Qlick Graphs with Crossing Number One

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Abstract: In this paper, we deduce a necessary and sufficient condition for graphs whose qlick graphs have crossing number one. We also obtain a necessary and sufficient condition for qlick graphs to have crossing number one in terms of forbidden subgraphs.

1 Introduction

A graph is *planar* if it can be drawn in the plane or on the sphere in such a way that no two of its edges intersect. The *crossing number* $cr(G)$ of a graph G is the least number of intersections of pairs of edges in any embedding of G in the plane. Obviously, G is planar if and only if $cr(G) = 0$. It is implicit that the edges in a drawing are Jordan arcs (hence, non-selfintersecting), and it is easy to see that a drawing with the minimum number of crossing (an optimal drawing) must be a *good* drawing, that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing.

All graphs considered here are finite undirected and without loops or multiple edges. We refer to [6] for unexplained terminology and notations. For a graph G , let $V(G)$, $E(G)$ and $U(G)$ denote its vertex set, edge set and block set respectively. If $B = \{u_1, u_2, \dots, u_r; r \geq 2\}$ is a block of a graph G , then we say that the vertex u_1 and a block B are incident with each other, as are u_2 and B and so on. If $B = \{e_1, e_2, \dots, e_s; s \geq 1\}$ is a block of G , then we say that edge e_1 and block B are incident with each other, as are e_2 and B and so on. If two distinct blocks B_1 and B_2 are incident with common cut-vertex, then they are adjacent blocks. The edges and blocks of a graph are called its members.

The *qlick graph* $Q(G)$ of a graph G is the graph whose set of vertices is the union of the set of edges $E(G)$ and set of blocks $U(G)$ of G and in which two vertices are adjacent if and only if the corresponding edges are adjacent or corresponding blocks are adjacent or an edge and a block are incident. The *plick graph* $P(G)$ of G is the graph whose vertex set is the set of edges $E(G)$ and set of blocks $U(G)$ of G and two vertices are adjacent if and only if the corresponding edges are adjacent or an edge and a block are incident. These concepts were introduced by Kulli in [10] and were studied, for example, in [1, 2, 11, 12].

In [8], Kulli introduced the concept of total-block graph of a graph. In [3], Basavanagoud et al. generalized the concept of total-block graph and introduced the block-transformation graphs and defined as follows: Let $G = (V, E)$ be a graph with block set $U(G)$, and let α, β, γ be three variables having values 0 or 1. The *block-transformation graph* $G^{\alpha\beta\gamma}$ is the graph having $V(G) \cup U(G)$ as the vertex set. For any two vertices x and $y \in V(G) \cup U(G)$ we define α, β, γ as follows:

(i) Suppose x, y are in $V(G)$. $\alpha=1$ if x and y are adjacent in G . $\alpha=0$ if x and y are nonadjacent in G .

(ii) Suppose x, y are in $U(G)$. $\beta=1$ if x and y are adjacent in G . $\beta=0$ if x and y are nonadjacent in G .

(iii) $x \in V(G)$ and $y \in U(G)$. $\gamma=1$ if x and y are incident with each other in G . $\gamma=0$ if x and y are nonincident with each other in G .

Thus, we obtain eight kinds of block -transformation graphs, in which G^{111} is the total-block graph of G [8], and G^{000} is its complement. Also, G^{001} , G^{010} and G^{011} are the complements of G^{110} , G^{101} and G^{100} respectively. In [4], Basavanagoud et al. studied a criterion for (non-) planarity of the block-transformation graph $G^{\alpha\beta\gamma}$ when $\alpha\beta\gamma = 101$.

The *line graph* $L(G)$ of a graph G is the graph whose vertices can be put in one to one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The *block graph* $B(G)$ of a graph G is the graph whose vertices are the blocks of G and in which two vertices are adjacent whenever corresponding blocks have a cut-vertex in common. For other definitions concerning crossing numbers, line graphs or block graphs see [6].

The following will be useful in the proof of our results.

Remark 1.1 [11]. For any graph G , $L(G)$ and $B(G)$ are line disjoint subgraphs of $Q(G)$.

Theorem 1.1 [13]. The line graph of a planar graph G is planar if and only if $\Delta(G) \leq 4$ and every vertex of degree 4 is a cut vertex.

