

## On the Line Degree Splitting Graph of a Graph

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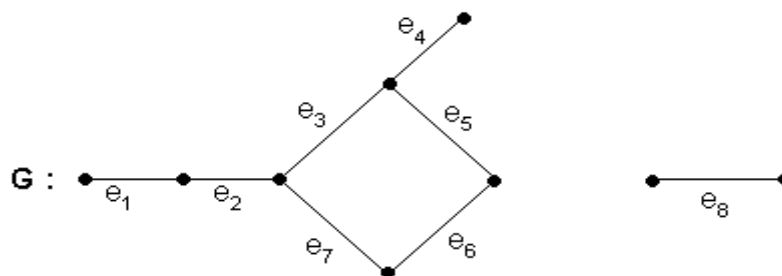
**Abstract:** In this paper, we introduce the concept of the line degree splitting graph of a graph. We obtain some properties of this graph. We find the girth of the line degree splitting graphs. Further, we establish the characterization of graphs whose line degree splitting graphs are eulerian, complete bipartite graphs and complete graphs.

### 1. Introduction

By a graph, we mean a finite, undirected graph without loops or multiple lines. For a graph  $G$ , let  $V(G)$ ,  $E(G)$  and  $L(G)$  denote its point set, line set and line graph respectively. We refer the terminology of [2]. The degree of a point  $v \in V(G)$  is the number of points adjacent to  $v$  and is denoted by  $\deg(v)$ . If  $u$  and  $v$  are two adjacent points of  $G$ , then the line connecting them will be denoted by  $uv$ . The degree of a line  $e$  in  $G$  is denoted by  $\deg(e)$ , and is defined by  $\deg(e) = \deg(u) + \deg(v) - 2$  with  $e = uv$ . A graph  $G$  is said to be the *line-regular* if all lines have same degree. The open-neighborhood  $N(e_i)$  of a line  $e_i$  in  $E(G)$  is the set of lines adjacent to  $e_i$ . For each line  $e_i$  of  $G$ , a new point  $e'_i$  is taken and the resulting set of points is denoted by  $A(G)$ .

The *line splitting graph*  $L_s(G)$  of a graph  $G$  is defined as the graph having point set  $E(G) \cup A(G)$  with two points are adjacent if they correspond to adjacent lines of  $G$  or one corresponds to an element  $e'_i$  of  $A(G)$  and the other to an element  $e_j$  of  $E(G)$  where  $e_j$  is in  $N(e_i)$ . This concept was introduced by Kulli and Biradar in [4]. Miscellaneous properties of line splitting graphs are studied by Basavanagoud and Mathad in [1].

Let  $G = (V, E)$  be a graph with  $E = S_1 \cup S_2 \cup \dots \cup S_t \cup T$  where each  $S_i$  is a set of lines of  $G$  having at least two lines and having the same line degree and  $T = E \setminus \cup S_i$ . For each set  $S_i$  of lines of  $G$ , a new point  $w_i$  is taken and the resulting set of points  $w_i$  is denoted by  $E_1(G)$ . The *line degree splitting graph*  $DL_s(G)$  of a graph  $G$  is defined as the graph having point set  $E(G) \cup E_1(G)$  with two points are adjacent if they correspond to adjacent lines of  $G$  or one correspond to a point  $w_i$  of  $E_1(G)$  and the other to a line  $e_j$  of  $G$  and  $e_j$  is in  $S_i$ . In Fig. 1, graph  $G$  and its line degree splitting graph are shown.



Here  $S_1 = \{e_4, e_6\}$ ,  $S_2 = \{e_2, e_5, e_7\}$ ,  $T = \{e_1, e_3, e_8\}$

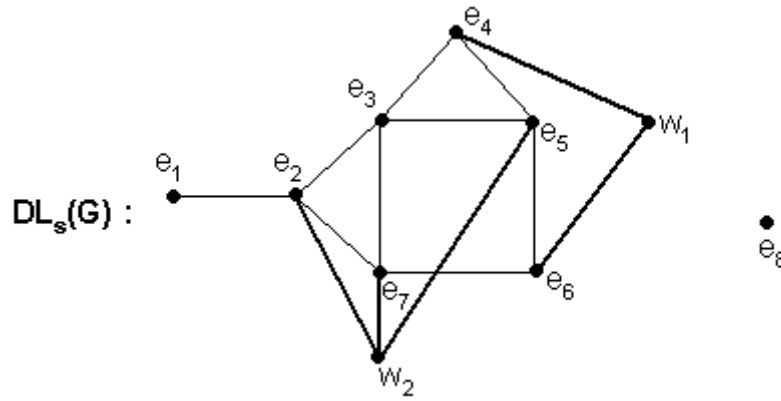


Figure 1.

**Notation 1.**  $\deg^*(e)$  denoted the degree of a point  $e$  in  $DL_s(G)$ . Clearly  $\deg(e) \leq \deg^*(e)$  for all  $e$  in  $E(G)$ .

**Notation 2.**  $\cup S_i$ ,  $T$  and  $E_1(G)$  are defined as in the definition of  $DL_s(G)$ .

**Observation 1.** For any graph,  $L(G)$  is an induced subgraph of  $DL_s(G)$ .

**Remark 1.** For any graph  $G$  without isolated points,  $L(G)$  is an induced subgraph of  $DL_s(G)$ . Clearly if  $G$  contains at least two lines, then  $G$  contains at least two lines of the same degree. Hence  $G = K_2$  is the only graph such that  $L(G) = DL_s(G)$ .

