Operations on Semigraphs

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Abstract. In this paper the structural equivalence of union, intersection ring sum and decomposition of semigraphs are explored by using the various types of isomorphisms such as isomorphism, ev-isomorphism, a-isomorphism and e- isomorphism for G_e, G_a and G_{ca}. We establish various types of binary operations in semigraphs.

1. Introduction

Many binary operations have been defined in Graph Theory [7] to derive its properties using set theoretical terminology. Boolean Operations on Graphs was introduced by Frank Harary and W.G. Wilcox [8]. Further many operations such as composition [9], Kronecker product of graphs and lexicographic products were enhanced by G. Sabidussi [10, 11] and P.M. Weichsel [17]. These Boolean operations provide an insight for investigating the structural properties of the newly generated graphs from a pair of simple graphs.

The concept of Semigraph was first introduced by E.Sampathkumar [6], which is enriched by variety of properties enjoyed not only by the vertices but also by the edges. Some of the concepts such as domination in semigraphs was introduced by S.S. Kamath, R.S. Bhat and S.R. Hebbar [21, 22], Bipartite theory of semigraphs and Hyper domination in semigraphs by Y.B.Venkatakrishnan and V. Swaminathan [23, 24], (m,e)-domination in semigraphs by S. Gomathi [20], Adjacency matrix and Incidence matrix of a semigraph by C.M.Deshpande, Y.S. Gaidhani and B.P.Athawale [3, 4], Factorization and Genus of semigraphs by P. Das and Surajith Kr. Nath [18, 19], Partial edge incidence matrix and Strong Circuit matrix and Strong Path matrix by P.R. Hampiholi and J.P. Kitturkar [15, 16]. The application of semigraph in DNA splicing was introduced by K. Thiagarajan, J. Padmashree and S. Jeyabharathi [12].

Semigraphs provide a wide scope for defining various types of fundamental operations and explore its hidden properties. Semigraphs mainly contains two types of edges called subedges and partial edges which play a very prominent role in the application of resolving traffic problems, representing family relationships. We employ some of these fundamental operations in Semigraphs with respect to partial edge, subedges and equal edges. The properties of partial edges, subedges and equal edges give rise to various types of union, intersection, decomposition and ring sum concepts in semigraphs. The main objective of this paper is to define full-edge operation, end-vertex operation and subedge operation. We can find applications of these operations in Eulerian semigraphs [5].

2. Preliminaries

Definition 1: [6] A semigraph G is a pair (V, X) where V is a nonempty set whose elements are called vertices of G, and X is a set of n-tuples, called edges of G, of distinct vertices $n \geq 2$, satisfying the following conditions:

SG1: Any two edges have atmost one vertex in common.

SG2: Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal if and only if

(1) $m = n$ and

(2) Either $u_i = v_i$ for $1 \le i \le n$, or $u_i = v_{n-i+1}$ for $1 \le i \le n$.

Thus, the edge $E = (u_1, u_2, \ldots, u_n)$ is same as the edge $(u_n, u_{n-1}, \ldots, u_1)$. u_1 and u_n are called the end vertices of E, while $u_2, u_3, \ldots, u_{n-1}$ are called the middle vertices of E. Also, a vertex which is a middle vertex for one edge and an end vertex of another edge is said to be an (m, e)-vertex. The middle vertex is drawn as a hollow circle and an (m, e)-vertex is drawn as a hollow circle with a small tangent drawn to it indicating it is an end vertex of the other edge.

Example 1: Let $G=(V, X)$ be a semigraph shown below in Figure 1 where $V = \{1,2,3,4,5,6,7,8\}$ and $X = \{(1,2,3), (3,4), (4,5,6,7), (7,8,3), (8,9)\}$ shown below in Figure 1. The vertices 1 and 3 are the end vertices of the edge (1,2,3) and 2 is the middle vertex of this edge.
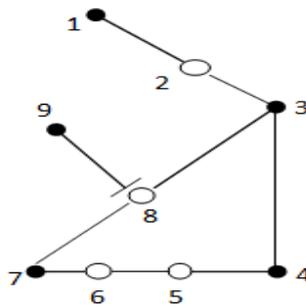

Figure 1.

Also vertex 8 is the (m, e) vertex of the edge (7,8,3). Vertex 4 is adjacent to the vertex 6 whereas vertex 4 is consecutively adjacent to vertex 5.

Definition 3: [6] A *partial edge* or fp edge E is a $j - i + 1$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$, where $1 \le i \le n$.

Definition 4: [6] A *subedge* or fs-edge $E = (v_1, v_2, \ldots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ where $1 \le i_1 < i_2 < \ldots < i_k \le n$ or $1 \le i_k < i_{k-1} < \ldots < i_1 \le n$.

Definition 5: [6] A *full edge* or f edge is any edge of a semigraph.

Definition 6: [6] Two edges are said to be *equal* if they have the same cardinality, that is if both the edges contain the same number of vertices.

Definition 7: [6] A semigraph $G' = (V', X')$ is a subsemigraph of a semigraph $G = (V, X)$ if $V' \subseteq V$ and the edges in G' are the subedges of the edges in G.

Definition 8: [1] A *clique* of a semigraph G is a complete subsemigraph of G.

Definition 9: [6] The *adjacency graph* denoted by G_a is a graph in which two vertices in G_a are adjacent if they are adjacent in G.

Definition 10: [6] The *consecutive adjacency graph* G_{ca} is a graph in which two vertices in G_{ca} are adjacent if they are consecutively adjacent vertices in G.

Definition 11: [6] The *end vertex graph* denoted by G_e is a graph in which two vertices in G_e are adjacent if they are end vertices of an edge in G.

Definition 12: [6] The union $G_1 \cup$ of two adjacency disjoint semigraphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ is a semigraph with vertex set $V_1 \cup$ and edge set $X_1 \cup X_2$.