Theorem 1.2 [7]. Let G be a non planar graph. Then $cr(L(G))=1$ if and only if the following conditions hold:

- (1) $cr(G)=1$;
- (2) $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex of G ;
- (3) There exists a drawing of G in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.

Theorem 1.3 [9]. The line graph of a planar graph G has crossing number one if and only if (1) or (2) holds:

- (1) $\Delta(G) = 4$ and there is unique non-cut-vertex of degree 4;
- (2) $\Delta(G) = 5$, every vertex of degree 4 is a cut-vertex, there is a unique vertex of degree 5 and it has at most 3 incident edges in any block.

Theorem 1.4 [11]. The *qlick graph* $Q(G)$ of a graph G is planar if and only if G satisfies the following conditions:

- (1) $\Delta(G) \leq 3$, and
- (2) if $deg(v)=3$ for every vertex v of G , then v is a cut-vertex.

Theorem 1.5 [5] A graph has a planar line graph if and only if it has no subgraph homeomorphic to $K_{3,3}$, $K_{1,5}$, $P_4 + K_1$ or $K_2 + \overline{K_3}$.

2 Results

The following theorem supports the main theorem.

Theorem 2.1 Let x be any edge of K_4 . If G is homeomorphic to $K_4 - x$, then $cr(Q(G))=1$.

Proof. We prove the theorem first for $K_4 - x$. One can see that the graph $Q(K_4 - x)$ has 6 vertices and 13 edges. But a planar graph with 6 vertices has at most 12 edges. This shows that $Q(K_4 - x)$ has

crossing number at least 1. Figure 1, being drawing of $Q(K_4 - x)$ concludes that $cr(Q(K_4 - x)) = 1$. Suppose now G is the graph as in the statement. Referring to Figure 1, it is immediate to see that $cr(Q(K_4 - x)) = 1$.

Figure 1

The following theorem gives a characterization of quick graphs with crossing number 1.

Theorem 2.2 *A graph G has a quick graph with crossing number 1 if and only if G is planar and one of the following holds:*

- (1) $\Delta(G) = 3$, G has exactly two adjacent non-cut-vertices of degree 3 and every other vertex of degree 3 is a cut-vertex.
- (2) $\Delta(G) = 4$, every vertex of degree 3 or 4 is a cut-vertex and there is a unique vertex v of degree 4 and v lies on 2 or 3 blocks.

Proof. Suppose $Q(G)$ has crossing number 1. Then by Remark 1.1, and by Theorem 1.2, G is planar. By Theorem 1.4, $\Delta(G) \leq 4$, then there exists a non-cut-vertex of degree 3 or 4 and more than one vertex of degree 4 and v lies on 2 or 3 blocks.

Suppose $\Delta(G) \leq 5$. Then $K_{1,5}$ is a subgraph of G . Clearly $L(K_{1,5}) = K_5$ and $B(K_{1,5}) = K_5$. It is known that $cr(K_5) = 1$. By remark 1.1, $2K_5$'s are line disjoint subgraphs of $Q(G)$ and hence $cr(Q(G)) > 1$, a contradiction. This implies that $\Delta(G) \leq 4$. If $\Delta(G) \leq 2$, then by Theorem 1.4, $Q(G)$ is planar, again a contradiction. Thus $\Delta(G) = 3$ or 4.

We now consider the following cases:

Case 1. Suppose $\Delta(G) = 3$. Then by Theorem 1.4 and since $cr(Q(G)) = 1$, G has a non-cut-vertex of degree 3. Clearly G contains a subgraph homeomorphic to $K_4 - x$, so that there exist at least two non-cut-vertex degree 3. More precisely there is an even number, say $2n$, of non-cut-vertex degree 3. Now suppose G has at least two diagonal edges. Then there are two subcases to consider depending on whether 2 diagonal edges exist in one cycle or in two different edge disjoint cycles.

Subcase 1.1. If two diagonal edges exist in one cycle of G . Then G has a subgraph homeomorphic to K_4 . The graph $Q(K_4)$ has 7 vertices and 18 edges. It is known that a planar graph with 7 vertices has at most 15 edges. This shows that $Q(K_4)$ must have crossing number exceeding 1 and hence $Q(G)$ has crossing number greater than 1, a contradiction.

