On the Line Degree Splitting Graph of a Graph

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Abstract: In this paper, we introduce the concept of the line degree splitting graph of a graph. We obtain some properties of this graph. We find the girth of the line degree splitting graphs. Further, we establish the characterization of graphs whose line degree splitting graphs are eulerian, complete bipartite graphs and complete graphs.

1. Introduction

By a graph, we mean a finite, undirected graph without loops or multiple lines. For a graph G, let $V(G)$, $E(G)$ and $L(G)$ denote its point set, line set and line graph respectively. We refer the terminology of [2]. The degree of a point $v \in V(G)$ is the number of points adjacent to v and is denoted by $\deg(v)$. If u and v are two adjacent points of G, then the line connecting them will be denoted by uv. The degree of a line e in G is denoted by $\deg(e)$, and is defined by $\deg(e) = \deg(u) + \deg(v)-2$ with $e = uv$. A graph G is said to be the *line-regular* if all lines have same degree. The open-neighborhood $N(e_i)$ of a line e_i in $E(G)$ is the set of lines adjacent to e_i. For each line e_i of G, a new point e_i' is taken and the resulting set of points is denoted by $A(G)$.

The *line splitting graph* $L_s(G)$ of a graph G is defined as the graph having point set $E(G) \cup A(G)$ with two points are adjacent if they correspond to adjacent lines of G or one corresponds to an element e_i' of $A(G)$ and the other to an element e_j of $E(G)$ where e_j is in $N(e_i)$. This concept was introduced by Kulli and Biradar in [4]. Miscellaneous properties of line splitting graphs are studied by Basavanagoud and Mathad in [1].

Let $G = (V, E)$ be a graph with $E = S_1 \cup S_2 \cup \ldots \cup S_t \cup T$ where each S_i is a set of lines of G having at least two lines and having the same line degree and $T = E \setminus \cup S_i$. For each set S_i of lines of G, a new point w_i is taken and the resulting set of points w_i is denoted by $E_1(G)$. The *line degree splitting graph* $DL_s(G)$ of a graph G is defined as the graph having point set $E(G) \cup E_1(G)$ with two points are adjacent if they correspond to adjacent lines of G or one correspond to a point w_i of $E_1(G)$ and the other to a line e_j of G and e_j is in S_i. In Fig. 1, graph G and its line degree splitting graph are shown.
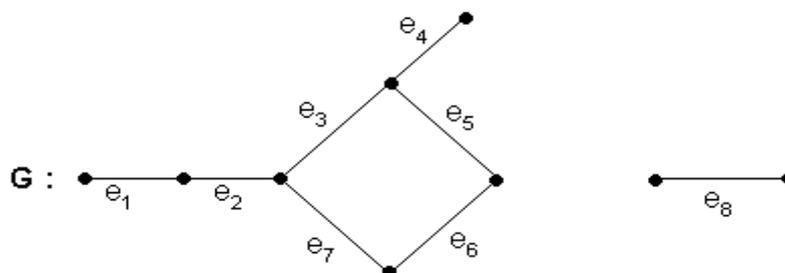
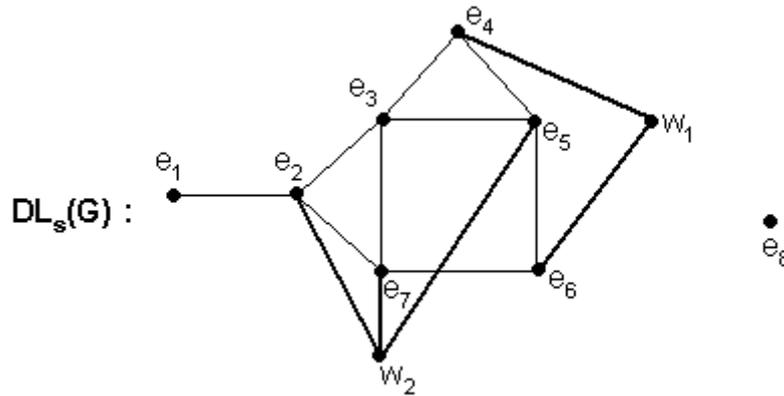Here $S_1 = \{e_4, e_6\}$, $S_2 = \{e_2, e_5, e_7\}$, $T = \{e_1, e_3, e_8\}$

Figure 1.

Notation 1. deg*(e) denoted the degree of a point e in $DL_s(G)$. Clearly deg(e) $\leq$ deg*(e) for all e in E(G).

Notation 2. $\cup S_i$, T and $E_1(G)$ are defined as in the definition of $DL_s(G)$.

Observation 1. For any graph, *L(G)* is an induced subgraph of *$DL_s(G)$*.