Definition 13: [6] A semigraph $G^c = (V, X)$ with vertex set V is called a complement of a semigraph $G = (V, X)$ if for any two vertices u and v in G, u and v are adjacent in G^c if they are non adjacent in G.

Definition 14: [6] Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs and f be a bijection from V_1 to V_2. Let $E = (v_1, v_2, \dots \dots \dots, v_n)$ be an edge in G_1. Then

1. f is an *isomorphism* if $(f(v_1), f(v_2), \dots \dots, f(v_n))$ is an edge in G_2.
2. f is an *end vertex isomorphism* (*ev-isomorphism*) if $(f(v_1), f(v_2), \dots \dots, f(v_n))$ is an edge in G_2 with end vertices $f(v_1)$ and $f(v_n)$.
3. f is an *edge isomorphism* (*e-isomorphism*) if the set $(f(v_1), f(v_2), \dots \dots, f(v_n))$ forms an edge in G_2.
4. f is an *adjacency isomorphism* (*a-isomorphism*) if adjacent vertices in G_1 are mapped onto adjacent vertices in G_2.

Theorem 15: [6] A semigraph G is p-eulerian if and only if, every vertex in G has an even degree.

3. Main Results

Now we define union, intersection and ring sum operations on two semigraphs having disjoint edges or equal edges.

3.1. Union of Two Semigraphs

Definition 3.1.1: If two semigraphs G_1 and G_2 are such that any edge in G_1 is either distinct with any edge in G_2 or the edges are equal then $G_1 \cup G_2$ is a semigraph with the vertex set $V_1 \cup V_2$ and edge set $X_1 \cup X_2$.
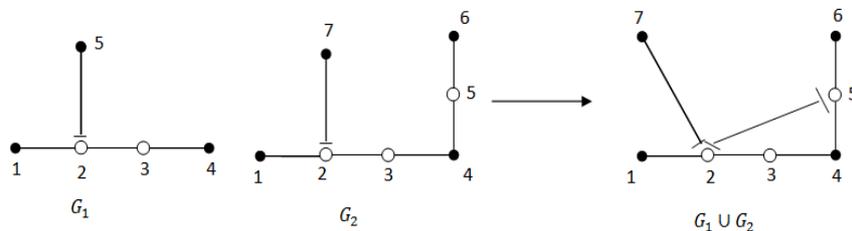
Example 3.1.2:

Figure 2.

The following observations are obvious.

Remarks 3.1.3:
   i) For any semigraph G, $G \cup G = G$.
   ii) For any semigraph G, $G \cup G^c \cong K(G)$ where $K(G)$ is a complete semigraph containing both G and G^c.
   iii) For any two semigraphs G_1 and G_2, $G_1 \cup G_2 = G_2 \cup G_1$.
   iv) For semigraphs G_1, G_2 and G_3, $G_1 \cup (G_2 \cup G_3) = (G_1 \cup G_2) \cup G_3$.

With this we proceed to prove the main results.

Theorem 3.1.4: If G_a, G_e and G_{ca} represent the adjacency graph, end vertex graph and consecutive adjacency graph respectively, then for any two semigraphs G' and G'', we have
   i) $G'_a \cup G''_a \cong (G' \cup G'')_a$
   ii) $G'_e \cup G''_e \cong (G' \cup G'')_e$
   iii) $G'_{ca} \cup G''_{ca} \cong (G' \cup G'')_{ca}$

Proof: i) By the definition 3.1, the vertex set of $G'_a \cup G''_a$ is same as the vertex set of $(G' \cup G'')_a$. Let, $E(u_1, u_n) = (u_1, u_2, \dots \dots \dots, u_n) \in E(G' \cup G'')$. This implies either $E \in E(G')$ or $E \in E(G'')$ or both. Without loss of generality, let $E \in E(G')$. Then, the edge E corresponds to the clique say C in

G'_a. By definition 2.9, this implies that for each edge $E \in E(G' \cup G'')$ there corresponds a clique C in G'_a or G''_a or both. Hence, clique $C \in G'_a \cup G''_a$. Again by the definition 2.9, for the same edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cup G'')$, there exists a clique $C \in (G' \cup G'')_a$. Hence, for each edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n)$ of $G' \cup G''$ there corresponds a clique $C \in G'_a \cup G''_a$ and $C \in (G' \cup G'')_a$ on vertices $u_1, u_2, \ldots \ldots \ldots, u_n$.

Now to show the existence of isomorphism between $G'_a \cup G''_a$ and $(G' \cup G'')_a$ we establish the following result. Let $E = (E_1, E_2, \ldots \ldots \ldots, E_k)$ be the edges of $G' \cup G''$ and $C = (C_1, C_2, \ldots \ldots \ldots, C_k)$ be the cliques in $G'_a \cup G''_a$.

Let, $f: E \to C$ be a function defined as $f(E_i) = C_i$. Then, we have the following
(i) $|E_i| = |C_i|$ (established above)
(ii) If E_i belongs to both $E(G'_a)$ and $E(G''_a)$ then, there exists one common clique $C_i \in C$
(iii) If, $E_i \neq E_j$ with $|E_i \cap E_j| = 1$, then $f(E_i) = C_i$ and $f(E_j) = C_j$ with $|C_i \cap C_j| = 1$. Also if $E_i \neq E_j$ then we claim that $C_i \neq C_j$ or $f(E_i) \neq f(E_j)$ for otherwise if $C_i = C_j$ then $|C_i| = |E| - 1$, which is a contradiction to (i). Therefore, by definition 2.14 f is an *adjacency-isomorphism*.
Hence, $G'_a \cup G''_a \cong (G' \cup G'')_a$.

ii) By the definition 3.1, the vertex set of $G'_e \cup G''_e$ is same as the vertex set of $(G' \cup G'')_e$

Let, $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cup G'')$. This implies either $E \in E(G')$ or $E \in E(G'')$ or both. Without loss of generality, let $E \in E(G')$. Let u_1 and u_n be the end vertices of the edge E in G'. Hence by the definition 2.11, u_1 and u_n are the end vertices of the edge E' in G'_e. This implies that for each edge $E \in E(G' \cup G'')$, there corresponds a complete graph of order 2 that is K_2 in G'_e or G''_e or both. Hence, for each edge E there corresponds $K_2 \in G'_e \cup G''_e$. Again by the definition 2.11, for the same edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cup G'')$, there corresponds complete graph $K_2 \in (G' \cup G'')_e$. Hence, for each edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n)$ of $(G' \cup G'')$, there corresponds a complete graph $K_2 \in G'_e \cup G''_e$ and $K_2 \in (G' \cup G'')_e$ with the end vertices u_1 and u_n.