**Remark 2.** If  $G$  is connected then  $DL_s(G)$  is connected. But not conversely.

For example, if  $G = nK_2$  then  $DL_s(G) = K_{1,n}$ .

**Remark 3.** A graph  $G$  is a cycle  $C_n$ ,  $n \geq 3$  if and only if  $DL_s(G)$  is a wheel  $W_{n+1}$ .

**Remark 4.** If  $G$  is a star  $K_{1,n}$ ,  $n \geq 2$  then  $DL_s(G) = K_{n+1}$ .

**Remark 5.** Let  $e = uv$  be a line of a graph  $G$ .

- i) If  $e$  belongs to  $\cup S_i$ , then  $\deg^*(e) = \deg(u) + \deg(v) - 1$ .
- ii) If  $e$  belongs to  $T$ , then  $\deg^*(e) = \deg(u) + \deg(v) - 2$ .

**Remark 6.** Let  $w_i$  be a point belonging to  $E_1(G)$ . Then  $\deg^*(w_i) = |E_1(G)|$ .

**Theorem A [2].** Unless  $p = 8$ , a graph  $G$  is the line graph of  $K_p$  if and only if

- 1)  $G$  has  $\binom{p}{2}$  points,
- 2)  $G$  is regular of degree  $2(p - 2)$ ,
- 3) Every two nonadjacent points are mutually adjacent to exactly four points,
- 4) Every two adjacent points are mutually adjacent to exactly  $p - 2$  points.

When  $p = 8$ , there are exactly three exceptional graphs satisfying the conditions.

**Theorem B [2].** Unless  $m = n = 4$ , a graph  $G$  is the line graph of  $K_{m,n}$  if and only if

- 1)  $G$  has  $mn$  points,
- 2)  $G$  is regular of degree  $m + n - 2$ ,
- 3) Every two nonadjacent points are mutually adjacent to exactly two points,

- 4) Among the adjacent pairs of points, exactly  $n \binom{m}{2}$  pairs are mutually adjacent to exactly  $m - 2$  points, and the other  $m \binom{n}{2}$  pairs to  $n - 2$  points.

There is only one exceptional graph satisfying these conditions. It has 16 points, is not  $L(K_{4,4})$ , and was found by Shrikhande [7] when he proved Theorem B for the case  $m = n$ .

## 2. Results

**Theorem 1.** Let  $G$  be a graph with  $p$  points and  $q$  lines whose points have degree  $d_i$  and let  $r$  be the number of lines in  $T$  and  $t$  be the number of points in  $E_1(G)$ .

Then  $|V(DL_s(G))| = q + t$  and

$$|E(DL_s(G))| = \frac{1}{2} \sum_{i=1}^p d_i^2 - r.$$

**Proof.**  $DL_s(G)$  of a graph  $G$  is obtained from  $L(G)$  by adding  $t$  new points which corresponds to the set  $S_i, 1 \leq i \leq t$ , where  $S_i$  is as in the definition of  $DL_s(G)$ . Therefore, the number of points in  $DL_s(G)$  is the sum of points of  $L(G)$  and  $t$ . Hence  $|V(DL_s(G))| = q + t$ .

It is known in [2, pp 72], that  $L(G)$  has  $-q + \frac{1}{2} \sum_{i=1}^p d_i^2$  lines. By the definition of  $DL_s(G)$ , the number of lines in  $DL_s(G)$  is the sum of number of lines in  $L(G)$  and the lines due to  $t$  new points  $w_1, w_2, \dots, w_t$ .

$$\begin{aligned} \text{Hence } |E(DL_s(G))| &= |E(L(G))| + \sum_{i=1}^t \deg^*(w_i) \\ &= -q + \frac{1}{2} \sum_{i=1}^p d_i^2 + q - r \\ &= \frac{1}{2} \sum_{i=1}^p d_i^2 - r \end{aligned} \tag{1}$$

**Theorem 2.** Let  $G$  be a connected graph of size  $q > 1$  or a disconnected graph with at least one component having size  $q > 1$ . Then  $DL_s(G)$  contains a cycle.

**Proof.** By the definition of  $DL_s(G)$ , the proof is trivial. □

**Corollary 2.1.** Let  $G$  be any graph. Then  $DL_s(G)$  is acyclic if and only if the number of lines in every component of  $G$  is  $\leq 1$ , that is, every component of  $G$  is either  $K_1$  or  $K_2$ . Further,  $DL_s(G) = K_{1,n}$  when every component of  $G$  is  $K_2$ .

**Theorem 3.** Let  $G$  be a graph without isolated points. Then  $DL_s(G)$  is a cycle if and only if  $G$  is  $P_3$  or  $P_4$ .

**Proof.** Suppose  $DL_s(G)$  is a cycle. By Remark 1,  $L(G)$  is an induced subgraph of  $DL_s(G)$ . It is clear that  $L(G)$  is acyclic and hence  $G$  is acyclic. Assume  $G$  has at least one component containing a point of degree 3. Then  $K_{1,3}$  is a subgraph of  $G$  and hence  $K_3$  is subgraph of  $L(G)$ , a contradiction. Hence, it follows that every component of  $G$  is a tree whose points are of degree  $\leq 2$ . It is clear that every component of  $G$  is a path. Further if  $G$  is a path  $P_n, n > 4$  then  $L(G)$  is  $P_{n-1}$ .  $DL_s(G)$  contains  $(n - 3)$  number of cycles, a contradiction. Assume every component of  $G$  is  $K_2$ , then by Corollary 2.1,  $DL_s(G) = K_{1,n}$ , a contradiction. Hence, every component of  $G$  is a path  $P_n, 2 \leq n \leq 3$ . Now assume  $G$

has at least two such paths as its components. Then by Theorem 2,  $DL_s(G)$  contains more than one cycle. Hence,  $G$  is connected and is either  $P_3$  or  $P_4$ .