Remark 1. For any graph G without isolated points, L(G) is an induced subgraph of $DL_s(G)$. Clearly if G contains at least two lines, then G contains at least two lines of the same degree. Hence $G = K_2$ is the only graph such that $L(G) = DL_s(G)$.

Remark 2. If G is connected then $DL_s(G)$ is connected. But not conversely.

For example, if $G = nK_2$ then $DL_s(G) = K_{1,n}$.

Remark 3. A graph G is a cycle C_n, n $\geq$ 3 if and only if $DL_s(G)$ is a wheel W_{n+1}.

Remark 4. If G is a star $K_{1,n}$, n $\geq$ 2 then $DL_s(G) = K_{n+1}$.

Remark 5. Let e $=$ uv be a line of a graph G.

    i)   If e belongs to $\cup S_i$, then deg*(e) = deg(u) + deg(v) – 1.

    ii)   If e belongs to T, then deg*(e) = deg(u) + deg(v) – 2.

Remark 6. Let w_i be a point belonging to $E_1(G)$. Then deg*(w_i) = | $E_1(G)$ |.

Theorem A [2]. Unless $p = 8$, a graph G is the line graph of K_p if and only if

1) G has $\binom{p}{2}$ points,

2) G is regular of degree $2(p – 2)$,

3)   Every two nonadjacent points are mutually adjacent to exactly four points,

4)   Every two adjacent points are mutually adjacent to exactly $p – 2$ points.

    When $p = 8$, there are exactly three exceptional graphs satisfying the conditions.

Theorem B [2]. Unless $m = n = 4$, a graph G is the line graph of $K_{m,n}$ if and only if

1) G has mn points,

2) G is regular of degree $m + n – 2$,

3)   Every two nonadjacent points are mutually adjacent to exactly two points,

4) Among the adjacent pairs of points, exactly $n\binom{m}{2}$ pairs are mutually adjacent to exactly

$m - 2$ points, and the other $m\binom{n}{2}$ pairs to $n - 2$ points.

There is only one exceptional graph satisfying these conditions. It has 16 points, is not $L(K_{4,4})$, and was found by Shrikhande [7] when he proved Theorem B for the case $m = n$.

2. Results

Theorem 1. Let G be a graph with p points and q lines whose points have degree d_i and let r be the number of lines in T and t be the number of points in $E_1(G)$.

Then $|V(DL_s(G))| = q + t$ and

$$|E(DL_s(G))| = \frac{1}{2}\sum_{i=1}^{p} d_i^2 - r.$$

Proof. $DL_s(G)$ of a graph G is obtained from $L(G)$ by adding t new points which corresponds to the set S_i, $1 \le i \le t$, where S_i is as in the definition of $DL_s(G)$. Therefore, the number of points in $DL_s(G)$ is the sum of points of $L(G)$ and t. Hence $|V(DL_s(G))| = q + t$.

It is known in [2, pp 72], that $L(G)$ has $-q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$ lines. By the definition of $DL_s(G)$, the number of lines in $DL_s(G)$ is the sum of number of lines in $L(G)$ and the lines due to t new points w_1, w_2, …, w_t.

Hence $|E(DL_s(G))| = |E(L(G))| + \sum_{i=1}^{t} \deg^*(w_i)$

$$= -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2 + q - r \tag{1}$$

$$= \frac{1}{2}\sum_{i=1}^{p} d_i^2 - r$$

$\square$

Theorem 2. Let G be a connected graph of size $q > 1$ or a disconnected graph with at least one component having size $q > 1$. Then $DL_s(G)$ contains a cycle.

Proof. By the definition of $DL_s(G)$, the proof is trivial. $\square$

Corollary 2.1. Let G be any graph. Then $DL_s(G)$ is acyclic if and only if the number of lines in every component of G is ≤ 1, that is, every component of G is either K_1 or K_2. Further, $DL_s(G) = K_{1,n}$ when every component of G is K_2.

Theorem 3. Let G be a graph without isolated points. Then $DL_s(G)$ is a cycle if and only if G is P_3 or P_4.

Proof. Suppose $DL_s(G)$ is a cycle. By Remark 1, $L(G)$ is an induced subgraph of $DL_s(G)$. It is clear that $L(G)$ is acyclic and hence G is acyclic. Assume G has at least one component containing a point of degree 3. Then $K_{1,3}$ is a subgraph of G and hence K_3 is subgraph of $L(G)$, a contradiction. Hence, it follows that every component of G is a tree whose points are of degree ≤ 2. It is clear that every component of G is a path. Further if G is a path P_n, $n > 4$ then $L(G)$ is P_{n-1}. $DL_s(G)$ contains $(n - 3)$ number of cycles, a contradiction. Assume every component of G is K_2, then by Corollary 2.1, $DL_s(G) = K_{1,n}$, a contradiction. Hence, every component of G is a path P_n, $2 \le n \le 3$. Now assume G

has at least two such paths as its components. Then by Theorem 2, $DL_s(G)$ contains more than one cycle. Hence, G is connected and is either P_3 or P_4.