Let, $E = (E_1, E_2, \ldots \ldots \ldots, E_k)$ be the set of edges and $M = \{m_1, m_2, \ldots \ldots, m_l\}$ be the set of middle vertices $G' \cup G''$. Also, let $E_e = (E_{e_1}, E_{e_2}, \ldots \ldots \ldots, E_{e_k})$ be the set of edges and $M_e = (m_{e_1}, m_{e_2}, \ldots \ldots \ldots, m_{e_l})$ be the set of isolated vertices in $G'_e \cup G''_e$. Clearly, $|M| = |M_e|$ and $|E| = |E_e|$.

Now an isomorphism between $G'_e \cup G''_e$ and $(G' \cup G'')_e$ will be established.
Let $f: (E \cup M) \to (E_e \cup M_e)$ be the function defined as
$f(E) = E_e$, if $|E| = 2$
$\quad = E_e \cup \left(\bigcup_{m_i \in M} m_i \right), 1 \leq i \leq l$ if $|E| > 2$
We claim that, if $E_i \neq E_j$ then $f(E_i) \neq f(E_j)$ for otherwise, if $E_i \neq E_j$ and $f(E_i) = f(E_j)$ then $|E_e| = |E| - 1$. Therefore, by definition 2.14 f is an *end vertex isomorphism*.
Hence, $G'_e \cup G''_e \cong (G' \cup G'')_e$.

iii) By the definition 3.1, the vertex set of $G'_{ca} \cup G''_{ca}$ is same as the vertex set of $(G' \cup G'')_{ca}$

Let, $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cup G'')$. This implies either $E \in E(G')$ or $E \in E(G'')$ or both. Without loss of generality, let $E \in E(G')$. Then, the edge E corresponds to a path P_n in G'_{ca}. This implies that for each edge $E \in E(G' \cup G'')$ there is a path P_n in G'_{ca} or G''_{ca} or both. Hence, by the definition 2.10, path $P_n \in G'_{ca} \cup G''_{ca}$. Similarly, for the edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cup G'')$, there exists a path $P_n \in (G' \cup G'')_{ca}$. Hence, for each edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n)$ of $(G' \cup G'')$, there corresponds a path $P_n \in G'_{ca} \cup G''_{ca}$ and by the definition 2.10, the same path $P_n \in (G' \cup G'')_{ca}$ with vertices $u_1, u_2, \ldots \ldots \ldots, u_n$ in order. By the definition 2.10 and 2.14, the consecutive adjacency graph G_{ca} of a semigraph G resembles a graph. Hence the establishment of isomorphism between $G'_{ca} \cup G''_{ca}$ and $(G' \cup G'')_{ca}$ is trivial.
Hence, $G'_{ca} \cup G''_{ca} \cong (G' \cup G'')_{ca}$.

Definition 3.1.5: Intersection of Two Semigraphs

  If two semigraphs G_1 and G_2 are such that any edge in G_1 has either atmost one vertex in common with any edge in G_2 or the edges are equal then $G_1 \cap G_2$ is a semigraph with vertex set $V_1 \cap V_2$ and edge set $X_1 \cap X_2$.
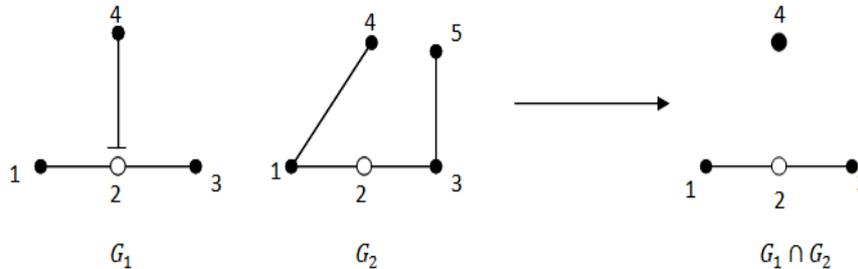
Example 3.1.6:

Figure 3.

Remarks 3.1.7:

  i) For any semigraph G, $G \cap G = G$.

  ii) For any semigraph G, $G \cap G^c$ is a null graph.

  iii) For any two semigraphs G_1 and G_2, $G_1 \cap G_2 = G_2 \cap G_1$.

  iv) For any semigraphs G_1, G_2 and G_3, $G_1 \cap (G_2 \cap G_3) = (G_1 \cap G_2) \cap G_3$.

  v) For any semigraphs G_1, G_2 and G_3,

   a) $G_1 \cap (G_2 \cup G_3) = (G_1 \cap G_2) \cup (G_1 \cap G_3)$

   b) $G_1 \cup (G_2 \cap G_3) = (G_1 \cup G_2) \cap (G_1 \cup G_3)$.

Theorem 3.1.8: If G_a, G_e and G_{ca} are the adjacency graph, end vertex graph and consecutive adjacency graph respectively of a semigraph G, then for any two semigraphs G' and G'' we have

  i) $G'_a \cap G''_a \cong (G' \cap G'')_a$

  ii) $G'_e \cap G''_e \cong (G' \cap G'')_e$

  iii) $G'_{ca} \cap G''_{ca} \cong (G' \cap G'')_{ca}$

Proof: i) By the definition 3.5, the vertex set of $G'_a \cap G''_a$ is same as the vertex set of $(G' \cap G'')_a$.

