 2, \ldots, \left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right)$, $j = 1, 2, \ldots, \left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right)$ in the edges E_1, E_2. Clearly, the i^{th} position in E_1 can be selected in $\left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right)$ ways and for each i, the j^{th} position in the edge E_2 edge can be selected in $\left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right)$ ways, hence the number of dendroids is $\left(\left\lceil \frac{k_1}{2} \right\rceil - 1\right) \left(\left\lceil \frac{k_2}{2} \right\rceil - 1\right)$.

For each of these dendroids, the third edge can be added by choosing one of the vertices from the (k_1+k_2-1) vertices of E_1, E_2 and one vertex from $\lceil \frac{k_3}{2} \rceil$ vertices of the edge E_3.

Hence, the number of non-isomorphic dendroids with v at the non-centre position is $=\left(\lceil \frac{k_1}{2} \rceil - 1\right)\left(\lceil \frac{k_2}{2} \rceil - 1\right)(k_1+k_2-1)\lceil \frac{k_3}{2} \rceil$.

b) For $i = \lceil \frac{k_1}{2} \rceil$, $j = 1,2,3,\ldots,\left(\lceil \frac{k_2}{2} \rceil - 1\right)$ that is v be at the centre position of E_1 and the non-centre positions of E_2.

Let the common vertex v be located at (i,j), for $i = \lceil \frac{k_1}{2} \rceil$, $j = 1,2,3,\ldots,\left(\lceil \frac{k_2}{2} \rceil - 1\right)$ in the edges E_1, E_2. Clearly, i^{th} position in E_1 can be selected in one way and the j^{th} position in the edge E_2 edge can be selected in $\left(\lceil \frac{k_2}{2} \rceil - 1\right)$ ways, hence the number of dendroids is $\left(\lceil \frac{k_2}{2} \rceil - 1\right)$.

If v is at the centre position of edge E_1 then the dendroids obtained by adding third edge by choosing common vertices as $u_1, u_2, \ldots u_{(\lceil \frac{k_1}{2} \rceil)}$ of edge E_1, is isomorphic to the dendroids obtained by adding third edge by choosing common vertex as $u_{(\lceil \frac{k_1}{2} \rceil)}, u_{(\lceil \frac{k_1}{2} \rceil+1)}, \ldots u_{k_1}$ of edge E_1. Hence, the non-isomorphic dendroids are obtained by adding third edge by choosing one common vertex from either first $\lceil \frac{k_1}{2} \rceil$ vertices of edge E_1 or k_2 vertices of E_2 that is $\left(\lceil \frac{k_1}{2} \rceil + k_2 - 1\right)$ vertices of E_1, E_2 and one vertex from $\lceil \frac{k_3}{2} \rceil$ vertices of the edge E_3.

Therefore, the number of non-isomorphic dendroids with v at the centre position of E_1 and the non-centre position of E_2 is $=\left(\lceil \frac{k_2}{2} \rceil - 1\right)\left(\lceil \frac{k_1}{2} \rceil + k_2 - 1\right)\lceil \frac{k_3}{2} \rceil$.

c) For $i = 1,2,3,\ldots,\left(\lceil \frac{k_2}{2} \rceil - 1\right)$, $j = \lceil \frac{k_1}{2} \rceil$ that is the common vertex v be at the non-centre position of E_1 and the centre positions of E_2.

Interchanging the role of E_1 and E_2 in the above sub-case (b), we will get the number of non-isomorphic dendroids with v at centre position of E_2 and at non-centre position of E_1 is $=\left(\lceil \frac{k_1}{2} \rceil - 1\right)\left(k_1 + \lceil \frac{k_2}{2} \rceil - 1\right)\lceil \frac{k_3}{2} \rceil$.

d) For $i = \lceil \frac{k_1}{2} \rceil$, $j = \lceil \frac{k_1}{2} \rceil$ that is v be at the centre position of E_1 and E_2.