   Conversely, suppose $G = P_3$ or P_4. Then it is easy to see that $DL_s(G)$ is C_3 or C_4 respectively. Hence $DL_s(G)$ is a cycle.   □

Theorem 4. For any connected graph G of size q ≥ 2,

$$g(DL_s(G)) = \begin{cases} 4 & \text{if } G = P_4 \\ 3 & \text{otherwise} \end{cases}$$

Proof. Let G be a connected graph of size q ≥ 2. If $G = P_4$, then by Theorem 3, $DL_s(G)$ is C_4 and hence its girth is 4. Otherwise

   (i)   If either K_3 or $K_{1,3}$ is a subgraph of G, then $DL_s(G)$ contains a triangle.

   (ii) If G is any cycle, then by Remark 3, $DL_s(G)$ is a wheel.

   (iii) If G is a path P_n, n ≥ 5, then $DL_s(G)$ contains (n − 4) number of triangles. And if $G = P_3$, then by Theorem 3, $DL_s(G)$ is K_3.

Hence (i), (ii) and (iii) implies that $g(DL_s(G)) = 3$ for all G other than P_4.   □

Theorem 5. $DL_s(G)$ is bipartite if and only if every component of G is either K_2 or P_4.

Proof. Suppose $DL_s(G)$ is bipartite. Assume G has at least one point of degree ≥ 3. Then $L(G)$ contains K_3 as a subgraph and by Remark 1, $DL_s(G)$ contains an odd cycle, a contradiction. Hence every point of G is of degree ≤ 2. Clearly every component of G is either a cycle or a path. We consider the following cases.

Case 1. Assume one of the component of G is a cycle. By Remark 3, $DL_s(G)$ has a wheel as its subgraph, a contradiction.

Case 2. Suppose one of the component of G is a path P_n, n ≥ 5. Then $L(G)$ has P_{n-1} as a component and $DL_s(G)$ has at least (n − 4) number of triangles, a contradiction. Hence, every component of G is a path P_n, n ≤ 4. Assume G has at least one component as P_3. Then by Theorem 3, $DL_s(G)$ contains a triangle, a contradiction. Hence, every component of G is either P_4 or K_2.
Conversely, suppose G is a graph such that each of its component is either K_2 or P_4. We consider the following cases.

Case 1. Assume every component of G is K_2. Then by Corollary 2.1, $DL_s(G)$ is $K_{1,n}$, which is a bipartite graph.

Case 2. Assume every component of G is P_4. Suppose G has exactly one component. Then by Theorem 2, $DL_s(G)$ is C_4, which is a bipartite graph. Now suppose G has more than one component. Then $L(G)$ contains more than one components each of which is $K_{1,2}$. We construct $DL_s(G)$ as follows. Since each P_4 contains one line of line degree 4 and the other line of line degree 3. Hence $DL_s(G)$ contains two points w_1 and w_2 corresponding to the set S_1 of lines of line degree 4 and set S_2 of lines of line degree 3 respectively. Now partition the points of $DL_s(G)$ such that $V_1(DL_s(G))$ contains w_1 and all those points of $L(G)$ of degree 1 and $V_2(DL_s(G))$ contains w_2 and all those points of $L(G)$ of degree 2. And w_1 is adjacent to all points of $V_2(DL_s(G))$ except w_2 and w_2 is adjacent to all points of $V_1(DL_s(G))$ except w_1. Clearly the resulting graph $DL_s(G)$ is bipartite.

Case 3. Assume G has at least two components one of which is P_4 and the other is K_2. Then line graph of each P_4 is $K_{1,2}$ and each K_2 is K_1. It is easy to see that line degree of lines of P_4 is different from the line degree of lines of K_2. Hence no point of $E_1(G)$ of $DL_s(G)$ which corresponds to the set of lines of same line degree is adjacent to the points of both $K_{1,2}$ and K_1 in $L(G)$. By Case 1 and Case 2 of the converse part of this theorem, $DL_s(G)$ contains two components each of which is bipartite. Hence $DL_s(G)$ is bipartite.   □

Theorem 6. For any graph G, $\omega(DL_s(G)) \leq \omega(G)$, where $\omega(G)$ denotes the number of components of G.