Subcase 1.2. If two diagonal edges exist in two different edges-disjoint cycles of G . Then by Theorem 2.1, we see that for every subgraph of G homeomorphic to $K_4 - x$, there corresponds at least one crossing of G . Hence $Q(G)$ has at least 2 crossings, a contradiction. Hence G has exactly two non-cut-vertices of degree 3 and every other vertex of degree 3 is a cut-vertex.

Suppose G has two non-cut-vertices of degree 3 and they are not adjacent. Then G contains a subgraph homeomorphic to $K_{2,3}$. On drawing $Q(K_{2,3})$ in a plane one can see that $cr(Q(K_{2,3})) = 2$. Since $Q(K_{2,3})$ is a subgraph of $Q(G)$, $Q(G)$ has crossing number exceeding 1, a contradiction. (see Figure 2)

Therefore, we conclude that G contains exactly two non-cut-vertices of degree 3 and these are adjacent. This proves(1).

Figure 2

Case 2. Suppose $\Delta(G) = 4$. Then every vertex of degree 3 is a cut-vertex, otherwise, by case 1, $Q(G)$ has crossing number exceeding 1.

Next we show that every vertex of degree 4 is a cut-vertex. On the contrary suppose that G has non-cut-vertex v of degree 4. Then by Theorem 1.3, $cr(L(G)) \geq 1$. The vertex u_1 in $Q(G)$ corresponding to the block which contains a non-cut-vertex of degree 4 is adjacent to every vertex of $L(G)$. We obtain the drawing of $Q(G)$ with 3 crossings.

Assume v lies on at least 4 blocks. The vertex v and four of its neighboring vertices lie in a subgraph of G isomorphic to $K_{1,4}$. On drawing $Q(K_{1,4})$ in a plane, one can see that $cr(Q(K_{1,4})) > 1$. Since $Q(K_{1,4})$ is a subgraph of $Q(G)$, $Q(G)$ has crossing number exceeding 1, a contradiction.

Figure 3

Suppose G has at least two cut-vertices u_1 and u_2 which are of degree 4. Then the cut-vertices u_1 and u_2 together with their corresponding incident four edges form two subgraphs as K_5 in $Q(G)$. Hence $cr(Q(G)) > 1$, a contradiction.

Conversely, suppose G is a planar graph satisfying (1) or (2). Then by Theorem 1.4, $Q(G)$ has crossing number at least 1. We now show that its crossing number is at most 1. First assume (1) holds. Then G has exactly one subgraph H_1 , homeomorphic to $K_4 - x$ which contains exactly two adjacent non-cut-vertices of degree 3. By Theorem 2.1, $cr(Q(H_1)) = 1$. By Theorem 1.4, all other remaining blocks of G have a planar qlick graphs. Hence $Q(G)$ has crossing number 1.

Now assume (2) holds, and let v be a unique cut-vertex of degree 4 and it lies on 2 or 3 blocks. We have the following cases.

Case 1. Suppose v lies on 2 blocks. Then each block of G is a cycle. On embedding of $Q(G)$ in a plane which gives $K_{3,3}$ as a subgraph in $Q(G)$ as shown in Fig 4. Clearly $Q(G)$ has crossing number 1.

Figure 4.

Case 2. Suppose v lies on 3 blocks. Then one of these blocks is a cycle and each of the remaining blocks is a K_2 . The qlick graph $Q(G)$ can be constructed in the plane which gives exactly one subgraph K_5 (see Figure 5). Hence $cr(Q(G))=1$.

Figure 5.

3 Forbidden Subgraphs

By using Theorem 2.2, we now characterize graphs whose qlick graphs have crossing number one in terms of forbidden subgraphs.

Theorem 3.1 *The qlick graph $Q(G)$ of a connected graph G has crossing number one if and only if G has no induced subgraph $K_{1,4}$ or a subgraph homeomorphic from any one of the graphs Figure 6.*

Figure 6

Proof. Suppose G is a connected plane graph such that $Q(G)$ has crossing number one. By Theorem 2.2, we have (1) $\Delta(G) = 3$, G has exactly two adjacent non-cut-vertices of degree 3 and every other vertex of degree 3 is a cut-vertex or (2) $\Delta(G) = 4$, every vertex of degree 3 or 4 is a cut-vertex and there is unique vertex v of degree 4 lies on 2 or 3 blocks. From (1) and (2) it follows that G has no induced subgraph $K_{1,4}$ or a subgraph homeomorphic from any one of the graphs of Fig 6.