Let the common vertex v be located at (i,j), for $i = \lceil \frac{k_1}{2} \rceil, j = \lceil \frac{k_1}{2} \rceil$. In this case, the dendroids obtained by adding a third edge by choosing the common vertices as $u_1, u_2, u_3, \ldots u_{(\lceil \frac{k_1}{2} \rceil)}$ of edge E_1 (or $v_1, v_2, \ldots v_{(\lceil \frac{k_2}{2} \rceil)}$ of edge E_2) is isomorphic to the dendroids obtained by adding a third edge by choosing the common vertex as $u_{(\lceil \frac{k_1}{2} \rceil)}, u_{(\lceil \frac{k_1}{2} \rceil+1)}, \ldots u_{k_1}$ of edge E_1 (or $v_{(\lceil \frac{k_2}{2} \rceil)}, v_{(\lceil \frac{k_2}{2} \rceil+1)}, \ldots v_{k_2}$ of edge E_2). Hence, the non-isomorphic dendroids are obtained by adding a third edge by choosing one common vertex from either first $\lceil \frac{k_1}{2} \rceil$ vertices of edge E_1 or $\lceil \frac{k_2}{2} \rceil$ vertices of E_2 that is $\left(\lceil \frac{k_1}{2} \rceil + \lceil \frac{k_2}{2} \rceil - 1\right)$ vertices of E_1, E_2 and one vertex from the $\lceil \frac{k_3}{2} \rceil$ vertices of the edge E_3.

Therefore, the number of non-isomorphic dendroids with v at the centre position of E_1 and E_2 is $=\left(\lceil \frac{k_1}{2} \rceil + \lceil \frac{k_2}{2} \rceil - 1\right)\lceil \frac{k_3}{2} \rceil$.

Combining all the cases, the number of non-isomorphic dendroids with both k_1 and k_2 $(k_1 \neq k_2)$ are odd is

$$=\left(\left\lceil \tfrac{k_1}{2}\right\rceil - 1\right)\left(\left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)(k_1+k_2-1)\left\lceil \tfrac{k_3}{2}\right\rceil + \left(\left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)\left(\left\lceil \tfrac{k_1}{2}\right\rceil + k_2 - 1\right)\left\lceil \tfrac{k_3}{2}\right\rceil +$$
$$+\left(\left\lceil \tfrac{k_1}{2}\right\rceil - 1\right)\left(k_1 + \left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)\left\lceil \tfrac{k_3}{2}\right\rceil + \left(\left\lceil \tfrac{k_1}{2}\right\rceil + \left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)\left\lceil \tfrac{k_3}{2}\right\rceil$$
$$=\left\{\left(\left\lceil \tfrac{k_1}{2}\right\rceil - 1\right)\left(\left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)(k_1+k_2-1) + \left(\left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)\left(\left\lceil \tfrac{k_1}{2}\right\rceil + k_2 - 1\right) +\right.$$
$$\left.+\left(\left\lceil \tfrac{k_1}{2}\right\rceil - 1\right)\left(k_1 + \left\lceil \tfrac{k_2}{2}\right\rceil - 1\right) + \left(\left\lceil \tfrac{k_1}{2}\right\rceil + \left\lceil \tfrac{k_2}{2}\right\rceil - 1\right)\right\}\left\lceil \tfrac{k_3}{2}\right\rceil.$$
$$=\left\{(k_1+k_2+1)\left\lceil \tfrac{k_1}{2}\right\rceil\left\lceil \tfrac{k_2}{2}\right\rceil - k_1\left\lceil \tfrac{k_2}{2}\right\rceil - k_2\left\lceil \tfrac{k_1}{2}\right\rceil\right\}\left\lceil \tfrac{k_3}{2}\right\rceil.$$

Theorem 14. *The number of non-isomorphic dendroid on n-vertices with exactly three edges E1, E2, E3 of cardinality $k_1 \geqslant 2$, $k_2 \geqslant 2$ and $k_3 \geqslant 2$ $(k_1 = k_2 = k \neq k_3)$ with $5 \leqslant k_1 + k_2 + k_3 - 2 \leqslant n$ is equal to*

(i) $\frac{1}{2}\left\{(2k+1)\left\lceil \frac{k}{2}\right\rceil - (2k-1)\right\}\left\lceil \frac{k}{2}\right\rceil\left\lceil \frac{k_3}{2}\right\rceil$, *if k is odd,*

(ii) $\frac{1}{2}\left\{(2k-1)\left\lceil \frac{k}{2}\right\rceil + 1\right\}\left\lceil \frac{k}{2}\right\rceil\left\lceil \frac{k_3}{2}\right\rceil$, *if k is even.*

Proof. Let G be a dendroid on n-vertices containing three edges $E_1 = (u_1, u_2, \ldots, u_{k_1-1}, u_{k_1})$ and $E_2 = (v_1, v_2, \ldots, v_{k_2-1}, v_{k_2})$ and $E_3 = (w_1, w_2, \ldots, w_{k_3-1}, w_{k_3})$ of cardinalities $k_1 \geqslant 2$, $k_2 \geqslant 2$ and $k_3 \geqslant 2$ $(k_1 = k_2 = k \neq k_3)$ with $5 \leqslant k_1 + k_2 + k_3 - 2 \leqslant n$.