Proof. Let $V(L(G)) = \{e_1, e_2, \ldots, e_q\}$ and

$$V(DL_s(G)) \setminus V(L(G)) = \{w_1, w_2, \ldots, w_t\}.$$

Case 1. Let G be connected. Then by Remark 2, $DL_s(G)$ is connected. Therefore, $\omega(G) = 1 = \omega(DL_s(G))$.

Case 2. Let G be disconnected. Then $L(G)$ is disconnected.

Let $G_1, G_2, \ldots, G_k, k \geq 2$ be the components of G. If G_i and G_j have lines of same degree for some i, j. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ be such that $\deg(e_1) = \deg(e_2)$. By the definition of $DL_s(G)$, $DL_s(G)$ contains a new point w such that w is adjacent to e_1 and e_2 of $L(G)$ in $DL_s(G)$. Hence, $\omega(DL_s(G)) \leq k - 1 < k = \omega(G)$.

If there is no pair i and j such that G_i and G_j have lines of same degree, then $\omega(DL_s(G)) = k = \omega(G)$. Hence $\omega(DL_s(G)) \leq \omega(G)$.   □

Theorem 7. The $DL_s(G)$ of a nontrivial graph G is complete if and only if G is either K_3 or $K_{1,n}$, $n \geq 1$.

Proof. Suppose $DL_s(G)$ is complete graph of order ≥ 2. Then every pair of points of $DL_s(G)$ are adjacent. Thus all the lines of G are mutually adjacent and hence have same line degree. Clearly there exists exactly one new point in $DL_s(G)$ which is adjacent to all the points of $L(G)$. Suppose $G \neq K_3$ and G has at least one cycle. Then it is easy to observe that there exist at least two nonadjacent lines. It implies that $L(G)$ is not complete. By Remark 1, $DL_s(G)$ is not complete, which is a contradiction. Suppose next $G \neq K_{1,n}$, $n \geq 1$ and G is a tree with at least two nonadjacent lines in G. Then again $L(G)$ is not complete. By Remark 1, $DL_s(G)$ is not complete a contradiction.

Conversely, suppose $G = K_3$. It has three mutually adjacent lines which are of same line degree. Then these three lines are the points of $DL_s(G)$ together with a new point corresponding to the lines of same line degree of G in $DL_s(G)$. These four points are mutually adjacent points in $DL_s(G)$. Thus $DL_s(G)$ is K_4. Now suppose G is $K_{1,n}$, $n \geq 2$. By Remark 4, $DL_s(G)$ is K_{n+1}. Thus $DL_s(G)$ is complete.   □

Theorem 8.   If G ($\neq K_2$ and K_3) is regular graph with p points and q lines, then $DL_s(G)$ is not regular.

Proof. If G is regular of degree k, then $L(G)$ is also regular of degree $2(k - 1)$ with q points and $\dfrac{pk^2}{2} - q$ lines. Let $e \in V(L(G))$. Then $\deg(e) = 2(k - 1) \neq q - 1$.

Clearly $2(k - 1) < q - 1$, except for $G = K_2$ and K_3 (2)

Let $V(DL_s(G)) \setminus V(L(G)) = \{w\}$. Since w is adjacent to all the lines of G in $L(G)$, $\deg^*(w) = q > 2(k - 1) + 1 = 2k - 1$. (by (2)).

But $\deg^*(e) = \deg(e) + 1 = 2(k - 1) + 1 = 2k - 1$. Hence $DL_s(G)$ is not regular.   □

Corollary 8.1. If G is k-regular graph, then $DL_s(G) = L(G) + K_1$.

Corollary 8.2. $DL_s(G)$ is line-regular if and only if G is $K_{1,n}$ or K_3.

Theorem 9. Let G be any graph. A necessary and sufficient condition for $DL_s(G)$ to be eulerian is that each of the following holds:

> 1)   Each even degree line of G occurs exactly once.
>
> 2)   Each odd degree line of G occurs even number of times.

Proof. Suppose $DL_s(G)$ is eulerian. Let x be a point of $DL_s(G)$. Then x is a line of G or a point of $E_1(G)$. Suppose x = uv is a line of G. Remark 5 implies that if x belongs to T, then $\deg^*(x) = \deg(u) + \deg(v) - 2$. If x belongs to $\cup S_i$, then $\deg^*(x) = \deg(u) + \deg(v) - 1$. Suppose x belongs to T. Then both deg u and deg v must be even or odd. This implies that x is adjacent to even number of lines of G.