  Let, $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cap G'')$. This implies $E \in E(G')$ and $E \in E(G'')$. Then, by the definition 2.9 the edge E corressponds to a clique C in $(G' \cap G'')_a$. This implies that for each edge $E \in E(G' \cap G'')$ there corressponds a clique C in $(G' \cap G'')_a$. Hence clique $C \in (G' \cap G'')_a$. Again by the definition 2.9, for the same edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n) \in E(G' \cap G'')$, there corressponds a clique $C \in G'_a \cap G''_a$. Hence for each edge $E(u_1, u_n) = (u_1, u_2, \ldots \ldots \ldots, u_n)$ of $G' \cap G''$ there corresponds a clique $C \in G'_a \cap G''_a$ and the same clique $C \in (G' \cap G'')_a$ of vertices $u_1, u_2, \ldots \ldots \ldots, u_n$.

  It remains to establish the isomorphism between $G'_a \cap G''_a$ and $(G' \cap G'')_a$.

  Let $E = \{E_i \text{ and } E_i = E_j / E_i \in G', E_j \in G'', |E_i \cap E_j| > 1\}$ be the edge set and $U = \{u_i / u_i \in E_i \text{ and } E_j, E_i \in G', E_j \in G'' \text{ and } E_i \neq E_j\}$ be the set of vertices. Let $C = \{C_i / C_i \in (G' \cap G'')_a\}$ be the set of cliques and $U' = \{u'_i / u'_i \in (G' \cap G'')_a \text{ with } u_i \in U\}$ be the set of isolated vertices. Let, $f: (E \cup U) \rightarrow (C \cup U')$ be a function defined as $f(E_i) = C_i$ and $f(u_i) = u_i'$. Clearly, $|E_i| = |C_i|$ and $|u_i| = |u_i'|$.

  Now, $|E \cup U| = |C \cup U'|$. We claim that, $f(E_i) \neq f(E_j)$ if $E_i \neq E_j$. For otherwise, if $f(E_i) = f(E_j)$ and $E_i \neq E_j$ then $|C_i| = |E| - 1$ which is a contradiction. Also, if u_r is the common vertex to E_r, $E_s \in G'$ and E_r', $E_s' \in G''$ then u_r' will be the common vertex to the cliques C_r and $C_s \in (G' \cap G'')_a$. Hence adjacency of two edges in G' and G'' is preserved as adjacency of two cliques in $(G' \cap G'')_a$. Hence, $G'_a \cap G''_a \cong (G' \cap G'')_a$.

  Proofs of (ii) and (iii) are similar to (i).

Definition 3.1.9: Ring Sum of Two Semigraphs

If two semigraphs G_1 and G_2 are such that any edge in G_1 has either atmost one vertex in common with any edge in G_2 or any one of the edge in G_1 is equal to any one of the edge in G_2 (equal edges) then the ring sum of semigraphs G_1 and G_2 denoted by $G_1 \oplus G_2$ is a semigraph with the vertex set $V_1 \cup V_2$ and the edges are the full edges present either in G_1 or G_2 but not in both.

That is $G_1 \oplus G_2 = \{(V_1 \cup V_2), [(E_1 \cup E_2) - (E_1 \cap E_2)]\}$
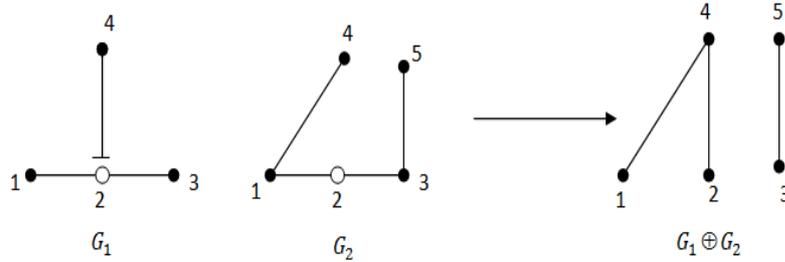
Example 3.1.10:

Figure 4.

Remarks 3.1.11:

i) For any semigraph G on n vertices, $G \oplus G = \overline{K_n}$, null graph.

ii) For any semigraph G on n vertices, $G \oplus \overline{K_n} = G$.

iii) For any two semigraphs G_1 and G_2, $G_1 \oplus G_2 = G_2 \oplus G_1$.

iv) For any semigraphs G_1, G_2 and G_3, $G_1 \oplus (G_2 \oplus G_3) = (G_1 \oplus G_2) \oplus G_3$.

Theorem 3.1.12: If G_a, G_e and G_{ca} are the adjacency graph, end vertex graph and consecutive adjacency graph respectively of a semigraph G, then for any two semigraphs G' and G'' we have

i) $G'_a \oplus G''_a \cong (G' \oplus G'')_a$

ii) $G'_e \oplus G''_e \cong (G' \oplus G'')_e$

iii) $G'_{ca} \oplus G''_{ca} \cong (G' \oplus G'')_{ca}$

Proof: i) By the definition 3.9, the vertex set of $G'_a \oplus G''_a$ is same as the vertex set of $(G' \oplus G'')_a$.

Let $E_I = \{E_i / E_i \in G', i = 1,2, \ldots, k\}$ and $E_J = \{E_j / E_j \in G', j = 1,2, \ldots, m\}$ be the set of edges. By definition 2.9, for each edge E_i in G' there is a clique say, C_i in G'_a and for each edge E_j in G'' there is a clique say, C_j in G''_a provided $E_i \neq E_j$. This implies $(C_i \cup C_j) \in (G'_a \oplus G''_a)$. Also, if any edge E_i in G' is equal to some edge E_j in G'', then $C_i = C_j$ and hence $(C_i \cap C_j) \notin (G'_a \oplus G''_a)$. Again by the definition 2.9, for the same set of edges E_i and E_j, the corresponding cliques C_i and C_j exists in $G' \oplus G''$ and also in $(G' \oplus G'')_a$ for $E_i \neq E_j$. Thus, for each set of edges E_i and E_j there corresponds cliques $(C_i \cup C_j) \in (G'_a \oplus G''_a)$ and $(C_i \cup C_j) \in (G' \oplus G'')_a$.