As dendroid with exactly three edges, can be obtained by adding a third edge to the dendroid of two edges, therefore the number of dendroid on three edges will depend on the number of dendroid on two edges and also the position of the common vertex in both edges. Let (i, j) represents the location of the common vertex v as i^{th} in E_1 and j^{th} in the edge E_2 . Therefore, we consider the following different cases based on (i, j) and cardinality $k_1 = k_2 = k$:

Case (i) Let both k_1 and k_2 $(k_1 = k_2 = k)$ are odd.

Here, we consider the following different cases based (i, j).

a) For $i = j = 1, 2, \ldots \left(\left\lceil \frac{k}{2}\right\rceil - 1\right)$

Let the common vertex v is located at (i, j), for $i = j = 1, 2, \ldots \left(\left\lceil \frac{k}{2}\right\rceil - 1\right)$ in the edge E_1, E_2.

Hence, the number of non-isomorphic dendroids corresponding to (i, j), for $i = j = 1, 2, \ldots \left\lceil \frac{k}{2}\right\rceil - 1$, is $\left(\left\lceil \frac{k}{2}\right\rceil - 1\right)$.

In this case , the dendroids obtained by adding a third edge by choosing the common vertex as one of the vertices amongst $u_1, u_2, \ldots, u_{k-1}, u_k$ of edge E_1 is isomorphic to the dendroids obtained by adding a third edge by choosing the common vertex as one of the vertices amongst $v_1, v_2, \ldots, v_{k-1}, v_k$ of edge E_2. Hence, the non-isomorphic dendroids are obtained by adding a third edge by choosing one common vertex from either k vertices of edge E_1 or k vertices of E_2 and one vertex from the $\left\lceil \frac{k_3}{2}\right\rceil$ vertices of the edge E_3.

Therefore, the number of non-isomorphic dendroids is

$= k\left(\left\lceil \frac{k}{2}\right\rceil - 1\right)\left\lceil \frac{k_3}{2}\right\rceil$.

b) For $i = j = \left\lceil \frac{k}{2}\right\rceil$ that is located at the centre of both the edges E_1 and E_2.

Let the common vertex v be located at $(i, j) = \left(\left\lceil \frac{k}{2}\right\rceil, \left\lceil \frac{k}{2}\right\rceil\right)$. Then the number of such dendroids on two edges E_1, E_2 of cardinality k_1 and k_2 for $k_1 = k_2 = k$ is exactly ONE.

In this case , the dendroids obtained by adding a third edge by choosing the common vertex as $u_i, i = 1, 2, \ldots \left\lceil \frac{k}{2}\right\rceil$ of edge E_1 is isomorphic to the dendroids obtained by adding a third edge by choosing the common vertex as $u_{k-i}, i = 1, 2, \ldots \left\lceil \frac{k}{2}\right\rceil$ of edge E_1 OR $v_i, i = 1, 2, \ldots \left\lceil \frac{k}{2}\right\rceil$ edge E_2 OR $v_{k-i}, i = 1, 2, \ldots \left\lceil \frac{k}{2}\right\rceil$ edge E_2. Hence, the non-isomorphic dendroids are obtained by adding a third edge by choosing either one common vertex from

$\left\lceil\frac{k}{2}\right\rceil$ vertices of edge E_1 or one common vertex from $\left\lceil\frac{k}{2}\right\rceil$ vertices of edge E_2 and then one vertex from the $\left\lceil\frac{k_3}{2}\right\rceil$ vertices of the edge E_3.

Therefore, the number of non-isomorphic dendroids is
$= \left\lceil\frac{k}{2}\right\rceil \left\lceil\frac{k_3}{2}\right\rceil$.

c) For $i, j = 1, 2, \ldots \left(\left\lceil\frac{k}{2}\right\rceil - 1\right), i \neq j$

Let the common vertex v be located at (i, j), for $i, j = 1, 2, \ldots \left(\left\lceil\frac{k}{2}\right\rceil - 1\right)$, $i \neq j$. In this case the dendroids corresponding to $i, j = 1, 2, \ldots \left(\left\lceil\frac{k}{2}\right\rceil - 1\right)$, $i \neq j$ is isomorphic to the dendroids corresponding to the common vertex v located at (j, i) for $i, j = 1, 2, \ldots \left(\left\lceil\frac{k}{2}\right\rceil - 1\right), i \neq j$.