Hence all such x which belongs to T are even degree lines. Thus (1) holds. Suppose next x belongs to $\cup S_i$. Then either deg(u) or deg(v) is odd. This implies that x is adjacent to odd number of lines. Hence all such x are odd degree lines and since they belong to $\cup S_i$, the point w_i belonging to $E_1(G)$ in $DL_s(G)$ contributes one to the degree of x in $DL_s(G)$. Thus degree of x in $DL_s(G)$ is even. If x is a point of $E_1(G)$, then Remark 6 implies that deg*(x) = | $E_1(G)$ |. Since deg*(x) is even, | $E_1(G)$ | is even. Thus, each S_i contains even number of lines of odd degree. Hence (2) holds.

Conversely, assume the condition (1) and (2) on G hold. Suppose x is a point of $DL_s(G)$. Then x is a line of G or x belongs to $E_1(G)$. Assume x is a line of G. If x belongs to T, then by (1), deg*(x) is even. And if x belongs $\cup S_i$, then by (2) and definition of $DL_s(G)$, deg*(x) is even. Now if x belongs to $E_1(G)$, then by condition (2), each S_i contains even number of lines of odd degree and hence deg*(x) is even. Hence $DL_s(G)$ is eulerian.   □

Theorem 10. Let G be a nontrivial graph. If $DL_s(G)$ is eulerian, then G is not eulerian.

Proof. Suppose $DL_s(G)$ is eulerian. Theorem 2 implies that there exists at least one component in G of size q > 1. Then by Remark 1, G has at least two lines x and y of same line degree. Let x' and y' be the points of $DL_s(G)$ corresponding to the lines x and y of G respectively. Since $DL_s(G)$ is eulerian deg*(x') and deg*(y') are even. Then by Remark 5, one of the end points of both the lines x and y in G is odd. Thus, G is not eulerian.   □

Theorem 11. If G is eulerian, then $DL_s(G)$ is hamiltonian.

The converse of the Theorem 11 is need not be true. This is easily seen in Fig. 2.
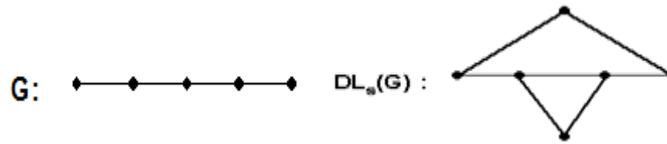
Figure 2.

In the next results we present characterization of line degree splitting graph of complete bipartite graphs and complete graphs. The line degree splitting graph of complete bipartite graphs and complete graphs are characterized by immediate observations involving adjacencies and degree of lines in $K_{m,n}$ and K_p.

In the following theorem, we establish the characterization for line degree splitting graphs of complete bipartite graphs.

Properties of $DL_s(K_{m,n})$: If we suppose that $m \geq n \geq 1$ it can be seen that $DL_s(K_{m,n})$ has the following four properties.

1) The graph has $mn + 1$ points,
2) Of the $mn + 1$ points, mn points are of degree $m + n - 1$ and one point is of degree mn,
3) Every two nonadjacent points are mutually adjacent to three points,
4) $n\binom{m}{2}$ of the pairs of adjacent points of same degree are mutually adjacent to $m - 1$ other points, $m\binom{n}{2}$ of the remaining pairs of adjacent points of same degree are mutually adjacent to $n - 1$ points and every two adjacent points of different degree are mutually adjacent to exactly $m + n - 2$ points.

The object of this note is to show that if any graph satisfies the four conditions (1) – (4), then it is isomorphic to $DL_s(K_{m,n})$ except possibly when $(m, n) = (4, 4)$. Here we have tried to give a generalization for line degree splitting graphs, which is similar to the characterization given for line graphs which was studied by Moon [5] and Hoffman [3].

Before proving that if any graph satisfies the conditions (1) – (4) then it is isomorphic to $DL_s(K_{m,n})$ except for $(m, n) = (4, 4)$, we state and prove the following lemma.

Lemma 1. Let there be given a graph G satisfying conditions (1), (2), (3) and (4) where $m \geq n \geq 1$, but $(m, n) \neq (4, 4), (5, 4)$ or $(4, 3)$. Let p_{11} and p_{12} be two adjacent points of same degree of G which are mutually adjacent to each of the $m - 1$ points in $A = \{p_{13}, \ldots, p_{1m}, w\}$. Let $C_1 = \{p_{21}, \ldots, p_{n1}\}$ be the set of $n - 1$ points which are adjacent to p_{11} but not to p_{12}. Furthermore let there be at least $m - 1$ points in $A \cup C_1$ such that each of these points and p_{11} are mutually adjacent to $m - 1$ other points. Then $A \cup p_{11} \cup p_{12}$ and $C_1 \cup p_{11}$ are the point set of complete graphs of $m + 1$ and n points respectively, and no point of A is adjacent to any of C_1 whose degree is same as that of the point of A.