Conversely, suppose  $G = P_3$  or  $P_4$ . Then it is easy to see that  $DL_s(G)$  is  $C_3$  or  $C_4$  respectively. Hence  $DL_s(G)$  is a cycle.  $\square$

**Theorem 4.** For any connected graph  $G$  of size  $q \geq 2$ ,

$$g(DL_s(G)) = \begin{cases} 4 & \text{if } G = P_4 \\ 3 & \text{otherwise} \end{cases}$$

**Proof.** Let  $G$  be a connected graph of size  $q \geq 2$ . If  $G = P_4$ , then by Theorem 3,  $DL_s(G)$  is  $C_4$  and hence its girth is 4. Otherwise

- (i) If either  $K_3$  or  $K_{1,3}$  is a subgraph of  $G$ , then  $DL_s(G)$  contains a triangle.
- (ii) If  $G$  is any cycle, then by Remark 3,  $DL_s(G)$  is a wheel.
- (iii) If  $G$  is a path  $P_n$ ,  $n \geq 5$ , then  $DL_s(G)$  contains  $(n - 4)$  number of triangles. And if  $G = P_3$ , then by Theorem 3,  $DL_s(G)$  is  $K_3$ .

Hence (i), (ii) and (iii) implies that  $g(DL_s(G)) = 3$  for all  $G$  other than  $P_4$ .  $\square$

**Theorem 5.**  $DL_s(G)$  is bipartite if and only if every component of  $G$  is either  $K_2$  or  $P_4$ .

**Proof.** Suppose  $DL_s(G)$  is bipartite. Assume  $G$  has at least one point of degree  $\geq 3$ . Then  $L(G)$  contains  $K_3$  as a subgraph and by Remark 1,  $DL_s(G)$  contains an odd cycle, a contradiction. Hence every point of  $G$  is of degree  $\leq 2$ . Clearly every component of  $G$  is either a cycle or a path. We consider the following cases.

**Case 1.** Assume one of the component of  $G$  is a cycle. By Remark 3,  $DL_s(G)$  has a wheel as its subgraph, a contradiction.

**Case 2.** Suppose one of the component of  $G$  is a path  $P_n$ ,  $n \geq 5$ . Then  $L(G)$  has  $P_{n-1}$  as a component and  $DL_s(G)$  has at least  $(n - 4)$  number of triangles, a contradiction. Hence, every component of  $G$  is a path  $P_n$ ,  $n \leq 4$ . Assume  $G$  has at least one component as  $P_3$ . Then by Theorem 3,  $DL_s(G)$  contains a triangle, a contradiction. Hence, every component of  $G$  is either  $P_4$  or  $K_2$ .

Conversely, suppose  $G$  is a graph such that each of its component is either  $K_2$  or  $P_4$ . We consider the following cases.

**Case 1.** Assume every component of  $G$  is  $K_2$ . Then by Corollary 2.1,  $DL_s(G)$  is  $K_{1,n}$ , which is a bipartite graph.

**Case 2.** Assume every component of  $G$  is  $P_4$ . Suppose  $G$  has exactly one component. Then by Theorem 2,  $DL_s(G)$  is  $C_4$ , which is a bipartite graph. Now suppose  $G$  has more than one component. Then  $L(G)$  contains more than one components each of which is  $K_{1,2}$ . We construct  $DL_s(G)$  as follows. Since each  $P_4$  contains one line of line degree 4 and the other line of line degree 3. Hence  $DL_s(G)$  contains two points  $w_1$  and  $w_2$  corresponding to the set  $S_1$  of lines of line degree 4 and set  $S_2$  of lines of line degree 3 respectively. Now partition the points of  $DL_s(G)$  such that  $V_1(DL_s(G))$  contains  $w_1$  and all those points of  $L(G)$  of degree 1 and  $V_2(DL_s(G))$  contains  $w_2$  and all those points of  $L(G)$  of degree 2. And  $w_1$  is adjacent to all points of  $V_2(DL_s(G))$  except  $w_2$  and  $w_2$  is adjacent to all points of  $V_1(DL_s(G))$  except  $w_1$ . Clearly the resulting graph  $DL_s(G)$  is bipartite.

**Case 3.** Assume  $G$  has at least two components one of which is  $P_4$  and the other is  $K_2$ . Then line graph of each  $P_4$  is  $K_{1,2}$  and each  $K_2$  is  $K_1$ . It is easy to see that line degree of lines of  $P_4$  is different from the line degree of lines of  $K_2$ . Hence no point of  $E_1(G)$  of  $DL_s(G)$  which corresponds to the set of lines of same line degree is adjacent to the points of both  $K_{1,2}$  and  $K_1$  in  $L(G)$ . By Case 1 and Case 2 of the converse part of this theorem,  $DL_s(G)$  contains two components each of which is bipartite. Hence  $DL_s(G)$  is bipartite.  $\square$

**Theorem 6.** For any graph  $G$ ,  $\omega(DL_s(G)) \leq \omega(G)$ , where  $\omega(G)$  denotes the number of components of  $G$ .