Conversely, suppose G is a connected plane graph and it does not contain $K_{1,4}$ as induced subgraph or a subgraph homeomorphic from any one of the graphs of Fig 6. We now show that G satisfies the condition (1) or (2) and hence by Theorem 2.1, $Q(G)$ has crossing number one. Suppose there exist at least three non-cut-vertices of degree 3 in G . Then G has at least two diagonal edges. Here we consider two cases depending on whether 2 diagonal edges exist in one block or in two different blocks.

Case 1. If two diagonal edges exist in one block of G , then G has a subgraph homeomorphic from G_1 or G_2 .

Case 2. If two diagonal edges exist in two different blocks of G , then G has a subgraph homeomorphic from G_3 .

In each case we arrived at a contradiction. Hence G has exactly two non-cut-vertices of degree 3.

Suppose G has exactly two nonadjacent non-cut-vertices of degree 3. Then there exist 3 disjoint paths between these two non-cut-vertices of degree 3. Clearly G contains a subgraph homeomorphic from G_4 , a contradiction. Thus G has exactly two adjacent non-cut-vertices of degree 3.

Suppose $\Delta(G) \geq 5$. Then G contains a subgraph homeomorphic from G_5 , a contradiction. Hence $\Delta(G) = 4$.

Let v be a vertex of G and $\deg v = 4$. We prove that v is a cut-vertex. On the contrary, let a, b, c and d be the vertices of G adjacent to v . Then there exist paths between every pair of vertices of a, b, c and d not containing v . Then it is proved in Theorem 1.5, G has a subgraph homeomorphic from $P_4 + K_1$ or $K_2 + \overline{K_3}$, since $P_4 + K_1$ contains G_7 as a subgraph and $K_2 + \overline{K_3}$ contains G_6 as a subgraph, this is a contradiction. Thus v is a cut-vertex and every vertex of degree 4 is a cut-vertex.

Suppose every cut-vertex of degree 4 lies on 4 blocks of G . Let v be the cut-vertex of degree 4 and it lies on 4 blocks. Then $K_{1,4}$ is an induced subgraph of G , a contradiction. Hence v lies on 2 or 3 blocks of G .

Suppose there are at least two cut-vertices of degree 4, each which lies 2 or 3 blocks. Let v_1 and v_2 be the two cut-vertices of degree 4 such that v_1 and v_2 are connected by a path P and let a_i , $i = 1, 2, 3$ and b_j , $j = 1, 2, 3$ be the vertices of adjacent to v_1 and v_2 respectively.

We consider the following cases.

Case 1. Assume v_1 and v_2 both lies on 3 blocks. If there exists a path between a vertex of a_i and a vertex of b_j , then G has a subgraph homeomorphic from G_6 .

Case 2. Assume v_1 and v_2 both lies on 2 blocks. If there exists a path between a vertex of a_i and a vertex of b_j and there is a path between remaining two vertices of a_i and two vertices of b_j , then G contains G_6 as a subgraph.

We have exhausted all the cases and we arrive at the conclusion that G has exactly one cut-vertex of degree 4 which lies on either 2 or 3 blocks of G .

Suppose $\deg v = 4$ and v lies on 2 or 3 blocks of G in which either of the blocks is neither a cycle nor a K_2 . Then G has a subgraph homeomorphic from G_7 or G_8 .

Further suppose $\deg v_1 = 3$ for any other vertex $v_1 \neq v$ of G and v_1 is not a cut-vertex. The proof of the condition (2) of Theorem 2.2 will be completed if we disprove the above statement by deducing a contradiction.

We consider two cases.

Case 1. v is adjacent to v_1 .

Let v_2 and v_3 be the vertices of G adjacent to v_1 . Since v_1 is not a cut-vertex, there exist two shortest paths $Z_1(v-v_3)$ and $Z_2(v-v_2)$ which do not contain v_1 . The minimality of the paths assures that either $v_2 \notin Z_1$ or $v_3 \notin Z_2$. Let x be the first vertex of intersection (starting from v_2) of Z_2 with Z_1 . If $x = v_2$ or v_3 , then in either case G has a subgraph homeomorphic from G_8 [see Fig.7(b_1) or 7(b_2)]. If $x = v$, then G contains a subgraph homeomorphic from G_5 or G_7 [see Fig.7(c)]. If $x \neq v_2, v_3$ and v then G has a subgraph homeomorphic from G_8 [see Fig.7(a)].