Proof. We consider first the case in which $m \geq n \geq 5$. No point in C_1 can be adjacent to more than two points in A without violating condition (3) with respect to the point p_{12}. Therefore, each point in C_1 is adjacent to at least $n - 3$ of the remaining $n - 2$ points in C_1 in order to satisfy condition (4) with respect to the point p_{11}. Then, if there exists two nonadjacent points in C_1 they must be mutually adjacent to the remaining $n - 3$ points in C_1 as well as to p_{11} and w. This contradicts condition (3) since $n - 2 \geq 3$ in the case being considered. Hence, every point in C_1 is adjacent to every other point in C_1. If some point in C_1, p_{21} say, is adjacent to some point in A whose degree is same as p_{21} then there are n points adjacent to both p_{21} and p_{11}. Thus n must equal $m - 1$, by (4), or $m = n + 1$.

If $m = n + 1$ suppose that some point in A, p_{13} say, is adjacent to some points of C_1. It is easily seen that p_{13} cannot be adjacent to more than one point of C_1 but not to all points of C_1 without violating condition (3). But if p_{13} is adjacent to all points of C_1 then the number of points which are adjacent to both p_{11} and p_{13} is, including p_{12}, at least n which contradicts condition (4). Hence, p_{13} can be adjacent to at most one point in C_1. From condition (4) it follows that there is at least one other point in A, p_{14} say, which is not adjacent to p_{13}. But p_{13} and p_{14} are each adjacent to at least $m - 4$ of the remaining $m - 3$ points of A and if $m > 6$ there will be at least two of these points which is adjacent to both p_{13} and p_{14}. This, however, contradicts condition (3) since p_{13} and p_{14} are both adjacent to p_{11} and p_{12}.

The only alternative remaining to be treated, under the assumption that $m = n + 1$ and that some lines join points in A to points in C_1, is when $m = n + 1 = 6$. In this case it is not difficult to see that the only configuration which can satisfy condition (4) without implying a contradiction of the type just described is one in which each point of A except w is adjacent to p_{15}, say, and p_{14} is adjacent to p_{16}. Suppose that p_{21} and p_{31} are the different points in C_1 which are adjacent to p_{13} and p_{15}, respectively. Then p_{21} and p_{15} are not adjacent to each other but are mutually adjacent to w, p_{31}, p_{13} and p_{11}, contradicting condition (3). Hence, no point of A is adjacent to any point of C_1 under the given assumptions. This and the fourth sentence of the hypothesis of the lemma implies that each point in A is adjacent to every other point in A which suffices to complete the proof of the lemma when $m \geq n \geq 5$.

Next consider the case in which $n = 4$ and $m \geq 6$. No point in C_1 can be adjacent to $m - 1$ other points of $A \cup C_1$ by an earlier remark and the fact that $m - 1 \geq 5$. Hence, from the hypothesis, each of the $m - 1$ points of A must be adjacent to $m - 2$ other points of $A \cup C_1$. Using again the fact that no point in C_1 can be adjacent to more than two point in A it follows that there is at least one point of A which is not adjacent to any point in C_1 and hence is adjacent to each of the remaining $m - 2$ points of A. To avoid contradicting condition (4) it must be that $A \cup p_{11} \cup p_{12}$ is the point set of a complete graph of $m + 1$ points. Condition (4) now implies that no point of A is adjacent to any point of C_1 and that $C_1 \cup p_{11}$ is the point set of complete graph of 4 points, which completes the proof of the lemma for this case.

An entirely analogous argument proves the lemma when $n = 3$ and $m \geq 5$ (see Figure 3).

By the construction of $DL_s(G)$ we have considered that its validity when $n = m = 3$ follows from the similar result to that of Shrikhande [8] for line graphs and the remaining cases, when $n = 1$ or 2, are also easily established.