To establish the isomorphism we proceed in the following way.

Let, $f: \{(E_I \cup E_J) - (E_I \cap E_J)\} \to \{(C_I \cup C_J) - (C_I \cap C_J)\}$ be a function defined as $f(E_k) = C_k$.

The remaining proof is similar as in Theorem 3.1.4.

Proofs of (ii) and (iii) are similar to (i).

Definition 3.1.13: Decomposition of Two Semigraphs

A semigraph G is said to be *decomposed* into two subsemigraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \emptyset$.

Definition 3.1.14: A circuit of a semigraph G which contains a strong cycle is called a *strong circuit* and is denoted by C_s.

Similarly we define a *weak circuit* of a semigraph.

Theorem 3.1.15: A semigraph G is p-eulerian, if and only if it can be decomposed into strong circuits.

Proof: Let G be a p-eulerian semigraph. Consider the consective adjacency graph G_{ca} of G. Then the graph G_{ca} is also an Eulerian graph containing a closed walk. By theorem 2.15 it follows that every vertex of G and its G_{ca} is of even degree. Hence, the walk traversed from the vertex v has to end at the same vertex v, thus forming a strong circuit C_s in G. On removing this strong circuit C_s from G, the remaining vertices must also be of even degree. Again the remaining vertices are traversed in a similar manner until all the strong circuits are exhausted. This process is repeated until all the partial edges are traversed which forms a strong circuit. Hence, a semigraph G can be decomposed into strong circuits.

    Conversely, suppose a semigraph G can be decomposed into strong circuits. That is, G_{ca} is a union of edge disjoint circuits. It follows that the degree of every vertex in the circuit is even. Hence, G_{ca} is an Eulerian graph. Thus, semigraph G is is p-eulerian.

3.2. End-vertex Union of Semigraphs (ev-union)

Definition 3.2.1: If $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs then the end-vertex union of G_1 and G_2 denoted by $G_1 \cup_{ev} G_2$ is a semigraph with the vertex set $V_1 \cup V_2$ and the edge set satisfies the following the conditions:

1) If $|E_{X_1} \cap E_{X_n}| \leq 1$, then E_{X_1} and $E_{X_2} \in G_1 \cup_{ev} G_2$ where $E_{X_1} \in G_1$ and $E_{X_2} \in G_2$.
2) If E_{X_1} and E_{X_2} are edges having the same end vertices and with the same order of common middle vertices with $|E_{X_1} \cap E_{X_n}| \geq 2$ then $E = E_{X_1} \cup E_{X_2} \in G_1 \cup_{ev} G_2$ is an edge whose end vertices are the end vertices of E_{X_1} and E_{X_2} and the permutations of middle vertices of E retain the order of vertices in E_{X_1} and E_{X_2}.

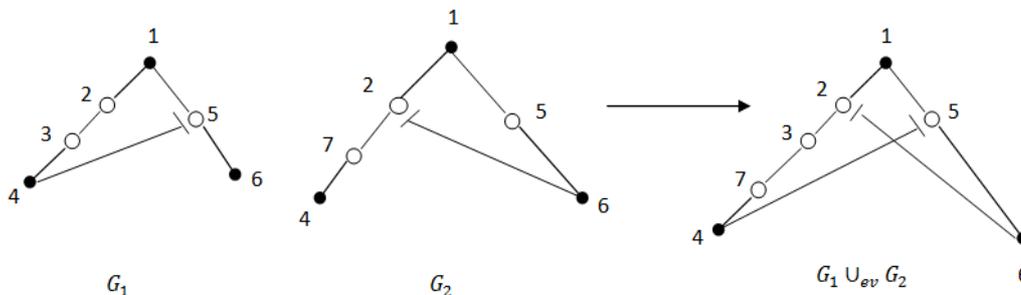   Clearly, the end-vertex union of semigraphs is not unique.

Example 3.2.2:

Figure 5.

Remarks 3.2.3:
    i) For any semigraph G, $G \cup_{ev} G \cong_{ev} G$.
    ii) For any semigraph G, $G \cup_{ev} G^c \cong_{ev} K(G)$ where $K(G)$ is complete semigraph containing both G and G^c.
    iii) For any two semigraphs G_1 and G_2, $G_1 \cup_{ev} G_2 \cong_{ev} G_2 \cup_{ev} G_1$.
    iv) For any semigraphs G_1, G_2 and G_3, $G_1 \cup_{ev} (G_2 \cup_{ev} G_3) \cong_{ev} (G_1 \cup_{ev} G_2) \cup_{ev} G_3$

Theorem 3.2.4: If G_a and G_e are the adjacency graph and the end vertex graphs respectively of a semigraph G, then for any two semigraphs G' and G'' we have
    i) $G'_a \cup_{ev} G''_a \cong (G' \cup_{ev} G'')_a$

    ii) $G'_e \cup_{ev} G''_e \cong (G' \cup_{ev} G'')_e$

Proof: i) By the definition 4.1, the vertex set of $G'_a \cup_{ev} G''_a$ is same as the vertex set of $(G' \cup_{ev} G'')_a$. Let us consider two edges $E_{X_1} = \{u_1, x, y, u_n\} \in G'$ and $E_{X_2} = \{u_1, x, y, w, u_n\} \in G''$

having the same end vertices u_1 and u_n, with atleast one different middle vertex. Then, the end vertex union of $G'_a \cup_{ev} G''_a$ contains the same clique as that of $G' \cup_{ev} G''$, retaining the order of vertices. Hence for each $E_{X_1} \cup E_{X_2}$, there corressponds a unique clique denoted by $C_{E_{X_1} \cup E_{X_2}}$ in $G'_a \cup_{ev} G''_a$. Similarly, for the same edges E_{X_1} and E_{X_2} with the same end vertices there corressponds a unique clique $C_{E_{X_1} \cup E_{X_2}}$ in $(G' \cup_{ev} G'')_a$, with the same vertex order. Hence, $C_{E_{X_1} \cup E_{X_2}} \in G'_a \cup_{ev} G''_a$ and also $C_{E_{X_1} \cup E_{X_2}} \in (G' \cup_{ev} G'')_a$ with the same end vertices.