Case 2. v is not adjacent to v_1 .

Since G is connected, there exists a path $Z_1(v-v_1)$. Let v_2 and v_3 be the vertices of G adjacent to v_1 and let v_4 be the vertex on Z_1 which divides Z_1 into subpaths $Z'_1(v-v_4)$ and $Z''_1(v_4-v_1)$. Since v_1 is not a cut-vertex, there are two paths $Z_2(v_4-v_2)$ and $Z_3(v_4-v_3)$ which do not contain v_1 . There are 4 subcases to consider depending on whether or not v is on Z_2 and Z_3 .

Subcase 2.1. v is on Z_2 and Z_3 .

Let x be the last but one vertex of Z_2 which also belongs to Z_3 . If $x = v_2$ or v_3 , then in either case we have a subgraph of G that is homeomorphic from G_8 [see Fig.7(e_1) or 7(e_2)]. If $x = v$, then G has a subgraph homeomorphic from G_5 , G_7 or G_4 [see Fig.7(f)]. If $x \neq v_2, v_3$ and v_4 , then G contains a subgraph homeomorphic from G_8 [see Fig.7(d)].

Subcase 2.2. v lies on Z_2 or Z_3 . In either case we have subgraph homeomorphic to G_8 [see Fig.7(g_1) or 7(g_2)].

Subcase 2.3. v does not lies on Z_2 and Z_3 . Again we consider two subcases of subcase 2.3.

Subcase 2.3.1. v lies on three blocks.

a] There exists a path (other than Z'_1) $v-y$ where $y \in Z'_1$.

I] Assume Z_2 and Z_3 are not disjoint. Let x be the last but one vertex of Z_2 which also belongs to Z_3 . If $x = v_2$ or v_3 , then in either case G has a subgraph homeomorphic from G_8 or G_2 [see *Fig.7(i₁)* or *7(i₂)*]. If $x = v_4$, then G contains a subgraph homeomorphic from G_8, G_7 or G_4 [see *Fig.7(j)*]. If $x \neq v_2, v_3$ and v_4 , then G has a subgraph homeomorphic from G_8, G_2 or G_4 [see *Fig.7(h)*]. If $y = v_1$ and v_1, v_3 are not adjacent, then G has a subgraph homeomorphic from G_8 [see *Fig.7(k)*]. If $y \neq v_1$ and v_1, v_3 are not adjacent, then G contains a subgraph homeomorphic from G_8 [see *Fig.7(l)*].

II] Assume Z_2 and Z_3 are disjoint. If $y = v_1$, then G contains a subgraph homeomorphic from G_8 or G_4 [see *Fig.7(m)*]. If $y \neq v_1$, then G has a subgraph homeomorphic from G_7, G_8 or G_4 [see *Fig.7(n)*].

b] There exists a path (other than Z''_1) $v-y$ where $y \in Z''_1$ or there exists a path between pair of adjacent vertices of v .

I] Assume Z_2 and Z_3 are not disjoint. Let x be the last but one vertex of Z_2 which also belongs to Z_3 . If $x = v_2$ or v_3 , then in either case G has a subgraph homeomorphic from G_9 [see *Fig.7(p₁)* or *7(p₂)*]. If $x \neq v_2, v_3$ and v_4 , then G has a subgraph homeomorphic from G_9 or G_4 [see *Fig.7(o)*].

II] Assume Z_2 and Z_3 are adjacent. Then $x = v_4$. In this case G has a subgraph homeomorphic from G_7 [see *Fig.7(q)*].

We have completed all cases. In each case, we found that G contains a subgraph homeomorphic from one of the forbidden subgraphs of Fig. 6. Hence v_1 is a cut-vertex. Thus Theorem 2.2 implies that G has a quick graph with crossing number one.

Figure 7

4 Conclusion

In this paper two characterizations are given for graphs whose qlick graphs have crossing number 1, one in terms of degrees of vertices of a graph and the second in terms of forbidden subgraph. The crossing number $cr(G)$ of G is the least number of intersections of pairs of edges in any embedding of G in the plane. Obviously G is planar if and only if $cr(G)=0$.

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