Theorem 12. Let there be given a graph G satisfying conditions (1), (2), (3) and (4), where $m \geq n \geq 1$ but $(m, n) \neq (4, 4)$. Then G is isomorphic to $DL_s(K_{m,n})$.
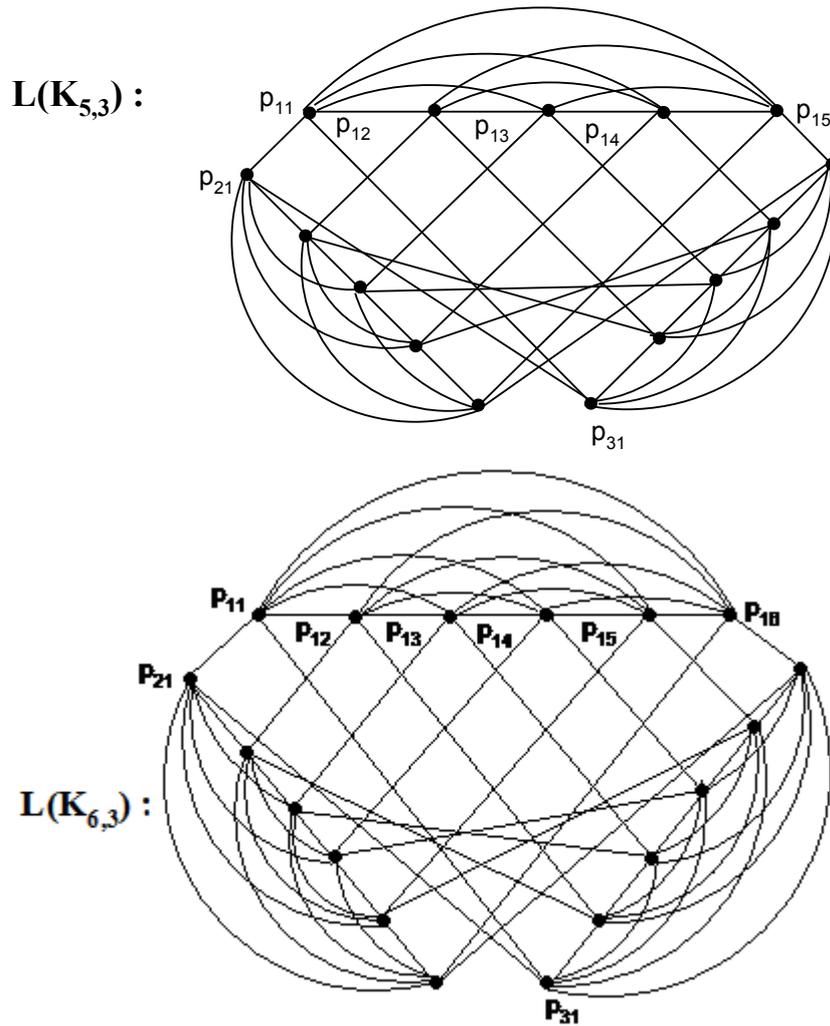

Figure 3.

Proof. We prove the theorem by considering the following cases.

Case 1. When $(m, n) \neq (5, 4)$ or $(4, 3)$.

Condition (4) implies that there are $2n\binom{m}{2}$ points p of same degree in G, counting multiplicities, for which there exists another point q of G such that p is adjacent to q, and p_1 and q are mutually adjacent to $m - 1$ other points of G. Since $2n\binom{m}{2} \big/ mn = m - 1$ it follows that there exists two points, p_{11} and p_{12} say, which satisfy the hypothesis of the lemma. Retaining the notation of the lemma let $C_2 = \{p_{22}, \ldots, p_{n2}\}$ be the set of $n - 1$ points which are adjacent to p_{12} but not to p_{11}; by symmetry it follows that $C_2 \cup p_{12}$ is the point set of a complete graph of n points and no point in C_2 is adjacent to any point in A whose degree is same as the degree of the considered point in C_2. By applying condition (3) to the points of C_1 with respect to p_{12} and to the points of C_2 with respect to p_{11} we see that each point of C_1 is adjacent to one, and only one, point of C_2 and vice versa. We may assume that the points are labeled in such a way that p_{j1} is adjacent to p_{j2}, for $j = 2, \ldots, n$.

The hypothesis of the lemma are now satisfied with any pair of distinct points, p_{1i} and p_{1j}, playing the roles of the points earlier labeled as p_{11} and p_{12}. Hence we may assert that for each point $p_{1j}, j = 1, \ldots, m$, there exists a set of $n - 1$ points, $C_j = \{p_{2j}, \ldots, p_{nj}\}$, such that $C_j \cup p_{1j}$ is the point set of a complete graph of n points and no point of C_j is adjacent to any point p_{1i}, where $i \neq j$. Also there are

no points common to C_i and C_j and each point of C_i is adjacent to one and only one point of C_j, for i, $j = 1, \ldots, m, i \neq j$. This exhausts the points and lines of G.