**Proof.** Let  $V(L(G)) = \{e_1, e_2, \dots, e_q\}$  and

$$V(DL_s(G)) \setminus V(L(G)) = \{w_1, w_2, \dots, w_t\}.$$

**Case 1.** Let  $G$  be connected. Then by Remark 2,  $DL_s(G)$  is connected. Therefore,  $\omega(G) = 1 = \omega(DL_s(G))$ .

**Case 2.** Let  $G$  be disconnected. Then  $L(G)$  is disconnected.

Let  $G_1, G_2, \dots, G_k$ ,  $k \geq 2$  be the components of  $G$ . If  $G_i$  and  $G_j$  have lines of same degree for some  $i, j$ . Let  $e_1 \in E(G_1)$  and  $e_2 \in E(G_2)$  be such that  $\deg(e_1) = \deg(e_2)$ . By the definition of  $DL_s(G)$ ,  $DL_s(G)$  contains a new point  $w$  such that  $w$  is adjacent to  $e_1$  and  $e_2$  of  $L(G)$  in  $DL_s(G)$ . Hence,  $\omega(DL_s(G)) \leq k - 1 < k = \omega(G)$ .

If there is no pair  $i$  and  $j$  such that  $G_i$  and  $G_j$  have lines of same degree, then  $\omega(DL_s(G)) = k = \omega(G)$ . Hence  $\omega(DL_s(G)) \leq \omega(G)$ . □

**Theorem 7.** The  $DL_s(G)$  of a nontrivial graph  $G$  is complete if and only if  $G$  is either  $K_3$  or  $K_{1,n}$ ,  $n \geq 1$ .

**Proof.** Suppose  $DL_s(G)$  is complete graph of order  $\geq 2$ . Then every pair of points of  $DL_s(G)$  are adjacent. Thus all the lines of  $G$  are mutually adjacent and hence have same line degree. Clearly there exists exactly one new point in  $DL_s(G)$  which is adjacent to all the points of  $L(G)$ . Suppose  $G \neq K_3$  and  $G$  has at least one cycle. Then it is easy to observe that there exist at least two nonadjacent lines. It implies that  $L(G)$  is not complete. By Remark 1,  $DL_s(G)$  is not complete, which is a contradiction. Suppose next  $G \neq K_{1,n}$ ,  $n \geq 1$  and  $G$  is a tree with at least two nonadjacent lines in  $G$ . Then again  $L(G)$  is not complete. By Remark 1,  $DL_s(G)$  is not complete a contradiction.

Conversely, suppose  $G = K_3$ . It has three mutually adjacent lines which are of same line degree. Then these three lines are the points of  $DL_s(G)$  together with a new point corresponding to the lines of same line degree of  $G$  in  $DL_s(G)$ . These four points are mutually adjacent points in  $DL_s(G)$ . Thus  $DL_s(G)$  is  $K_4$ . Now suppose  $G$  is  $K_{1,n}$ ,  $n \geq 2$ . By Remark 4,  $DL_s(G)$  is  $K_{n+1}$ . Thus  $DL_s(G)$  is complete. □

**Theorem 8.** If  $G (\neq K_2$  and  $K_3)$  is regular graph with  $p$  points and  $q$  lines, then  $DL_s(G)$  is not regular.

**Proof.** If  $G$  is regular of degree  $k$ , then  $L(G)$  is also regular of degree  $2(k - 1)$  with  $q$  points and  $\frac{pk^2}{2} - q$  lines. Let  $e \in V(L(G))$ . Then  $\deg(e) = 2(k - 1) \neq q - 1$ .

Clearly  $2(k - 1) < q - 1$ , except for  $G = K_2$  and  $K_3$  (2)

Let  $V(DL_s(G)) \setminus V(L(G)) = \{w\}$ . Since  $w$  is adjacent to all the lines of  $G$  in  $L(G)$ ,  $\deg^*(w) = q > 2(k - 1) + 1 = 2k - 1$ . (by (2)).

But  $\deg^*(e) = \deg(e) + 1 = 2(k - 1) + 1 = 2k - 1$ . Hence  $DL_s(G)$  is not regular. □

**Corollary 8.1.** If  $G$  is  $k$ -regular graph, then  $DL_s(G) = L(G) + K_1$ .

**Corollary 8.2.**  $DL_s(G)$  is line-regular if and only if  $G$  is  $K_{1,n}$  or  $K_3$ .

**Theorem 9.** Let  $G$  be any graph. A necessary and sufficient condition for  $DL_s(G)$  to be eulerian is that each of the following holds:

- 1) Each even degree line of  $G$  occurs exactly once.
- 2) Each odd degree line of  $G$  occurs even number of times.

**Proof.** Suppose  $DL_s(G)$  is eulerian. Let  $x$  be a point of  $DL_s(G)$ . Then  $x$  is a line of  $G$  or a point of  $E_1(G)$ . Suppose  $x = uv$  is a line of  $G$ . Remark 5 implies that if  $x$  belongs to  $T$ , then  $\deg^*(x) = \deg(u) + \deg(v) - 2$ . If  $x$  belongs to  $\cup S_i$ , then  $\deg^*(x) = \deg(u) + \deg(v) - 1$ . Suppose  $x$  belongs to  $T$ . Then both  $\deg u$  and  $\deg v$  must be even or odd. This implies that  $x$  is adjacent to even number of lines of  $G$ .

