Homotopy Analysis Method for Conformable Burgers-Korteweg-de Vries Equation

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Abstract. The main goal of this paper is finding the approximate analytical solution of Burgers-Korteweg-de Vries with newly defined conformable derivative by using homotopy analysis method (HAM). Then the approximate analytical solution is compared with the exact solution and comparative tables are given.

Introduction

The adventure of fractional calculus started with the L'Hospital's letter to Leibnitz in 1695. In this letter the original question which caused the name fractional question was: Can a derivative of integer order $\frac{d^n y}{dx^n}$ be extended to have the meaning when n is a fractional number? This question has a golden value for the development of fractional calculus. After that time fractional calculus has been a valuable tool for expressing the nonlinear phenomena in the nature. Thus scientists presented various applications of fractional derivatives and integrals for the physical interpretation the real world circumstances. For this interpretation most of scientists use popular and known derivative formulas such as Riemann-Liouville, Caputo [1, 2, 3]. As time progressed researchers found some insufficiencies at these definitions. It is understood that these definitions do not satisfy basic properties of the Newtonian concept, for instance the product, quotient, the Chain rules and etc. Researchers who study in the field of fractional calculus has been worried about these problems. To overcome this issue, R. Khalil *et al.* [4] introduced a new derivative and its anti-derivative called 'conformable derivative and integral'.

Definition. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. The α^{th} order "conformable derivative" of f is defined by,

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0, 1)$. If f is α -differentiable in some $(0, a), a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ and the "conformable integral" of a function f starting from $a \geq 0$ is defined as:

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

To test the efficiency and the accuracy of the conformable derivative, researchers made huge number of scientific articles on it. For example T. Abdeljawad [5] has presented conformable versions of the chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions and Laplace transform. Iyiola *et al.* [6] expressed the analytical solution of space-time fractional Fornberg-Whitham equation, Hesameddini *et al.* [7] demonstrated the numerical solution

of multi-order fractional differential equations via the sinc collocation method and K. R. Prasad *et al.* [8] discussed prove the existence of multiple positive solutions for a coupled system of iterative type boundary value problems involving new conformable fractional order derivative. Therewithal to these studies O. Acan *et al.* [9] introduced conformable fractional reduced differential transform method using Conformable Calculus and reduced differential transform method. A. Kurt *et al.* [10] used conformable derivative while obtaining the analytical and approximate solution of Burgers' equation. Finally A. Atangana *et al.* [11] introduced the new properties of conformable derivative. Hence, there are many physical applications can be considered in this new area.

In the following theorem some properties of this new definition is given [4].

Theorem 1 Let $\alpha \in (0, 1]$ and f, g α -differentiable at point $t > 0$. Then

1. $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$ for all $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
6. If in addition to f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

Homotopy Analysis Method

In this section, the authors applied HAM to the processed problem. To implement the fundamentals of the method we assume the following differential equation,

$$\mathcal{N}[u(x, t)] = 0$$

where $\mathcal{N}$ is a nonlinear operator, x and t show independent variables and $u(x, t)$ is an unknown function. By using the generalization of HAM, a zero-order deformation equation has been builded by Liao [12, 13]

$$(1 - p)\mathcal{L}[\phi(x, t; p) - u_0(x, t)] = p\hbar\mathcal{N}[\phi(x, t; p)] \quad (1)$$

where $p \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x, t)$ is an initial guess of $u(x, t)$, $\phi(x, t; p)$ is an unknown function, respectively. In this way, it would be thinkable to choose auxiliary parameters and operators in HAM. When p is selected as $p = 0$ and $p = 1$ then

$$\phi(x, t; 0) = u_0(x, t), \phi(x, t; 1) = u(x, t)$$

are gained successively. Hence, as long as the embedding parameter p increases from 0 to 1, the solutions $\phi(x, t; p)$ show a change from the initial value $u_0(x, t)$ to the solution $u(x, t)$. If $\phi(x, t; p)$ is expanded in Taylor series with respect to the embedding parameter p , we have:

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)p^m$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0}. \quad (2)$$

If auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are selected properly, the series which are introduced above converges at $p = 1$, and

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t).$$

it is one of the solutions of the original nonlinear equation, as expressed by Liao [13, 14]. Fittingly the (2), the governing equation can be changed from the zero-order deformation equation (1). Define the vector:

$$\mathbf{u}_n = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}.$$

By differentiating Eq. (1) m times with respect to the embedding parameter p , then adjusting $p = 0$ and dividing by $m!$, we get the m th-order deformation equation:

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar R_m(\mathbf{u}_{m-1}) \quad (3)$$

where

$$R_m(\mathbf{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(x, t; p)]}{\partial p^{m-1}} \Big|_{p=0}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

We emphasize that $u_m(x, t)$ for $m \geq 1$ is governed by Eq. (3) with the boundary condition that comes from the problem. Then it can be solved easily with using symbolic computation software such as Mathematica.