Hence, the number of non-isomorphic dendroids with common vertex v located at (i, j) as discussed above is $\frac{1}{2}\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(\left\lceil\frac{k}{2}\right\rceil - 2\right)$.

Now the dendroids containing three edges can be obtained by adding a third edge to the $\frac{1}{2}\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(\left\lceil\frac{k}{2}\right\rceil - 2\right)$ dendroids of two edges, by choosing any one of the $(2k-1)$ vertices of 'E_1 or E_2' in $(2k-1)$ ways.

Therefore, in this case, the number of non-isomorphic dendroids is
$=\frac{1}{2}\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(\left\lceil\frac{k}{2}\right\rceil - 2\right)(2k-1)\left\lceil\frac{k_3}{2}\right\rceil$.

d) For $i = \left\lceil\frac{k}{2}\right\rceil, j = 1, 2, \ldots \left(\left\lceil\frac{k}{2}\right\rceil - 1\right)$ that is located at the centre of the edges E_1 but not the centre of the edge E_2.

Let the common vertex v of E_1, E_2 be located at $\left(\left\lceil\frac{k}{2}\right\rceil, j\right)$, for $j = 1, 2, \ldots \left(\left\lceil\frac{k}{2}\right\rceil - 1\right)$.

Hence, the number of non-isomorphic dendroids corresponding to v located at $\left(\left\lceil\frac{k}{2}\right\rceil, j\right)$, for $j = 1, 2, \ldots \left\lceil\frac{k}{2}\right\rceil - 1$ is $\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)$.

In this case, the dendroids obtained by adding a third edge by choosing the common vertices as $u_i, i = 1, 2, \ldots \left\lceil\frac{k}{2}\right\rceil$ of edge E_1 is isomorphic to the dendroids obtained by adding a third edge by choosing the common vertex as $u_{k-i}, i = 1, 2, \ldots \left\lceil\frac{k}{2}\right\rceil$ of edge E_2. Hence, the non-isomorphic dendroids are obtained by adding a third edge by choosing one common vertex from $\left\lceil\frac{k}{2}\right\rceil$ vertices of edge E_1 or k vertices of E_2 that is $\left(k + \left\lceil\frac{k}{2}\right\rceil - 1\right)$ vertices of E_1 ,E_2 and one vertex from the $\left\lceil\frac{k_3}{2}\right\rceil$ vertices of the edge E_3.

Therefore, the number of non-isomorphic dendroids is
$= \left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(k + \left\lceil\frac{k}{2}\right\rceil - 1\right)\left\lceil\frac{k_3}{2}\right\rceil$.

e) For $j = \left\lceil\frac{k}{2}\right\rceil$, $i = 1, 2, \ldots k$, $i \neq \left\lceil\frac{k}{2}\right\rceil$ that is located at centre of the edges E_2 but not the centre of the edge E_1.

As $k_1 = k_2 = k$, the dendroid corresponding to the common vertex located at $\left(\left\lceil\frac{k}{2}\right\rceil, j\right)$ for $j = 1, 2, \ldots \left\lceil\frac{k}{2}\right\rceil - 1$ are isomorphic to the dendroid corresponding to the common vertex located at $\left(i, \left\lceil\frac{k}{2}\right\rceil\right)$ for $i = 1, 2, \ldots \left\lceil\frac{k}{2}\right\rceil - 1$.

Combining all the five results (a), (b), (c), (d), (e) of case (i), the number of non-isomorphic dendroids with $k_1 = k_2 = k$ as odd number is