Hence all such  $x$  which belongs to  $T$  are even degree lines. Thus (1) holds. Suppose next  $x$  belongs to  $\cup S_i$ . Then either  $\deg(u)$  or  $\deg(v)$  is odd. This implies that  $x$  is adjacent to odd number of lines. Hence all such  $x$  are odd degree lines and since they belong to  $\cup S_i$ , the point  $w_i$  belonging to  $E_1(G)$  in  $DL_s(G)$  contributes one to the degree of  $x$  in  $DL_s(G)$ . Thus degree of  $x$  in  $DL_s(G)$  is even. If  $x$  is a point of  $E_1(G)$ , then Remark 6 implies that  $\deg^*(x) = |E_1(G)|$ . Since  $\deg^*(x)$  is even,  $|E_1(G)|$  is even. Thus, each  $S_i$  contains even number of lines of odd degree. Hence (2) holds.

Conversely, assume the condition (1) and (2) on  $G$  hold. Suppose  $x$  is a point of  $DL_s(G)$ . Then  $x$  is a line of  $G$  or  $x$  belongs to  $E_1(G)$ . Assume  $x$  is a line of  $G$ . If  $x$  belongs to  $T$ , then by (1),  $\deg^*(x)$  is even. And if  $x$  belongs  $\cup S_i$ , then by (2) and definition of  $DL_s(G)$ ,  $\deg^*(x)$  is even. Now if  $x$  belongs to  $E_1(G)$ , then by condition (2), each  $S_i$  contains even number of lines of odd degree and hence  $\deg^*(x)$  is even. Hence  $DL_s(G)$  is eulerian.  $\square$

**Theorem 10.** Let  $G$  be a nontrivial graph. If  $DL_s(G)$  is eulerian, then  $G$  is not eulerian.

**Proof.** Suppose  $DL_s(G)$  is eulerian. Theorem 2 implies that there exists at least one component in  $G$  of size  $q > 1$ . Then by Remark 1,  $G$  has at least two lines  $x$  and  $y$  of same line degree. Let  $x'$  and  $y'$  be the points of  $DL_s(G)$  corresponding to the lines  $x$  and  $y$  of  $G$  respectively. Since  $DL_s(G)$  is eulerian  $\deg^*(x')$  and  $\deg^*(y')$  are even. Then by Remark 5, one of the end points of both the lines  $x$  and  $y$  in  $G$  is odd. Thus,  $G$  is not eulerian.  $\square$

**Theorem 11.** If  $G$  is eulerian, then  $DL_s(G)$  is hamiltonian.

The converse of the Theorem 11 is need not be true. This is easily seen in Fig. 2.

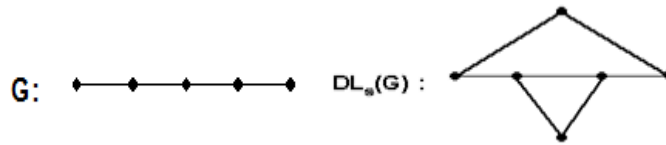


Figure 2.

In the next results we present characterization of line degree splitting graph of complete bipartite graphs and complete graphs. The line degree splitting graph of complete bipartite graphs and complete graphs are characterized by immediate observations involving adjacencies and degree of lines in  $K_{m,n}$  and  $K_p$ .

In the following theorem, we establish the characterization for line degree splitting graphs of complete bipartite graphs.

**Properties of  $DL_s(K_{m,n})$ :** If we suppose that  $m \geq n \geq 1$  it can be seen that  $DL_s(K_{m,n})$  has the following four properties.

- 1) The graph has  $mn + 1$  points,
- 2) Of the  $mn + 1$  points,  $mn$  points are of degree  $m + n - 1$  and one point is of degree  $mn$ ,
- 3) Every two nonadjacent points are mutually adjacent to three points,
- 4)  $n \binom{m}{2}$  of the pairs of adjacent points of same degree are mutually adjacent to  $m - 1$  other points,  $m \binom{n}{2}$  of the remaining pairs of adjacent points of same degree are mutually adjacent to  $n - 1$  points and every two adjacent points of different degree are mutually adjacent to exactly  $m + n - 2$  points.

The object of this note is to show that if any graph satisfies the four conditions (1) – (4), then it is isomorphic to  $DL_s(K_{m,n})$  except possibly when  $(m, n) = (4, 4)$ . Here we have tried to give a generalization for line degree splitting graphs, which is similar to the characterization given for line graphs which was studied by Moon [5] and Hoffman [3].

Before proving that if any graph satisfies the conditions (1) – (4) then it is isomorphic to  $DL_s(K_{m,n})$  except for  $(m, n) = (4, 4)$ , we state and prove the following lemma.