To show the applicability of the method on nonlinear CPDEs, we employ the method to the conformable Burgers-Korteweg-de Vries equation described in the next section. Some of the articles involving HAM can be seen in the Ref. [10, 15]

Application of the HAM

Consider the Burgers-KDV equation [16]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varpi u \frac{\partial u}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2} + s \frac{\partial^3 u}{\partial x^3} = 0 \quad (4)$$

with initial condition which are obtained by giving the values $s = 1$, $\beta = 1$, $\varpi = 1$, to the analytical solution in article [16]

$$u(x, 0) = -\frac{12 \left(\cosh\left(\frac{2}{5}x\right) - \sinh\left(\frac{2}{5}x\right) \right)}{25 \left(1 + \cosh\left(\frac{1}{5}x\right) - \sinh\left(\frac{1}{5}x\right) \right)^2} \quad (5)$$

where $\alpha \in (0, 1)$ and the derivative in the sense of conformable derivative.

To solve the equation (4) with initial condition (5) by aid of homotopy analysis method the consider the linear operator is chosen as

$$\mathcal{L}[\phi(x, t; p)] = D_t^\alpha \phi(x, t; p)$$

with the property

$$\mathcal{L}[s] = 0$$

where s is constant. From the equation (4) and for the given special values of $s = 1$, $\beta = 1$, $\varpi = 1$ the nonlinear operator can be defined as follows,

$$\mathcal{N}[\phi(x, t; p)] = \frac{\partial^\alpha \phi(x, t; p)}{\partial t^\alpha} + \phi(x, t; p) \frac{\partial \phi(x, t; p)}{\partial x} + \frac{\partial^2 \phi(x, t; p)}{\partial x^2} + \frac{\partial^3 \phi(x, t; p)}{\partial x^3}.$$

The property of this new definition which is given therein references [4] nonlinear operator can be written as follows,

$$\mathcal{N}[\phi(x, t; p)] = t^{1-\alpha} \frac{\partial \phi(x, t; p)}{\partial t} + \phi(x, t; p) \frac{\partial \phi(x, t; p)}{\partial x} + \frac{\partial^2 \phi(x, t; p)}{\partial x^2} + \frac{\partial^3 \phi(x, t; p)}{\partial x^3}.$$

Thus the zero-order deformation equation is established as:

$$(1 - p) \mathcal{L}[\phi(x, t; p) - u_0(x, t)] = p \hbar \mathcal{N}[\phi(x, t; p)]. \quad (6)$$

Exactly, by choosing $p = 0$ and $p = 1$ then we get

$$\phi(x, t; 0) = u_0(x, t) = u(x, 0), \phi(x, t; 1) = u(x, t).$$

Therefore, since the embedding parameter p differs from 0 to 1, the solution $\phi(x, t; p)$ differs from the initial value $u_0(x, t)$ to the solution $u(x, t)$. By expanding $\phi(x, t; p)$ in Taylor series with dependent to the embedding parameter p :

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) p^m \quad (7)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0}. \quad (8)$$

Regarding that the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are suitably chosen, the above series converges at $p = 1$, and

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)$$

which must be one of the solutions of the original nonlinear equations, as proven by Liao [13]. After then if we differentiate Eq. (6) m times with respect to the embedding parameter p , we have the m th-order deformation equation:

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar R_m(\mathbf{u}_{m-1}) \quad (9)$$

where

$$\begin{aligned} R_m(\mathbf{u}_{m-1}) &= t^{1-\alpha} \frac{\partial u_{m-1}(x, t)}{\partial t} + \frac{\partial^2 u_{m-1}(x, t)}{\partial x^2} + \frac{\partial^3 u_{m-1}(x, t)}{\partial x^3} \\ &+ \sum_{n=0}^{m-1} u_n(x, t) \frac{\partial u_{m-1-n}(x, t)}{\partial x} \end{aligned} \quad (10)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

The solutions of the m th-order deformation Eq. (9) for $m \geq 1$ result in

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar \mathcal{L}^{-1}[R_m(\mathbf{u}_{m-1})]. \quad (11)$$