Hence, $G'_a \cup_{ev} G''_a \cong (G' \cup_{ev} G'')_a$

ii) By the definition 4.1, the vertex set of $G'_e \cup_{ev} G''_e$ is same as the vertex set of $(G' \cup_{ev} G'')_e$.

Let us consider two edges $E_{X_1} = \{u_1, x, y, u_n\} \in G'$ and $E_{X_2} = \{y, r, s\} \in G''$, where y is an (m, e) vertex in E_{X_1}. For every pair of edges E_{X_1} and $E_{X_2} \in G'$ or G'', there exists adjacent complete graphs say K_2 s' in G'_e or G''_e. Hence for each pair of edges $E_{X_1}, E_{X_2} \in (G' \cup_{ev} G'')$, there corresponds adjacent complete graphs K_2 s' in $(G' \cup_{ev} G'')_e$.

Now for the same pair of edges E_{X_1} and $E_{X_2} \in (G' \cup_{ev} G'')$, there corresponds adjacent complete graphs K_2 s' which belong to $(G' \cup_{ev} G'')_e$. Hence for each pair of edges E_{X_1} and E_{X_2} of $G'_e \cup_{ev} G''_e$, there corresponds adjacent complete graphs K_2 s' which belongs to $G' \cup_{ev} G''$ and also K_2 s' belongs to $(G' \cup_{ev} G'')_e$ with the same end vertices (u_1, u_n) and (y, s).

Definition 3.2.5: End-vertex Intersection of Semigraphs (ev-intersection)

If $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs then the end-vertex intersection of semigraphs G_1 and G_2 denoted by $G_1 \cap_{ev} G_2$ is a semigraph with the vertex set $V_1 \cap V_2$ and the edge set satisfies the following the conditions:

1) If $|E_{X_1} \cap E_{X_n}| \le 1$, then E_{X_1} and $E_{X_2} \in G_1 \cap_{ev} G_2$ where $E_{X_1} \in G_1$ and $E_{X_2} \in G_2$.

2) If E_{X_1} and E_{X_2} are edges having the same end vertices and with the same order of common middle vertices with $|E_{X_1} \cap E_{X_n}| \ge 2$ then $E = E_{X_1} \cap E_{X_2} \in G_1 \cap_{ev} G_2$ is an edge whose end vertices are the end vertices of E_{X_1} and E_{X_2} and the permutations of middle vertices of E retain the order of vertices in E_{X_1} and E_{X_2}.

Clearly, the end-vertex intersection of semigraphs is not unique.
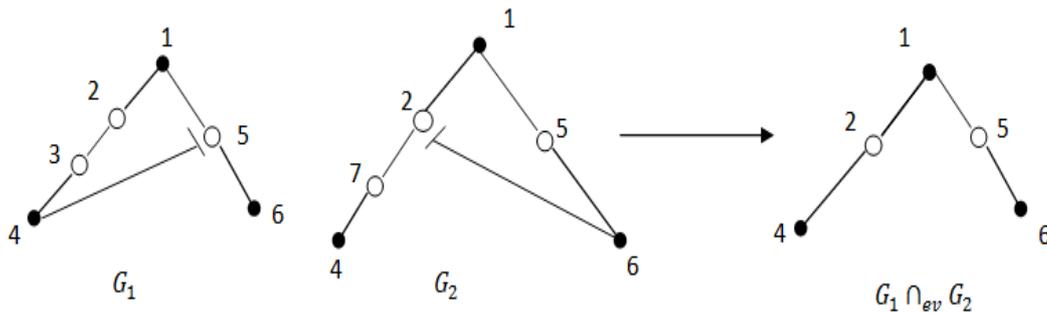
Example 3.2.6:

Figure 6.

Remarks 3.2.7:

   i) For any semigraph G, $G \cap_{ev} G \cong_{ev} G$.

   ii) For any semigraph G, $G \cap_{ev} G^c$ is a null graph.

   iii) For any two semigraphs G_1 and G_2, $G_1 \cap_{ev} G_2 \cong_{ev} G_2 \cap_{ev} G_1$.

   iv) For any semigraphs G_1, G_2 and G_3

      $G_1 \cap_{ev} (G_2 \cap_{ev} G_3) \cong_{ev} (G_1 \cap_{ev} G_2) \cap_{ev} G_3$

   v) For any semigraphs G_1, G_2 and G_3

      a) $G_1 \cap_{ev} (G_2 \cap_{ev} G_3) \cong_{ev} (G_1 \cap_{ev} G_2) \cup_{ev} (G_1 \cap_{ev} G_3)$

      b) $G_1 \cup_{ev} (G_2 \cap_{ev} G_3) \cong_{ev} (G_1 \cup_{ev} G_2) \cap_{ev} (G_1 \cup_{ev} G_3)$

Theorem 3.2.8: If G_a and G_e are the adjacency graph and the end vertex graph respectively of a semigraph G, then for any two semigraphs G' and G'' we have

    i) $G_a' \cap_{ev} G_a'' \cong (G' \cap_{ev} G'')_a$

    ii) $G_e' \cap_{ev} G_e'' \cong (G' \cap_{ev} G'')_e$

Proof: Proof is similar to Theorem 3.1.8.

Definition 3.2.9: End-vertex Decomposition

    A semigraph G on vertices n is said to have an *end vertex decomposition* if it can be decomposed into two subsemigraphs G_1 and G_2 such that $G_1 \cup_{ev} G_2 = G$ and $G_1 \cap_{ev} G_2 = \overline{K_n}$.