**Lemma 1.** Let there be given a graph  $G$  satisfying conditions (1), (2), (3) and (4) where  $m \geq n \geq 1$ , but  $(m, n) \neq (4, 4), (5, 4)$  or  $(4, 3)$ . Let  $p_{11}$  and  $p_{12}$  be two adjacent points of same degree of  $G$  which are mutually adjacent to each of the  $m - 1$  points in  $A = \{p_{13}, \dots, p_{1m}, w\}$ . Let  $C_1 = \{p_{21}, \dots, p_{n1}\}$  be the set of  $n - 1$  points which are adjacent to  $p_{11}$  but not to  $p_{12}$ . Furthermore let there be at least  $m - 1$  points in  $A \cup C_1$  such that each of these points and  $p_{11}$  are mutually adjacent to  $m - 1$  other points. Then  $A \cup p_{11} \cup p_{12}$  and  $C_1 \cup p_{11}$  are the point set of complete graphs of  $m + 1$  and  $n$  points respectively, and no point of  $A$  is adjacent to any of  $C_1$  whose degree is same as that of the point of  $A$ .

**Proof.** We consider first the case in which  $m \geq n \geq 5$ . No point in  $C_1$  can be adjacent to more than two points in  $A$  without violating condition (3) with respect to the point  $p_{12}$ . Therefore, each point in  $C_1$  is adjacent to at least  $n - 3$  of the remaining  $n - 2$  points in  $C_1$  in order to satisfy condition (4) with respect to the point  $p_{11}$ . Then, if there exists two nonadjacent points in  $C_1$  they must be mutually adjacent to the remaining  $n - 3$  points in  $C_1$  as well as to  $p_{11}$  and  $w$ . This contradicts condition (3) since  $n - 2 \geq 3$  in the case being considered. Hence, every point in  $C_1$  is adjacent to every other point in  $C_1$ . If some point in  $C_1$ ,  $p_{21}$  say, is adjacent to some point in  $A$  whose degree is same as  $p_{21}$  then there are  $n$  points adjacent to both  $p_{21}$  and  $p_{11}$ . Thus  $n$  must equal  $m - 1$ , by (4), or  $m = n + 1$ .

If  $m = n + 1$  suppose that some point in  $A$ ,  $p_{13}$  say, is adjacent to some points of  $C_1$ . It is easily seen that  $p_{13}$  cannot be adjacent to more than one point of  $C_1$  but not to all points of  $C_1$  without violating condition (3). But if  $p_{13}$  is adjacent to all points of  $C_1$  then the number of points which are adjacent to both  $p_{11}$  and  $p_{13}$  is, including  $p_{12}$ , at least  $n$  which contradicts condition (4). Hence,  $p_{13}$  can be adjacent to at most one point in  $C_1$ . From condition (4) it follows that there is at least one other point in  $A$ ,  $p_{14}$  say, which is not adjacent to  $p_{13}$ . But  $p_{13}$  and  $p_{14}$  are each adjacent to at least  $m - 4$  of the remaining  $m - 3$  points of  $A$  and if  $m > 6$  there will be at least two of these points which is adjacent to both  $p_{13}$  and  $p_{14}$ . This, however, contradicts condition (3) since  $p_{13}$  and  $p_{14}$  are both adjacent to  $p_{11}$  and  $p_{12}$ .

The only alternative remaining to be treated, under the assumption that  $m = n + 1$  and that some lines join points in  $A$  to points in  $C_1$ , is when  $m = n + 1 = 6$ . In this case it is not difficult to see that the only configuration which can satisfy condition (4) without implying a contradiction of the type just described is one in which each point of  $A$  except  $w$  is adjacent to  $p_{15}$ , say, and  $p_{14}$  is adjacent to  $p_{16}$ . Suppose that  $p_{21}$  and  $p_{31}$  are the different points in  $C_1$  which are adjacent to  $p_{13}$  and  $p_{15}$ , respectively. Then  $p_{21}$  and  $p_{15}$  are not adjacent to each other but are mutually adjacent to  $w$ ,  $p_{31}$ ,  $p_{13}$  and  $p_{11}$ , contradicting condition (3). Hence, no point of  $A$  is adjacent to any point of  $C_1$  under the given assumptions. This and the fourth sentence of the hypothesis of the lemma implies that each point in  $A$  is adjacent to every other point in  $A$  which suffices to complete the proof of the lemma when  $m \geq n \geq 5$ .

Next consider the case in which  $n = 4$  and  $m \geq 6$ . No point in  $C_1$  can be adjacent to  $m - 1$  other points of  $A \cup C_1$  by an earlier remark and the fact that  $m - 1 \geq 5$ . Hence, from the hypothesis, each of the  $m - 1$  points of  $A$  must be adjacent to  $m - 2$  other points of  $A \cup C_1$ . Using again the fact that no point in  $C_1$  can be adjacent to more than two point in  $A$  it follows that there is at least one point of  $A$  which is not adjacent to any point in  $C_1$  and hence is adjacent to each of the remaining  $m - 2$  points of  $A$ . To avoid contradicting condition (4) it must be that  $A \cup p_{11} \cup p_{12}$  is the point set of a complete graph of  $m + 1$  points. Condition (4) now implies that no point of  $A$  is adjacent to any point of  $C_1$  and that  $C_1 \cup p_{11}$  is the point set of complete graph of 4 points, which completes the proof of the lemma for this case.

An entirely analogous argument proves the lemma when  $n = 3$  and  $m \geq 5$  (see Figure 3).

By the construction of  $DL_s(G)$  we have considered that its validity when  $n = m = 3$  follows from the similar result to that of Shrikhande [8] for line graphs and the remaining cases, when  $n = 1$  or  $2$ , are also easily established.

**Theorem 12.** Let there be given a graph  $G$  satisfying conditions (1), (2), (3) and (4), where  $m \geq n \geq 1$  but  $(m, n) \neq (4, 4)$ . Then  $G$  is isomorphic to  $DL_s(K_{m,n})$ .