$=k\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left\lceil\frac{k_3}{2}\right\rceil + \left\lceil\frac{k}{2}\right\rceil\left\lceil\frac{k_3}{2}\right\rceil + \frac{1}{2}\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(\left\lceil\frac{k}{2}\right\rceil - 2\right)(2k-1)\left\lceil\frac{k_3}{2}\right\rceil +$
$+ \left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(k + \left\lceil\frac{k}{2}\right\rceil - 1\right)\left\lceil\frac{k_3}{2}\right\rceil$
$=\left\{k\left(\left\lceil\frac{k}{2}\right\rceil - 1\right) + \left\lceil\frac{k}{2}\right\rceil + \frac{1}{2}\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(\left\lceil\frac{k}{2}\right\rceil - 2\right)(2k-1) + 2\left(\left\lceil\frac{k}{2}\right\rceil - 1\right)\left(k + \left\lceil\frac{k}{2}\right\rceil - 1\right)\right\}\left\lceil\frac{k_3}{2}\right\rceil$.
$=\frac{1}{2}\left\{(2k+1)\left\lceil\frac{k}{2}\right\rceil - (2k-1)\right\}\left\lceil\frac{k}{2}\right\rceil\left\lceil\frac{k_3}{2}\right\rceil$.

Case (ii) Let both k_1 and k_2 $(k_1 = k_2 = k)$ are even.

Here, we consider the following different cases based (i, j).

a) For $i = j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil$.

Let the common vertex v be located at (i, j), for $i = j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil$.

Hence, the number of non-isomorphic dendroids corresponding to (i, j) for $i = j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil$ is $\left\lceil \frac{k}{2} \right\rceil$.

In this case, the dendroids obtained by adding a third edge by choosing the common vertices as $u_1, u_2, \ldots, u_{k-1}, u_k$ of edge E_1 is isomorphic to the dendroids obtained by adding a third edge by choosing the common vertex as $v_1, v_2, \ldots, v_{k-1}, v_k$ of edge E_2. Hence, the non-isomorphic dendroids are obtained by adding a third edge by choosing one common vertex from either k vertices of edge E_1 or k vertices of E_2 and one vertex from the $\left\lceil \frac{k_3}{2} \right\rceil$ vertices of the edge E_3.

Therefore, the number of non-isomorphic dendroids is

$= k \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil.$

b) For $i, j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil, i \neq j$.

Let the common vertex v be located at (i, j) for $i, j = 1, 2, \ldots \left\lceil \frac{k}{2} \right\rceil, i \neq j$.

Hence, the number of non-isomorphic dendroids with common vertex v located at (i, j) as discussed above is $\frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right)$ in number.

Now, the dendroids containing three edges can be obtained by adding a third edge to the $\frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right)$ dendroids of the two edges, by choosing any one vertex of edge E_1 or E_2 in $(2k - 1)$ ways.

Therefore, in this case, the number of non-isomorphic dendroids is

$= \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right) (2k - 1) \left\lceil \frac{k_3}{2} \right\rceil.$

Combining both the results (a), (b) of case (ii), the number of non-isomorphic dendroids with $k_1 = k_2 = k$ as even number is

$$
\begin{aligned}
&= k \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil + \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right) (2k - 1) \left\lceil \frac{k_3}{2} \right\rceil \\
&= \left\{ k \left\lceil \frac{k}{2} \right\rceil + \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right) (2k - 1) \right\} \left\lceil \frac{k_3}{2} \right\rceil \\
&= \frac{1}{2} \left\{ (2k - 1) \left\lceil \frac{k}{2} \right\rceil + 1 \right\} \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil.
\end{aligned}
$$

The result of Theorem 13, can be easily generalized for the distinct even cardinalities of m-number of edges, as below:

Theorem 15. *The number of non-isomorphic dendroids on n-vertices with m-edges $E_1, E_2, \ldots, E_m$, each of even cardinalities $k_l \geqslant 2$ for $l = 1, 2, 3, \ldots, m$ such that $k_1 \neq k_2 \neq \ldots \neq k_m$ and $k_1 + k_2 + \ldots + k_m - (m - 1) \leqslant n$ is*

$$= (k_1 + k_2 - 1)(k_1 + k_2 + k_3 - 2) \ldots (k_1 + k_2 + \ldots + k_m - m + 1) \left\lceil \frac{k_1}{2} \right\rceil \left\lceil \frac{k_2}{2} \right\rceil \left\lceil \frac{k_3}{2} \right\rceil \ldots \left\lceil \frac{k_m}{2} \right\rceil.$$

Proof. Proof is similar to case (i) of Theorem 13

Conclusion

Theorem 12 , Theorem 13 and Theorem 14 establish the results on the enumeration of dendroids containing two and three edges. Also, results on the enumeration of dendroids containing m-edges of distinct even cardinalities are established in the Theorem 15, as a sort of generalization of Theorem 13.

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