Definition 3.2.10: End-vertex Ring Sum

    The *end-vertex ring sum* of two semigraphs G_1 and G_2 denoted by $G_1 \oplus_{ev} G_2$ such that the edges E_{X_1} and E_{X_2} having the same end vertices is defined as a semigraph with vertex set $V_1 \cup V_2$ and the edges are the edges present either in G_1 or G_2 but not both.

Remarks 3.2.11:

    i) For any semigraph G on n vertices, $G \oplus_{ev} G = \overline{K_n}$, a null graph.
    ii) For any semigraph G on n vertices, $G \oplus_{ev} \overline{K_n} = G$.
    iii) For any two semigraphs G_1 and G_2, $G_1 \oplus_{ev} G_2 = G_2 \oplus_{ev} G_1$.
    iv) For any semigraphs G_1, G_2 and G_3, $G_1 \oplus_{ev} (G_2 \oplus_{ev} G_3) = (G_1 \oplus_{ev} G_2) \oplus_{ev} G_3$.

Theorem 3.2.12: If G_a and G_e represent the adjacency graph and the end vertex graph respectively of a semigraph G, then for any two semigraphs G' and G'' we have

    i) $G_a' \oplus_{ev} G_a'' \cong (G' \oplus_{ev} G'')_a$

    ii) $G_e' \oplus_{ev} G_e'' \cong (G' \oplus_{ev} G'')_e$

Proof: Proof is similar to Theorem 3.1.12.

3.3. Subedge Operation

Definition 3.3.1: Subedge Union of Semigraphs

    If $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs then the subedge union of G_1 and G_2 denoted by $G_1 \cup_{se} G_2$ is a semigraph with vertex set $V_1 \cup V_2$ and edge set satisfying the following conditions:

1) If $\left| E_{X_1} \cap E_{X_n} \right| \leq 1$ then E_{X_1} and $E_{X_2} \in G_1 \cup_{se} G_2$ where $E_{X_1} \in G_1$ and $E_{X_2} \in G_2$.

2) If $\left| E_{X_1} \cap E_{X_n} \right| \geq 2$ such that no other subedge E_{X_i}, $i \neq 1$ and $i \neq 2$ contains common vertices of E_{X_1} and E_{X_2}, then $E = E_{X_1} \cup E_{X_2} \in G_1 \cup_{se} G_2$ is an edge which is a permutation of vertices of $V(E_{X_1}) \cup V(E_{X_2})$ retaining the vertex order of E_{X_1} and E_{X_2}.

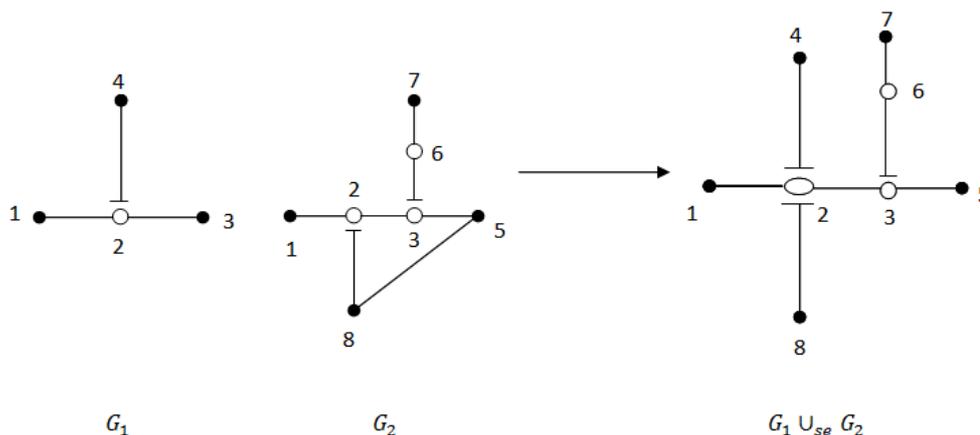   Clearly, the subedge union of semigraphs is not unique.

Example 3.3.2:

G_1 G_2 $G_1 \cup_{se} G_2$

Figure 7.

Remarks 3.3.3:

i) For any semigraph G on n vertices , $G \cup_{se} G \cong_e G$.

ii) For any semigraph G on n vertices, $G \cup_{se} G^c \cong_e K_n$ where K_n is a complete semigraph containing both G and G^c.

iii) For any two semigraphs G_1 and G_2, $G_1 \cup_{se} G_2 \cong_e G_2 \cup_{se} G_1$.

iv) For any semigraphs G_1, G_2 and G_3, $G_1 \cup_{se} (G_2 \cup_{se} G_3) \cong_e (G_1 \cup_{se} G_2) \cup_{se} G_3$

v) The definition of subedge union can be extended to k number of graphs for $k \geq 3$.

Theorem 3.3.4: If G_a represent the adjacency graph of a semigraph G, then for any two semigraphs G' and G'' we have $G'_a \cup_{se} G''_a \cong (G' \cup_{se} G'')_a$.

Proof: By definition 5.1, the vertex set of $G'_a \cup_{se} G''_a$ is same as the vertex set of $(G' \cup_{se} G'')_a$. Let, E_{X_1} and E_{X_2} belong to $E(G' \cup_{se} G'')$. This implies either $E_{X_1} \in E(G')$ and $E_{X_2} \in E(G'')$ or $E_{X_1} \in E(G'')$ and $E_{X_2} \in E(G')$. Without loss of generality, let $E_{X_1} \in E(G')$ and $E_{X_2} \in E(G'')$ such that no other subedge contains common vertices of E_{X_1} and E_{X_2}. Then the edge E_{X_1} corressponds to a clique C' in G'_a and the edge E_{X_2} corressponds to a clique C'' in G''_a. This implies that for each edge $E = E_{X_1} \cup E_{X_2} \in E(G' \cup_{se} G'')$ there corresponds a clique $C = C' \cup C'' \in (G'_a \cup_{se} G''_a)$.