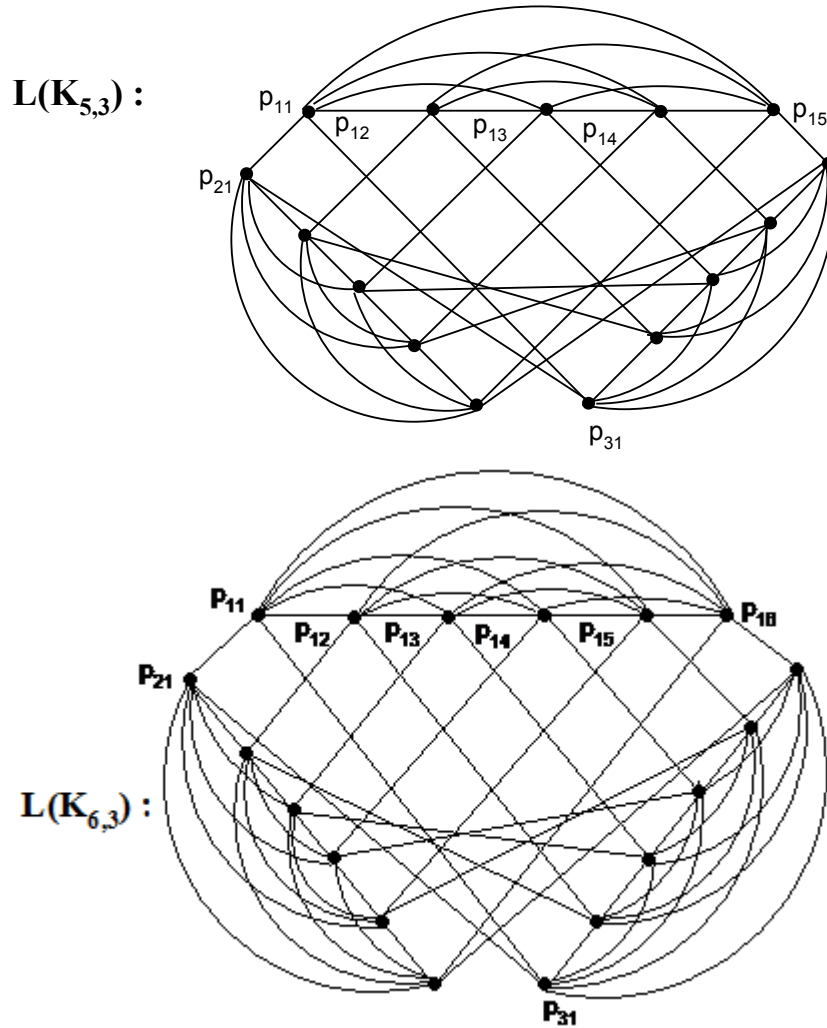


Figure 3.

**Proof.** We prove the theorem by considering the following cases.

**Case 1.** When  $(m, n) \neq (5, 4)$  or  $(4, 3)$ .

Condition (4) implies that there are  $2n \binom{m}{2}$  points  $p$  of same degree in  $G$ , counting multiplicities, for which there exists another point  $q$  of  $G$  such that  $p$  is adjacent to  $q$ , and  $p_1$  and  $q$  are mutually adjacent to  $m - 1$  other points of  $G$ . Since  $2n \binom{m}{2} / mn = m - 1$  it follows that there

exists two points,  $p_{11}$  and  $p_{12}$  say, which satisfy the hypothesis of the lemma. Retaining the notation of the lemma let  $C_2 = \{p_{22}, \dots, p_{n2}\}$  be the set of  $n - 1$  points which are adjacent to  $p_{12}$  but not to  $p_{11}$ ; by symmetry it follows that  $C_2 \cup p_{12}$  is the point set of a complete graph of  $n$  points and no point in  $C_2$  is adjacent to any point in  $A$  whose degree is same as the degree of the considered point in  $C_2$ . By applying condition (3) to the points of  $C_1$  with respect to  $p_{12}$  and to the points of  $C_2$  with respect to  $p_{11}$  we see that each point of  $C_1$  is adjacent to one, and only one, point of  $C_2$  and vice versa. We may assume that the points are labeled in such a way that  $p_{j1}$  is adjacent to  $p_{j2}$ , for  $j = 2, \dots, n$ .

The hypothesis of the lemma are now satisfied with any pair of distinct points,  $p_{1i}$  and  $p_{1j}$ , playing the roles of the points earlier labeled as  $p_{11}$  and  $p_{12}$ . Hence we may assert that for each point  $p_{1j}, j = 1, \dots, m$ , there exists a set of  $n - 1$  points,  $C_j = \{p_{2j}, \dots, p_{nj}\}$ , such that  $C_j \cup p_{1j}$  is the point set of a complete graph of  $n$  points and no point of  $C_j$  is adjacent to any point  $p_{1i}$ , where  $i \neq j$ . Also there are



no points common to  $C_i$  and  $C_j$  and each point of  $C_i$  is adjacent to one and only one point of  $C_j$ , for  $i, j = 1, \dots, m, i \neq j$ . This exhausts the points and lines of  $G$ .