Let p_{23} be that point of C_3 which is adjacent to p_{22}. If p_{23} is not adjacent to p_{21} suppose that p_{33} is the point of C_3 which is adjacent to p_{21} and that p_{31} is the point of C_1 which is adjacent to p_{23}. Then the nonadjacent points p_{21} and p_{23} are mutually adjacent to the distinct points p_{22}, p_{33}, p_{31} and w which contradicts condition (3). Hence p_{23} and p_{21} are adjacent. Letting p_{2j} be that point of C_j which is adjacent to p_{2j-1}, for $j = 4, \ldots, m$, and repeating this argument it can be seen that the points $p_{2j}, j = 1, \ldots, m$, form the point set of a complete graph of $m + 1$ points.

Next let p_{33} be the point of C_3 which is adjacent to p_{32} and repeat the above argument. Carrying through this procedure $n - 1$ times it is seen that the mn points of G may be labeled p_{ij}, $i = 1, \ldots, n, j = 1, \ldots, m$, in such a way that p_{ij} is adjacent to p_{rs} if, and only if, $i = r$ or $j = s$, but not both and one point w is adjacent to all other mn points of G. This shows that G is isomorphic to $DL_s(K_{m,n})$ under the given conditions.

Case 2. When $(m, n) = (5, 4)$ or $(4, 3)$.

By Theorem B it is clear that, the graph $H = L(K_{m,n})$ is a regular graph of degree $m + n - 2$. Hence in [3] Hoffman has given the spectral characterization for graph H to be $L(K_{m,n})$ when $(m, n) = (5, 4)$ or $(4, 3)$. $G = DL_s(K_{m,n})$ is obtained from H by introducing a new point in H and making this point adjacent to all the points of H. This is because, by Observation 1, H is an induced subgraph of $DL_s(K_{m,n})$ and since H is a regular graph of degree $m + n - 2$, every line of $K_{m,n}$ is of line degree $m + n$. Hence to obtain $G = DL_s(K_{m,n})$ a unique point w corresponding to the set of lines of $K_{m,n}$ is introduced in H and this point is made adjacent to all the points of H. Thus degree of every point in G which corresponds to the points of H are increased by one, and the degree of w is equal to the number of points in H. Hence, G is a irregular graph. So we felt it difficult to give the spectral characterization for $G = DL_s(K_{m,n})$. But by immediate observation of adjacencies of lines and degree of lines of G, it is clear that $G = DL_s(K_{m,n})$ satisfies conditions $(1) - (4)$.

Case 3. When $(m, n) = (4, 4)$.

This theorem does not hold when $(m, n) = (4, 4)$. This is because there exists just one counter example (see Fig. 4). This graph contains 17 points, but it is not isomorphic to $DL_s(K_{4,4})$. $\square$

Conclusion: Studying the above theorem, we conclude that unless $m = n = 4$, a graph G is the line degree splitting graph of $K_{m,n}$ if and only if it satisfies the conditions $(1) - (4)$ of properties of $DL_s(K_{m,n})$.
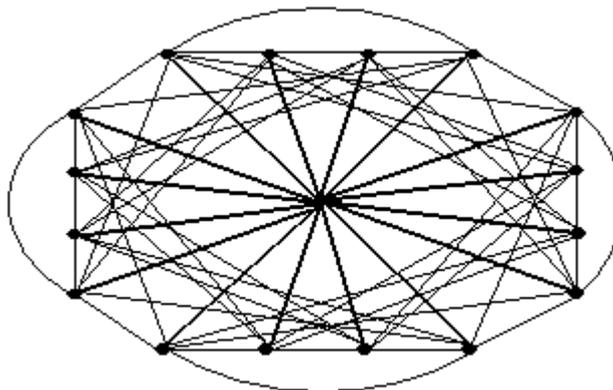
Figure 4.

In the following theorem, we establish the characterization for line degree splitting graphs of complete graphs.

Theorem 13. Unless $p = 8$, a graph G is the line degree splitting graph of K_p if and only if

1) G has $\binom{p}{2} + 1$ points,

2) G has $\binom{p}{2}$ points of degree $2p - 3$ and one point of degree $\binom{p}{2}$,

3) Every two nonadjacent points are mutually adjacent to exactly five points,

4) Every two adjacent points of same degree are mutually adjacent to exactly $p - 1$ points and every two adjacent points of different degree are mutually adjacent to exactly $2(p - 2)$ points.

Proof. The proof is similar to the proof of $DL_s(K_{m,n})$.

It is evident that $DL_s(K_p)$ has the above four properties. It is not at all obvious that when $p = 8$, there are exactly three exceptional graphs satisfying the above conditions.

By Observation 1, $L(G)$ is an induced subgraph of $DL_s(G)$. Hence, these three exceptional graphs are obtained by considering the exceptional graphs of Theorem A, and introducing a new point in it and making it adjacent to every other point in it. The graphs obtained thus are the exceptional graphs for Theorem 13.

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