Now for the same edges E_{X_1} and $E_{X_2} \in E(G' \cup_{se} G'')$ there exists a clique $C = C' \cup C'' \in (G'_a \cup_{se} G''_a)$. Hence for the edge $E = E_{X_1} \cup E_{X_2}$ of $G' \cup_{se} G''$ there exists a clique $C = C' \cup C'' \in (G'_a \cup_{se} G''_a)$ and $= C' \cup C'' \in (G' \cup_{se} G'')_a$, with no vertices common to other subedges.

To show the existence of isomorphism between $G'_a \cup_{se} G''_a$ and $(G' \cup_{se} G'')_a$ we establish the following result. Let $E = (E_1, E_2, \dots\dots, E_k)$ be the set of edges of $G' \cup_{se} G''$ and $C = (C_1, C_2, \dots\dots, C_k)$ be the cliques in $G'_a \cup_{se} G''_a$. Let $U = \{u_1, u_2, \dots\dots, u_i\}$ be the set of vertices which belongs to either E_{X_1} or E_{X_2} but does not belong to any other subedge of either G' or G''.

Let, $f: (E \cup u_i) \to (C \cup u_i)$ be a function defined as $f(E_i) = \left(E_{E_{X_1} \cup E_{X_2}}\right) \cup u_i$.

Then, we have the following

(i) $\left| E_{X_i} \cup E_{X_j} \right| = \left| C_i \cup C_j \right|, i \neq j$

(ii) If, $E_i \neq E_j$ with $\left| E_i \cap E_j \right| \geq 2$, then $f(E_i \cup E_j) = C_i \cup C_j$ with $\left| C_i \cap C_j \right| \geq 2$. For otherwise if $E_i \neq E_j$ and $f(E_i) = f(E_j)$ then $\left| E_{E_{X_1} \cup E_{X_2}} \right| = |E| - 1$, which is a contradiction to (i).

Therefore, by definition 2.14 f is an *e-isomorphism*.

Hence, $G'_a \cup_{se} G''_a \cong (G' \cup_{se} G'')_a$.

Definition 3.3.5: Subedge Intersection of Semigraphs

Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs then $G_1 \cap_{se} G_2$ is a semigraph with vertex set $V_1 \cap V_2$ and edge set satisfying the following the conditions:

1) If $\left| E_{X_1} \cap E_{X_2} \right| \leq 1$, then E_{X_1} and $E_{X_2} \in G_1 \cap_{se} G_2$ where $E_{X_1} \in G_1$ and $E_{X_2} \in G_2$.

2) If $\left| E_{X_1} \cap E_{X_2} \right| \geq 2$ such that no other subedge E_{X_i}, $i \neq 1$ and $i \neq 2$ contains common vertices of E_{X_1} and E_{X_2}, then $E = E_{X_1} \cap E_{X_2} \in G_1 \cap_{se} G_2$ is an edge which is a permutation of vertices of $V(E_{X_1}) \cap V(E_{X_2})$ retaining the vertex order of E_{X_1} and E_{X_2}.

Clearly the subedge intersection of semigraphs is not unique.

Remarks 3.3.6:

i) For any semigraph G, $G \cap_{se} G \cong_e G$.

ii) For any semigraph G, $G \cap_{se} G^c$ is a null graph.

iii) For any two semigraphs G_1 and G_2, $G_1 \cap_{se} G_2 \cong_e G_2 \cap_{se} G_1$.

iv) For any semigraphs G_1, G_2 and G_3, $G_1 \cap_{se} (G_2 \cap_{se} G_3) \cong_e (G_1 \cap_{se} G_2) \cap_{se} G_3$.

v) For any semigraphs G_1, G_2 and G_3

a) $G_1 \cap_{se} (G_2 \cap_{se} G_3) \cong_e (G_1 \cap_{se} G_2) \cap_{se} (G_1 \cap_{se} G_3)$

b) $G_1 \cup_{se} (G_2 \cap_{se} G_3) \cong_e (G_1 \cup_{se} G_2) \cap_{se} (G_1 \cup_{se} G_3)$

Theorem 3.3.7: If G_a is the adjacency graph of a semigraph G then for any two semigraphs G' and G'' we have $G'_a \cap_{se} G''_a \cong (G' \cap_{se} G'')_a$.

Proof: Proof is similar to Theorem 3.1.8

Definition 3.3.8: Subedge Decomposition

A semigraph G on n vertices is said to have a subedge decomposition into two subsemigraphs G_1 and G_2 if $G_1 \cup_{se} G_2 = G$ and $G_1 \cap_{se} G_2 = \overline{K_n}$.

Definition 3.3.9: Ring Sum of Subedge

The subedge ring sum of two semigraphs G_1 and G_2 denoted by $G_1 \oplus_{se} G_2$ is defined as a semigraph with vertex set $V_1 \cup V_2$ and the edges are the subedges present either in G_1 or G_2 but not in both.

Remarks 3.3.10:

  i)   For any semigraph G, $G \oplus_{se} G = \overline{K_n}$, is a null graph.
  ii) For any semigraph G, $G \oplus_{se} \overline{K_n} = G$.
  iii) For any two semigraphs G_1 and G_2, $G_1 \oplus_{se} G_2 = G_2 \oplus_{se} G_1$.
  iv) For any semigraphs G_1, G_2 and G_3, $G_1 \oplus_{se} (G_2 \oplus_{se} G_3) = (G_1 \oplus_{se} G_2) \oplus_{se} G_3$.

Conclusion

The concept of union, intersection, ring sum and decomposition is defined for semigraphs. The existence of isomorphism between a semigraph and its adjacency graph, consecutive adjacency graph, end-vertex graph is established. Relaxing the conditions imposed in all the definitions better generalizations are possible. The application of elementary operations to the p-Eulerian semigraph is also proved.

Acknowledgement

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