Let  $p_{23}$  be that point of  $C_3$  which is adjacent to  $p_{22}$ . If  $p_{23}$  is not adjacent to  $p_{21}$  suppose that  $p_{33}$  is the point of  $C_3$  which is adjacent to  $p_{21}$  and that  $p_{31}$  is the point of  $C_1$  which is adjacent to  $p_{23}$ . Then the nonadjacent points  $p_{21}$  and  $p_{23}$  are mutually adjacent to the distinct points  $p_{22}, p_{33}, p_{31}$  and  $w$  which contradicts condition (3). Hence  $p_{23}$  and  $p_{21}$  are adjacent. Letting  $p_{2j}$  be that point of  $C_j$  which is adjacent to  $p_{2j-1}$ , for  $j = 4, \dots, m$ , and repeating this argument it can be seen that the points  $p_{2j}, j = 1, \dots, m$ , form the point set of a complete graph of  $m + 1$  points.

Next let  $p_{33}$  be the point of  $C_3$  which is adjacent to  $p_{32}$  and repeat the above argument. Carrying through this procedure  $n - 1$  times it is seen that the  $mn$  points of  $G$  may be labeled  $p_{ij}, i = 1, \dots, n, j = 1, \dots, m$ , in such a way that  $p_{ij}$  is adjacent to  $p_{rs}$  if, and only if,  $i = r$  or  $j = s$ , but not both and one point  $w$  is adjacent to all other  $mn$  points of  $G$ . This shows that  $G$  is isomorphic to  $DL_s(K_{m,n})$  under the given conditions.

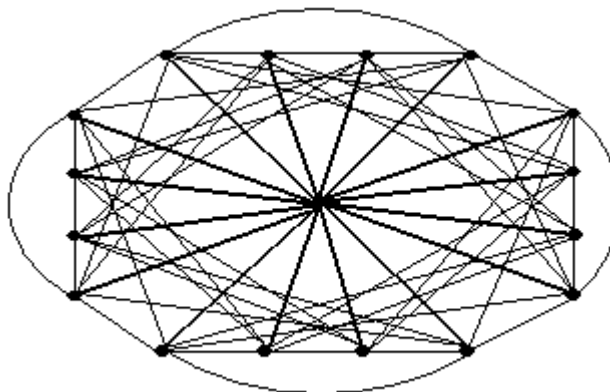
**Case 2.** When  $(m, n) = (5, 4)$  or  $(4, 3)$ .

By Theorem B it is clear that, the graph  $H = L(K_{m,n})$  is a regular graph of degree  $m + n - 2$ . Hence in [3] Hoffman has given the spectral characterization for graph  $H$  to be  $L(K_{m,n})$  when  $(m, n) = (5, 4)$  or  $(4, 3)$ .  $G = DL_s(K_{m,n})$  is obtained from  $H$  by introducing a new point in  $H$  and making this point adjacent to all the points of  $H$ . This is because, by Observation 1,  $H$  is an induced subgraph of  $DL_s(K_{m,n})$  and since  $H$  is a regular graph of degree  $m + n - 2$ , every line of  $K_{m,n}$  is of line degree  $m + n$ . Hence to obtain  $G = DL_s(K_{m,n})$  a unique point  $w$  corresponding to the set of lines of  $K_{m,n}$  is introduced in  $H$  and this point is made adjacent to all the points of  $H$ . Thus degree of every point in  $G$  which corresponds to the points of  $H$  are increased by one, and the degree of  $w$  is equal to the number of points in  $H$ . Hence,  $G$  is a irregular graph. So we felt it difficult to give the spectral characterization for  $G = DL_s(K_{m,n})$ . But by immediate observation of adjacencies of lines and degree of lines of  $G$ , it is clear that  $G = DL_s(K_{m,n})$  satisfies conditions (1) – (4).

**Case 3.** When  $(m, n) = (4, 4)$ .

This theorem does not hold when  $(m, n) = (4, 4)$ . This is because there exists just one counter example (see Fig. 4). This graph contains 17 points, but it is not isomorphic to  $DL_s(K_{4,4})$ . □

**Conclusion:** Studying the above theorem, we conclude that unless  $m = n = 4$ , a graph  $G$  is the line degree splitting graph of  $K_{m,n}$  if and only if it satisfies the conditions (1) – (4) of properties of  $DL_s(K_{m,n})$ .



**Figure 4.**

In the following theorem, we establish the characterization for line degree splitting graphs of complete graphs.

**Theorem 13.** Unless  $p = 8$ , a graph  $G$  is the line degree splitting graph of  $K_p$  if and only if

- 1)  $G$  has  $\binom{p}{2} + 1$  points,
- 2)  $G$  has  $\binom{p}{2}$  points of degree  $2p - 3$  and one point of degree  $\binom{p}{2}$ ,
- 3) Every two nonadjacent points are mutually adjacent to exactly five points,
- 4) Every two adjacent points of same degree are mutually adjacent to exactly  $p - 1$  points and every two adjacent points of different degree are mutually adjacent to exactly  $2(p - 2)$  points.

**Proof.** The proof is similar to the proof of  $DL_s(K_{m,n})$ .

It is evident that  $DL_s(K_p)$  has the above four properties. It is not at all obvious that when  $p = 8$ , there are exactly three exceptional graphs satisfying the above conditions.

By Observation 1,  $L(G)$  is an induced subgraph of  $DL_s(G)$ . Hence, these three exceptional graphs are obtained by considering the exceptional graphs of Theorem A, and introducing a new point in it and making it adjacent to every other point in it. The graphs obtained thus are the exceptional graphs for Theorem 13.

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