By using Eq. (11) with initial condition given by (5) we respectively obtain

$$\begin{aligned} u_0(x, t) &= -\frac{12 \left(\cosh \left(\frac{2}{5}x \right) - \sinh \left(\frac{2}{5}x \right) \right)}{25 \left(1 + \cosh \left(\frac{1}{5}x \right) - \sinh \left(\frac{1}{5}x \right) \right)^2}, \\ u_1(x, t) &= -\frac{144e^{x/5}ht^\alpha}{3125(1+e^{x/5})^3\alpha}, \\ u_2(x, t) &= \frac{144e^{x/5}ht^\alpha}{3125(1+e^{x/5})^3\alpha} - \frac{144e^{x/5}h^2t^\alpha \left((-3+6e^{x/5})t^\alpha + 125(1+e^{x/5})\alpha \right)}{390625(1+e^{x/5})^4\alpha^2}, \\ &\vdots \end{aligned}$$

So, the series solutions found out by HAM can be written in the form

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \quad (12)$$

The series solutions of $u(x, t)$ can be obtained by using the Eq. (12).

The auxiliary parameter $\hbar$, which is contained in our HAM solution series, supplies us with a simple way to arrange and control the convergence of the solution series. To obtain an proper range for $\hbar$, we consider the so-called $\hbar$ -curves for different values of α which is shown in Figure 1 to choose a appropriate value of $\hbar$ which provide that the solution series is convergent, as pointed by Liao[13], by finding the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis.

Fig. 1: The $\hbar$ -curves of 5th-order approximate solutions obtained by the HAM for different values of α .

To show the applicability of the method, the HAM solutions given by Eq. (12) of the conformable Burgers-KDV Equation are compared with its exact solution given in [16]

$$u(x, t) = -\frac{12 \left(\cosh \left(\frac{2}{5} \left(x - \frac{6t^\alpha}{25\alpha} \right) \right) - \sinh \left(\frac{2}{5} \left(x - \frac{6t^\alpha}{25\alpha} \right) \right) \right)}{25 \left(1 + \cosh \left(\frac{1}{5} \left(x - \frac{6t^\alpha}{25\alpha} \right) \right) - \sinh \left(\frac{1}{5} \left(x - \frac{6t^\alpha}{25\alpha} \right) \right) \right)^2}. \quad (13)$$

Here is the absolute error graphics are given for the values given in Figure 2 below.

Fig. 2: The absolute error graphics of 5th-order approximate solutions obtained by the HAM for, $\hbar = -0.001$ different values of α .

Conclusion

In this study authors handle the Burgers-Korteweg-de Vries equation with conformable derivative. While obtaining the approximate analytical solution of the problem, HAM which is an efficient and useful technique for the solution of nonlinear problems is used. Numerical solutions are compared with the analytical solutions which are given in the [16]. Finally the absolute error graphics are given.

References

- [1] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, NY, USA, 1993.
- [2] H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, San Diego, 2006.
- [3] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, Calif, USA, 1999.
- [4] R. Khalil et al., A new definition of fractional derivative, Journal of Computational and Applied Mathematics. 264 (2014) 65-70.
- [5] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math. 279 (2015) 57-66.
- [6] O.S. Iyiola, G.O. Ojo, On the analytical solution of Fornberg-Whitham equation with the new fractional derivative, Pramana Journal of Physics. 85 (2015) 567-575.
- [7] E. Hesameddini, E. Asadollahifard, Numerical solution of multi-order fractional differential equations via the sinc collocation method, Iranian Journal of Numerical Analysis and Optimization. 5 (2015) 37-48.
- [8] K. R. Prasad, B. M. B. Krushna, Existence of multiple positive solutions for a coupled system of iterative type fractional order boundary value problems, J. Nonlinear Funct. Anal. 2015 (2015), Article ID 11.

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- [9] O. Acan et al., Solution of Conformable Fractional Partial Differential Equations by Reduced Differential Transform Method, Selcuk Journal of Applied Mathematics, 2016 (In Press).
 - [10] A. Kurt, Y. Cenesiz, O. Tasbozan, On the Solution of Burgers-Equation with the New Fractional Derivative, Open Phys. 13 (2015) 355.
 - [11] A. Atangana, D. Baleanu, A. Alsaedi, New properties of conformable derivative, Open Math. 13 (2015) 889-898.
 - [12] S.J. Liao, Homotopy analysis method and its application, Ph.D thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.
 - [13] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.
 - [14] S.J. Liao, Notes on the homotopy analysis method: Some definitions and theorems, Commun. Nonlinear Sci. Numer. Simulat. 14(4) (2009) 983-997.
 - [15] A. Kurt, O. Tasbozan, Approximate Analytical Solution of the Time Fractional Whitham-Broer-Kaup Equation Using the Homotopy Analysis Method, Int. J. of Pure and Appl. Math. 98 (2015) 503.
 - [16] Y. Cenesiz et al., New Exact Solutions of Burgers' Type Equations with Conformable Derivative, Waves in Random and Complex Media. (2016) 1-14.